Quantum geometry I: Basics of loop quantum gravity. The quantum polyhedra

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General Relativity describes gravity in geometrical terms. This suggests that quantizing such theory is the same as quantizing geometry. The subject can therefore be called quantum geometry and one may think that mathematicians are responsible of this subject. Unfortunately, most mathematicians are not aware of this beautiful area of study. Here we give a basic introduction to what quantum geometry means to a community working in a theory known as loop quantum gravity. It is directed towards graduate or upper students of physics and mathematics. We do it from a point of view of a mathematician.

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1. Introduction

The two most important physical theories of the 20th century are, of course, general relativity and quantum mechanics. It is very well known to a physicist that classical physical theories have a quantum version. The question then is: what is the quantum version of general relativity? The thing is that there is no known correct answer to this question.

Physically the question is what is quantum gravity? It is impressive that such problem has been studied using many different directions; each one of these directions claims their theory is the solution to such problem.

Loop quantum gravity [1-5] is one of these many directions. Among all of these directions, loop quantum gravity is the second most studied one, just after string theory.

There is no easy way to start learning loop quantum gravity. It is a difficult theory, there is plenty of literature out there most of which is very technical, and in fact there are many different problems on which people are working.

In this paper we give a basic introduction to only one of the constructions of loop quantum gravity. We selected this particular problem because we personally think it is the easiest one of all and, in fact, it is very beautiful.

The problem is the following. It is known that general relativity is a theory of gravity which is described in geometrical terms. Therefore, quantizing general relativity must be equivalent to quantizing classical geometry.

We can now rephrase the question what is quantum geometry? And this is again a question with no good answer, because quantizing classical geometry may mean a different thing to different scientific communities. For example, it may mean something to a mathematician which is very different from what a physicist thinks. Mathematically speaking quantum geometry may refer to a theory known as noncommutative geometry [9] or in fact, it may refer to loop quantum gravity. There have been some studies relating some ideas of loop quantum gravity to noncommutative geometry. We do not explore this latter problem here.

The problem we introduce here is[i]: if we want to start understanding what quantum geometry may be, we should ask ourselves; is there a quantum version of a classical polyhedron.

It turns out that the answer is yes. Classical polyhedra such as the Platonic solids for example have a quantum version. This is very exciting in fact. However, unfortunately mathematicians are not aware of this fact and it may be because this idea emerged in loop quantum gravity which is a theory mostly invented by physicists. This is why our intention is to spread the idea to the mathematical community, and therefore give a basic introduction from the point of view of a mathematician.

This review is directed from a mathematician point of view towards advanced undergraduates or postgraduates in mathematics and in physics. The mathematics of Sec. 2 will be more familiar to mathematicians, whereas the mathematics of Sec. 3 will be more familiar to physicists.

We will start by recalling what classical polyhedra are and then will describe the quantum analogues.

2. Classical Polyhedra

This section is based on reference [10]; however it is all written as we understand it, that is, our own words. A classical polyhedron P is just a solid in three dimensional space $\mathbb{R}^3$, ($P \subset \mathbb{R}^3$) such that ∂P is composed of a finite number k of flat polygons[ii]. The polygons forming the polyhedron are called faces, and the sides and vertices of the faces are called edges and vertices. We denote the set of k faces of the polyhedron by $f_1, f_2, \dots, f_k$.

We will restrict ourselves to convex polyhedra. Through each flat face f_i of a classical polyhedron there exists a plane P_i that contains it. A convex polyhedron Π is a classical poly-

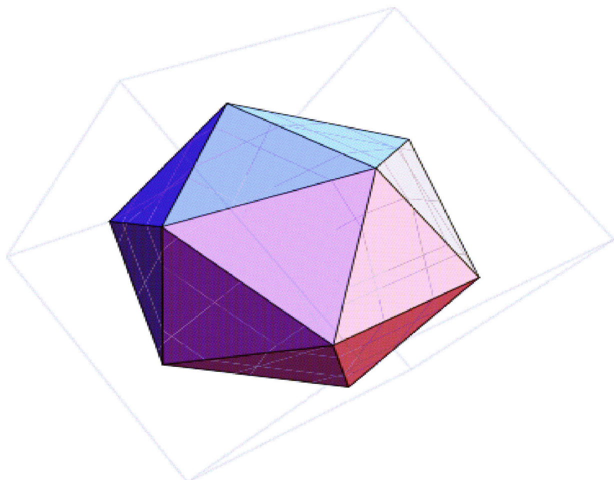

FIGURE 1. A Convex Polyhedron.

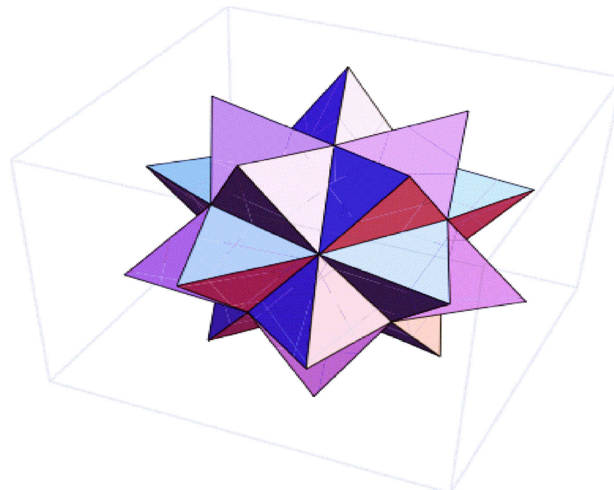

FIGURE 2. A Non Convex Polyhedron.

hedron such that any two polygonal faces $f_i \neq f_j$ are connected through other faces with common edges, and given a plane P_i which contains the f_i face, we have that $P_i \cap \Pi = f_i$ for all $i = 1, \ldots, k$. Figure 1 shows a convex polyhedron, whereas the polyhedron of Fig. 2 is not convex.

We will consider bounded convex polyhedra, that is, polyhedra with bounded faces.

Given a convex polyhedron Π, consider P_i, the plane which contains the face f_i. The unit vector $\mathbf{n}_i$ perpendicular to P_i and pointing to the side which does not contain any points of Π is called the outward normal of P_i relative to Π.

Now the most important theorem of this section^{iii}.

Theorem 1 (Minkowski) *Let $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$ be unit vectors, $k \geq 4$, such that any three different vectors $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_\ell$, are linearly independent.*

Let $A(f_1), A(f_2), ..., A(f_k) \in \mathbb{R}_{>0}$ such that

$$\sum_{i=1}^{k} A(f_i)\mathbf{n}_i = 0$$

Then there exists a closed convex polyhedron Π with faces f_i having areas $A(f_i)$ and outward normals $\mathbf{n}_i$.

This theorem implies that under given conditions there exist a convex polyhedron which satisfies the prescribed conditions^{iv}.

However, it also turns out that given a convex polyhedron whose faces f_i have areas $A(f_i)$ and whose outward unit normals to the faces are $\mathbf{n}_i$, the equation

$$\sum_{i=1}^{k} A(f_i)\mathbf{n}_i = 0 \tag{1}$$

is satisfied. This is easy to see.

Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. Consider the Euclidean inner product $< \mathbf{n} \, , \mathbf{n}_i >$ of the unit vector $\mathbf{n}$ with all of the unit normals to the faces of the convex polyhedron Π. The number $< \mathbf{n} \, , \mathbf{n}_i > A(f_i)$ is the area of the projection of the face f_i into a plane $\mathbf{n}^\perp$ whose all vectors are orthogonal to $\mathbf{n}$.

All of the unit vectors $\mathbf{n}, \mathbf{n}_1, \ldots, \mathbf{n}_k \in S^2$, where

$$S^2 = \{(z, x, y) \in \mathbb{R}^3 \mid z^2 + x^2 + y^2 = 1\}$$

is the unit sphere; therefore we have that some of the interior products $< \mathbf{n} \, , \mathbf{n}_i >$ will be positive and some others will be negative, since some of the $\mathbf{n}_i$ point in the same direction as $\mathbf{n}$, and some point in the opposite direction. The projection to the plane $\mathbf{n}^\perp$ of the faces whose normal vectors $\mathbf{n}_i$ point in the same direction as $\mathbf{n}$ and the projection of those whose normal vectors $\mathbf{n}_i$ point in the opposite direction as $\mathbf{n}$ cover the same area. Then

$$\sum_{i=1}^{k} < \mathbf{n} \, , \mathbf{n}_i > A(f_i) = 0$$

$\Rightarrow$

$$< \mathbf{n} \, , \sum_{i=1}^{k} A(f_i)\mathbf{n}_i >= 0$$

Since $\mathbf{n} \neq \vec{0}$, and it is an arbitrary vector, we have that

$$\sum_{i=1}^{k} A(f_i)\mathbf{n}_i = 0$$

Therefore equation (1) is proved.

3. Quantum Polyhedra

Classical physical theories have a quantum version^{v}. The question is can mathematics be quantized? Well, let us start by asking, is there a quantum version of a classical convex polyhedron described in the previous section?

Surprisingly there is, and loop quantum gravity has described these quantum versions [5-7]. But the idea can be generalised to quantizing any convex polyhedron. In this section we describe the quantum version of a classical convex polyhedron. We give a very basic introduction to this beautiful subject. It is in fact a hard thing to do since the subject is full of difficult and advanced mathematics. At least in this first introduction paper we keep it simple. This section is mathematically inspired in [11].

There is an action of the Lie group $SO(3)$ of rotations in the Euclidean space $\mathbb{R}^3$, and the areas $A(f_i)$ of the faces f_i of the convex polyhedron Π as well as its volume remain invariant under such rotations. The set of the normal unit vectors $\mathbf{n}_i$ to the faces obviously remain unit vectors.

Quantizing a convex polyhedron is defined by assigning a Hilbert space $\mathcal{H}_i$ to each of its faces f_i and the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$ to the polyhedron in the following way. This implies that the observables are related to measures on the quantum polyhedron faces. In classical geometry a polyhedron Π has faces f_i of certain area $A(f_i)$. Area in classical geometry is a classical observable. Therefore its quantum counterpart is called quantum area and it must be an operator defined on a Hilbert space. This is understood as follows.

As $SO(3)$ sends the unit sphere S^2 to itself, the Hilbert space associated to each face is in fact $\mathbf{L}^2(S^2)$, the space of complex valued squared-integrable functions[vi]. That is,

$$\mathbf{L}^2(S^2) = \left\{ \psi : S^2 \to \mathbb{C} \mid \int_{S^2} \mid \psi(\vec{x}) \mid^2 d\vec{x} < \infty \right\}$$

Just as the Lie group of rotations $SO(3)$ acts in S^2, it also acts in the Hilbert space $\mathbf{L}^2(S^2)$ by

$$R\,\psi(\vec{x}) := \psi(R^{-1}\,\vec{x})$$

where $R : S^2 \to S^2$ is a rotation and R^{-1} is its inverse.

To the polyhedron we assign the tensor product $\otimes_k\ \mathbf{L}^2(S^2)$, so that a vector in $\otimes_k\ \mathbf{L}^2(S^2)$ is given by $\psi_1(\vec{x_1}) \otimes \psi_2(\vec{x_2}) \otimes \cdots \otimes \psi_k(\vec{x_k})$ such that the $SO(3)$ action on this tensor product space is given by

$$R(\psi_1(\vec{x_1}) \otimes \psi_2(\vec{x_2}) \otimes \cdots \otimes \psi_k(\vec{x_k})) := \psi_1(R^{-1}\,\vec{x_1}) \otimes \psi_2(R^{-1}\,\vec{x_2}) \otimes \cdots \otimes \psi_k(R^{-1}\,\vec{x_k})$$

Physically, the wave function of a quantum polyhedron is a complex valued function defined on the tensor product $\otimes_k\ \mathbf{L}^2(S^2)$ Hilbert space, such that $\psi_1(\vec{x_1}) \otimes \psi_2(\vec{x_2}) \otimes \cdots \otimes \psi_k(\vec{x_k})$ is a unit vector in the Hilbert space $\otimes_k\ \mathbf{L}^2(S^2)$. Mathematically this is written

$$\int\limits_{S^2 \times S^2 \cdots \times S^2} \mid \psi_1(\vec{x_1}) \otimes \psi_2(\vec{x_2}) \otimes \cdots \otimes \psi_k(\vec{x_k}) \mid^2 \times d\vec{x_1}\, d\vec{x_2} \ldots d\vec{x_k} = 1$$

where the integral is over k products of S^2. In fact this latter integral is given by

$$\prod_{i=1}^{k} \int\limits_{S^2} \mid \psi_i(\vec{x_i}) \mid^2 d\vec{x_i}.$$

When we study quantum mechanics, we know that the wave function of a system is a superposition (linear combination of basis vectors in the Hilbert vector space) of states which are eigenvectors of an observable (self-adjoint operator in the Hilbert space). The eigenvalues of the observable are the observed quantities with a certain probability.

When we say that the quantum polyhedron has a wave function, or is in the state $\psi_1(\vec{x_1}) \otimes \psi_2(\vec{x_2}) \otimes \cdots \otimes \psi_k(\vec{x_k})$, we must understand that it is a superposition state. What are the observed quantities? What is an observable in this theory of quantum polyhedra? In order to answer these questions we should know some more things. Let us discuss these issues.

The Hilbert space $\mathbf{L}^2(S^2)$ of squared-integrable functions over S^2 has an inner product given by

$$< \psi(\vec{x}) \mid \chi(\vec{x}) > = \int\limits_{S^2} \overline{\psi(\vec{x})}\, \chi(\vec{x})\, d\vec{x}$$

If we introduce spherical coordinates in S^2

$$f(\theta, \phi) = (\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

such that $0 < \theta < \pi, 0 < \phi < 2\pi$. Then the functions $\psi(\vec{x})$ become functions of the spherical angles $\psi(\theta,\phi)$ and the inner product can be written explicitly as

$$\begin{aligned} &< \psi(\theta,\phi) \mid \chi(\theta,\phi) > \\ &= \frac{1}{4\pi} \int\limits_{S^2} \overline{\psi(\theta,\phi)}\, \chi(\theta,\phi) \sin\theta\, d\theta\, d\phi \end{aligned}$$

4π is the area of the unit sphere, or in other words[vii].

In this Hilbert space the observables include the self-adjoint operators $J_1, J_2, J_3 : \mathrm{Dom}(\mathbf{L}^2(S^2)) \to \mathbf{L}^2(S^2)$ given by

$$\begin{aligned} J_1 &= i \left(\sin\phi\, \frac{\partial}{\partial\theta} + \cos\phi\, \frac{\cos\theta}{\sin\theta}\, \frac{\partial}{\partial\phi} \right) \\ J_2 &= i \left(-\cos\phi\, \frac{\partial}{\partial\theta} + \sin\phi\, \frac{\cos\theta}{\sin\theta}\, \frac{\partial}{\partial\phi} \right) \\ J_3 &= -i\, \frac{\partial}{\partial\phi} \end{aligned}$$

and the commutation relations of these operators are given by

$$[J_1, J_2] = i\,J_3 \quad , \quad [J_2, J_3] = i\,J_1 \quad , \quad [J_3, J_1] = i\,J_2$$

There is also an operator known as the Casimir operator given by

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

Using the expressions for J_1, J_2, J_3 it can be seen that

$$J^2 = -\frac{1}{\sin\theta} \left(\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2}{\partial\phi^2} \right)$$

This latter expression is the minus Laplacian on the sphere which has eigenvectors given by the well known spherical harmonics functions $Y(\theta,\phi)$. This means that

$$J^2\, Y(\theta,\phi) = j(j+1)\, Y(\theta,\phi) \qquad (2)$$

where $j \in \mathbb{Z}_{\geq 0}$. Each eigenvalue $j(j+1)$ is of multiplicity $2j+1$ and therefore the eigenvectors of the operator J^2 with eigenvalue $j(j+1)$ generate a subspace H_j of $\mathbf{L}^2(S^2)$. This implies that the Hilbert space $\mathbf{L}^2(S^2)$ is a direct sum given by

$$\mathbf{L}^2(S^2) = \bigoplus_{j=0}^{\infty} H_j$$

It is customary to denote the orthogonal basis of eigenvectors with eigenvalue $j(j+1)$ that generate the subspace H_j by $Y_m^j(\theta,\phi)$ where m takes integer values $-j \leq m \leq j$.

In loop quantum gravity the observable J is the area operator[viii], and formula (2) is interpreted physically as the squared area of face f_i of the quantum polyhedron Π. Face f_i has therefore quantized area given by the numbers

$$A(f_i) = \sqrt{j_i(j_i+1)}$$

On the other hand, a general vector $\psi(\theta,\phi)$ (wave function) in the Hilbert space $\mathbf{L}^2(S^2)$ is a linear combination of bases vectors (superposition) given by

$$\psi(\theta,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c_m^j \, Y_m^j(\theta,\phi)$$

where $c_m^j \in \mathbb{C}$.

A wave function of a quantum polyhedron is given by

$$\psi_1(\theta_1,\phi_1) \otimes \psi_2(\theta_2,\phi_2) \otimes \cdots \otimes \psi_k(\theta_k,\phi_k)$$

where k is the number of faces of the classical polyhedron. It is of course a linear combination(superposition of states) of basis vectors which can be written as

$$\bigotimes_{i=1}^{k} \psi_i(\theta_i,\phi_i) = \sum_{j_i=0}^{\infty} \sum_{m_i=-j_i}^{j_i} \prod_{i=1}^{k} c_{m_i}^{j_i} \bigotimes_{i=1}^{k} Y_{m_i}^{j_i}(\theta_i,\phi_i)$$

After a measurement of the observable J the quantum polyhedron will be in a particular state

$$Y_{m_1}^{j_1}(\theta_1,\phi_1) \otimes Y_{m_2}^{j_2}(\theta_2,\phi_2) \otimes \cdots \otimes Y_{m_k}^{j_k}(\theta_k,\phi_k)$$

This implies that we have a quantum polyhedron which area faces are quantized and the total area of the quantum surface is[ix]

$$A(\Pi) = \ell_P^2 \sum_{i=1}^{k} \sqrt{j_i(j_i+1)}$$

where ℓ_P is the Planck length and it is introduced in the previous formula in order to have the correct dimensions.

4. Conclusions

This short review was intended to be a simple first introduction to one particular subject of loop quantum gravity; quantum polyhedra. It was directed to undergraduate or to first year postgraduate students in physics and mathematics. It was our intention to describe it from the perspective of a mathematician, and we hope we have succeeded in this task.

It is our intention to continue introducing loop quantum gravity to mathematicians, since most mathematicians are not aware of the beautiful subject called loop quantum gravity.

As this is a first introduction we have left so many things out; loop quantum gravity is a very extensive field and no first introduction will be satisfactory. Even dealing with quantum polyhedra requires more formal, and advanced mathematics we have not dealt with.

From what we studied in this first introduction, we have learnt that quantum polyhedra states are superposed and once we have performed a measure of its faces areas the superposition collapses to a polyhedron which faces have discrete areas. This means that the area operator is quantised and therefore we have a first glimpse of what quantum geometry is form the perspective of loop quantum gravity.

When quantizing geometry, area is not continuous but discrete. It happens the same when considering a volume operator and finding that its spectrum is discrete. We did not consider the volume operator here, since it is more complicated. But physicists of loop quantum gravity interpret the discrete spectrums as thinking of space formed by quantum entities called quanta of space.

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i. The one we consider is the simplest one in order to start understanding the idea behind loop quantum gravity.

ii. ∂P denotes the boundary of the polyhedron P.

iii. In this theorem we use loop quantum gravity notation when referring to face areas.

iv. For a proof of theorem 1 we refer the reader to [10].

v. For instance, a quantum version of space exists. See for example [8].

vi. $SO(3)$ not only acts on the Hilbert space $\mathbf{L}^2(S^2)$, it can also act on $\mathbf{L}^2(\mathbb{R}^3)$ for instance. However we choice the action restricted to $\mathbf{L}^2(S^2)$ since our equations will not depend on the radial coordinate.

vii. $\frac{1}{4\pi}\int_{S^2} \sin\theta \, d\theta \, d\phi = 1$

$viii$. The relation of the observable operator J and an area operator is a construction derived in Loop Quantum Gravity. This relation derivation is out of the scope of this review and we do not plan to deal with it at the moment. However it is our intention to have a new review in a future and it will be explained there.

ix. We have studied a very simplified problem. We have not dealt for instance with more complicated mathematics behind quantum polyhedra, like the theory of representations, including the quantum version of classical formula (1). We will deal with this in a future review.

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