Easy route to Tchebycheff polynomials

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Received 5 April 2018; accepted 23 April 2018

Tchebycheff Polynomials are obtained thorough linear algebra methods. A matrix corresponding to the Tchebycheff differential operator is found and its eigenvalues are obtained. The elements of the eigenvectors obtained correspond to the Tchebycheff polynomials.

Keywords: Tchebycheff; Tchebycheff polynomials; special functions

Se obtienen los polinomios de Tchebycheff usando métodos de álgebra lineal. Se encuentra la matriz correspondiente al operador diferencial de Tchebycheff y sus eigenvalores y eigenvectores son obtenidos. Los elementos de los eigenvectores obtenidos corresponden a los polinomios de Tchebycheff.

Descriptores: Tchebycheff, polinomios de Tchebycheff, funciones especiales.

PACS: 02.30.Gp; 02.10.Ud; 01.40.gb

1. Introduction

Tchebycheff differential equation and its solutions, *i.e*. Tchebycheff polynomials, are found in many important physics, mathematics, and engineering problems. A capacitor whose plates are two eccentric spheres is an interesting example [1], another one can be found in aircraft aerodynamics [2]. Many more applications are found in systems theory in connection with approximation theory and numerical analysis. Tchebycheff polynomials are studied in most science and engineering mathematics courses, mainly in those courses focused on differential equations or special functions. These polynomials are typically obtained as a result of the solution of Tchebycheff differential equation by power series. Usually, it is also shown that they can be obtained by a generating function, and also by Rodriguez formula for Tchebycheff polynomial. Finally, they can also be defined as a contour integral. Most mathematics courses also include a study of the properties of these polynomials such as: orthogonality, completeness, recursion relations, special values, asymptotic expansions, and relation to other polynomials and hypergeometric functions [3,4]. There is no doubt that this is a demanding subject that requires a great deal of attention from most students.

In this paper, Tchebycheff polynomials are obtained using basic concepts of linear algebra (which most students are already familiar with) and which contrast in simplicity with the standard methods as those described in the previously outlined syllabus. In the next section the Tchebycheff differential operator matrix is obtained as well as its eigenvalues and eigenvectors. From the eigenvectors found, the Tchebycheff polynomials follow. The method here presented has been applied to other polynomials such as Gegenbauer, Hermite and Laguerre [5-7].

2. Tchebycheff Polynomials

The algebraic polynomial of degree N,

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \tag{1}$$

with $a_0, a_1, \ldots, a_n \in \Re$, is represented by the vector:

$$A_n = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \tag{2}$$

Taking first derivative of the above polynomial (1) one obtains the polynomial:

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n\right) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} \tag{3}$$

This may be written as:

$$\frac{dA_n}{dx} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \tag{4}$$

Taking the second derivative of polynomial (1) one obtains:

$$\frac{d^2}{dx^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n\right) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots + n(n-1) a_n x^{n-2} \tag{5}$$

or,

$$\frac{d^2A_n}{dx^2} = \begin{bmatrix} 2a_2 \\ 6a_3 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Equation (4) may be written as:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \tag{7}$$

Therefore the first derivative operator of A_n may be written as:

$$\frac{d}{dx} \to \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{8}$$

In a similar manner, equation (6) may be written as:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Therefore the second derivative operator of A_n may be written as:

$$\frac{d^2}{dx^2} \to \begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{10}$$

The Tchebycheff differential operator is given by:

$$(1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} \tag{11}$$

Which using Eqs. (3) and (5) may be written as:

$$\begin{aligned} &[2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^4 + \cdots \\ &\quad + n(n-1)a_nx^{n-2}] - x^2[2a_2 + 6a_3x + 12a_4x^2 \\ &\quad + 20a_5x^4 + \cdots + n(n-1)a_nx^{n-2}] - x[a_1 + 2a_2x \\ &\quad + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1}] = [2a_2 + 6a_3x \\ &\quad + 12a_4x^2 + 20a_5x^3 + \cdots + n(n-1)a_nx^{n-2}] \\ &\quad - [2a_2x^2 + 6a_3x^3 + 12a_4x^4 + \cdots + n(n-1)a_nx^n] \\ &\quad - [a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \cdots + na_nx^n] \\ &\quad = 2a_2 + (6a_3 - a_1)x + (12a_4 - 4a_2)x^2 \\ &\quad + (20a_5 - 9a_3)x^3 + (30a_6 - 16a_4)x^4 + \cdots \end{aligned} \tag{12}$$

Which may be written as:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 6 & 0 & \cdots & 0 \\ 0 & 0 & -4 & 0 & 12 & \cdots & 0 \\ 0 & 0 & 0 & -9 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -2n \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 - a_1 \\ 12a_4 - 4a_2 \\ 20a_5 - 9a_3 \\ \vdots \\ -2na_n \end{bmatrix} \tag{13}$$

Therefore, for the sake of simplicity, as a 4×4 matrix, the Tchebycheff differential operator is represented by the following matrix:

$$(1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} \to \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix} \tag{14}$$

The eigenvalues of a matrix M are the values that satisfy the equation $Det(M - \lambda I) = 0$. However, since the matrix (14) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -4$, $\lambda_4 = -9$. The corresponding eigenvectors are the solutions of the equation $(M - \lambda_i I) \cdot v = 0$, where the eigenvector $v = [a_0, a_1, a_2, a_3, a_4]^T$.

$$\begin{bmatrix} 0-\lambda_i & 0 & 2 & 0 \\ 0 & -1-\lambda_i & 0 & 6 \\ 0 & 0 & -4-\lambda_i & 0 \\ 0 & 0 & 0 & -9-\lambda_i \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

Substituting in Eq. (1) the first eigenvalue $\lambda_1 = 0$, one obtains the eigenvector v_1:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

The elements of this eigenvector corresponds to the first Tchebycheff polynomial, $T_0(x) = 1$.

Substituting in Eq. (15) the second eigenvalue $\lambda_2 = -1$, one obtains the eigenvector v_2:

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

The elements of this eigenvector corresponds to the second Tchebycheff polynomial, $T_1(x) = x$.

Substituting in Eq. (15) the third eigenvalue $\lambda_3 = -4$, one obtains the eigenvector v_3:

$$v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \tag{18}$$

The elements of this eigenvector corresponds to the third Tchebycheff polynomial, $T_2(x) = -1 + 2x^2$.

Substituting in Eq. (15) the fourth eigenvalue $\lambda_4 = -9$, one obtains the eigenvector v_4:

$$v_4 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix} \tag{19}$$

The elements of this eigenvector corresponds to the fourth Tchebycheff polynomial, $T_3(x) = -3x + 4x^3$.

Using a larger matrix, higher order polynomials may be obtained.

3. Conclusion

Tchebycheff polynomials are obtained using basic linear algebra concepts, such as the eigenvalue and eigenvector of a matrix. Once the corresponding matrix of the Tchebycheff differential operator is obtained, the eigenvalues of this matrix are found and the elements of its eigenvectors correspond to the Tchebycheff Polynomials.

Acknowledgments

The author acknowledge with gratitude support and useful conversations with Prof. Ernst Wintner from TU-Wien and Dr. Klaus Huber from Berlin.

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