Electrostatic simulation of the Jackiw-Rebbi zero energy state

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Received 25 April 2018; accepted 7 June 2018

We present an analogy between the one dimensional Poisson equation in inhomogeneous media, and the Dirac equation in one space dimension with a Lorentz scalar potential for zero energy. We illustrate how the zero energy state in the Jackiw-Rebbi model can be implemented in a simple one dimensional electrostatic setting by using an inhomogeneous electric permittivity and an infinite charged sheet. Our approach provides a novel insight into the Jackiw-Rebbi zero energy state, and provides a helpful way to visualize and teach this important quantum field theory model using basic electrostatics.

Keywords: Poisson equation; Dirac equation; Jackiw-Rebbi model.

PACS: 42.25.Bs; 42.82.Et; 42.50.Xa; O3.65.Pm

1. Introduction

The Dirac equation is one of the fundamental equations in theoretical physics that accounts fully for special relativity in the context of quantum mechanics for elementary spin-1/2 particles [1]. The Dirac equation plays a key role to many exotic physical phenomena such as graphene, [2] topological insulators [3] and superconductors [4]. These systems proved to be ideal testing grounds for theories of the coexistence of quantum and relativistic effects in condensed matter physics.

Recently, a significant number of studies have addressed the problem of simulating relativistic quantum mechanics using different physical platforms such as optical structures, [5, 6] metamaterials [7] and ion traps [8]. These studies are based on the mathematical analogies found between different physical theories, which provide a way to explore at a macroscopic level many quantum phenomena which are currently inaccessible in microscopic quantum systems. Among the wide variety of quantum-classical analogies investigated so far it appears that the most fruitful one is given by the analogy between optics with quantum phenomena due naturally to the duality between matter and optical waves. The study of quantum-optical analogies is based on the formal similarity between the paraxial optical wave equation in dielectric media and the single particle Schrödinger equation [9]. Among the wide variety of quantum-optical analogies we can mention the Bloch oscillations and Zener tunneling, dynamic localization, Anderson localization, quantum Zeno effect, Rabi flopping and coherent population trapping. All this progress has led to the area of research of how relativistic quantum systems can be mimic by optical waves. More recently, optical systems governed by the relativistic Dirac equation have been investigated experimentally such as Klein Tunneling, Zitterbewegung and the Jackiw-Rebbi model.

The purpose of this article is to demonstrate that electrostatics can provide a laboratory tool where physical phenomena described by the Dirac equation can be explored. In particular, we demonstrate that the Poisson equation in one dimensional inhomogeneous media can be mapped into the zero energy state of the Dirac equation in one dimension with a Lorentz scalar potential. By tailoring the electric permittivity we propose an electrostatic experiment that simulates a historically important relativistic model known as the Jackiw-Rebbi model [10]. Since the derivation of this important model many useful variations of the Jackiw-Rebbi model have been investigated, such as the Ramajaran-Bell model [11], the massive Jackiw-Rebbi model [12], the coupled fermion-kink model [13] and the Jackiw-Rebbi model in distinct kink like backgrounds [14].

The article is organized as follows. First, we will start with a brief review of the Jackiw-Rebbi model and how one can obtain the zero energy state of the JR model. Then we will show how the Poisson equation can be mapped into a Dirac-like equation, and illustrate how the zero energy state in the Jackiw-Rebbi model can be implemented in a simple one dimensional electrostatic setting by using an infinite charged sheet separating two different media. The conclusions are summarized in the last section.

2. Jackiw-Rebbi model in one dimension

The Jackiw-Rebbi model describes a one dimensional Dirac field coupled to a static background soliton field, and is known as one of the earliest theoretical description of a topological insulator where the zero energy mode can be understood as the edge state. In particular, the Jackiw-Rebbi model has been studied by Su, Shrieffer and Heeger in the continuum limit of polyacetylene [15]. The one dimensional Dirac equation in the presence of an external field $\varphi(x)$ and with $\hbar = c = 1$ is given by

$$\hat{H}_D \Psi(x) = [\sigma_y \hat{p} + \sigma_x \varphi(x)]\, \Psi(x) = \mathcal{E}\Psi(x) \tag{1}$$

where

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{and} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2}$$

We use the Pauli matrices σ_x and σ_y in order to have a real two component spinor $\Psi(x)$. From Eq. (2) it follows that the Dirac Hamiltonian possesses a *chiral* symmetry defined by the operator σ_z, which anticommutes with the Dirac Hamiltonian, *i.e.* $\{\hat{H}_D, \sigma_z\} = 0$. The *chiral* symmetry implies that eigenstates come in pairs with positive and negative energy $\pm\mathcal{E}$, respectively. It is possible, however, for an eigenstate to be its own partner for $\mathcal{E} = 0$, if this is the case then the state is topologically protected. The resulting zero energy state is protected by the topology of the scalar field, whose existence is guaranteed by the index theorem, which is localized around the soliton [10].

The Jackiw-Rebbi model uses $\varphi(x) = m \tanh(\lambda x)$ for the external scalar field, with $m > 0$ and $\lambda > 0$. For simplicity we will consider a external scalar field given by

$$\varphi(x) = m\frac{x}{|x|} \tag{3}$$

forming a domain wall at $x = 0$ where $\varphi(x = 0) = 0$. The scalar field given by Eq. (3) is a simplification of the Jackiw-Rebbi model first proposed by Rajaraman-Bell [11]. The precise form of the external scalar potential is not important as long as it asymptotically approaches an opposite sign at $x \to \pm\infty$. The wave function may change corresponding to a particular form of the external scalar potential, but the existence of the zero energy state is determined solely by the fact that the mass is positive on one side and negative on the other. Therefore, the solution is very robust against the external scalar potential.

The solution of the Dirac equation at exactly zero energy for the scalar field given by Eq. (3) is obtained by solving the following equation

$$\begin{pmatrix} 0 & -\partial_x + \varphi(x) \\ \partial_x + \varphi(x) & 0 \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = 0 \tag{4}$$

which gives

$$\psi_i = C_\mp \exp\left[\mp m|x|\right], \quad \text{for } i = 1, 2. \tag{5}$$

where $C_\mp$ is a normalization constant and the double sign in Eq. (5) is $-(+)$ for $i = 1(2)$. Note that $\psi_{1,2}$ cannot be both normalized. If we impose that $\lim_{x\to\pm\infty} \psi_i(x) \to 0$ we need to make $C_+ = 0$ in order to have a properly normalized state. In Fig. (2) we show the wave function for the zero energy state of the Jackiw-Rebbi model.

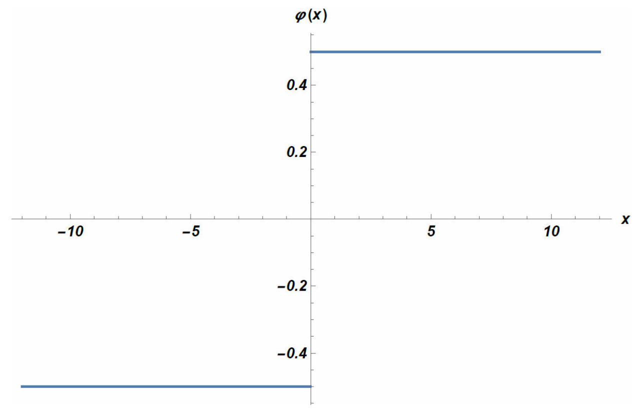

FIGURE 1. The figure shows the external scalar potential $\varphi(x)$ which changes sign at the interface $x = 0$.

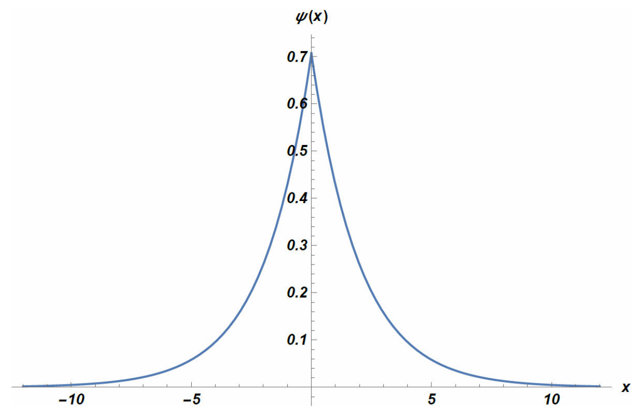

FIGURE 2. The figure shows the Jackiw-Rebbi zero energy mode given by Eq. (5) for the external scalar field depicted in Fig. (1). Note how the zero energy state is localized around the interface $x = 0$.

3. Electrostatic analog of the Jackiw-Rebbi model

In this section we show that the zero energy Jackiw-Rebbi state can be generated at the interface of two dielectric materials separated by a infinite charged sheet. The use of infinite charged sheets for emulating physical systems has been used extensively in the past for a wide range of applications such as a simple parallel plate capacitor [16] or the study of the one dimensional Coulomb gas [17–19]. Since we will be working with a planar charge distribution we will consider only the one dimensional Poisson equation with an inhomogeneous electric permittivity, *i.e.* $\epsilon(x)$, which is given by

$$\frac{d}{dx}\left(\epsilon(x)\frac{dV}{dx}\right) = -\rho(x), \tag{6}$$

where $V(x)$ is the electrostatic potential and $\rho(x)$ is the volume charge distribution. Expanding the left hand side of Eq. (6) and multiplying by $1/\epsilon$ we have

$$\frac{d^2V}{dx^2} + \frac{\epsilon'}{\epsilon}\frac{dV}{dx} = -\frac{\rho}{\epsilon}, \tag{7}$$

where ϵ' represents the total derivative with respect to the space coordinate x. Let us now make the following transformation

$$V(x) = V_0 \ln\left(\psi_1(x)/A\right), \tag{8}$$

where V_0 and A are constants to ensure dimensional consistensy, and $\psi_1(x)$ is an arbitrary function. Substituting Eq. (8) into Eq. (7) we have

$$V_0\frac{\psi_1''}{\psi_1} - \frac{\epsilon'}{\epsilon}E_x - \frac{1}{V_0}E_x^2 = -\frac{\rho}{\epsilon} \tag{9}$$

where we have used the identity $E_x = -dV/dx$. If we use Eq. (6) in the right hand side of Eq. (9) we end up with the following equation

$$V_0\frac{\psi_1''}{\psi_1} - \frac{dE_x}{dx} - \frac{1}{V_0}E_x^2 = 0, \tag{10}$$

note that Eq. (10) does not depend on the electric permittivity anymore. Multiplying Eq. (10) by ψ_1/V_0 and adding and subtracting the term $E_x\psi_1'/V_0$ to the left hand side of Eq. (10) we have

$$\lim_{\mathcal{E}\to 0}\left\{\left[\psi_1'+\frac{E_x}{V_0}\psi_1\right]' - \frac{E_x^2}{V_0^2}\psi_1 - \frac{E_x}{V_0}\psi_1' + \mathcal{E}^2\psi_1\right\} = 0 \tag{11}$$

where $\mathcal{E}$ is an auxiliary constant that we will set to zero at the end of our calculations. If we make the following substitution $\psi_1' + E_x\psi_1/V_0 = \mathcal{E}\psi_2$ into Eq. (11), we end up with the following equation $-\psi_2' + E_x\psi_2/V_0 = \mathcal{E}\psi_1$. These two coupled differential equations can be written in the same mathematical form as the Dirac equation with $c = \hbar = 1$, *i.e.*

$$\hat{H}_D\Psi = \left[\sigma_y\hat{p} + \sigma_x\left(\frac{E_x}{V_0}\right)\right]\Psi = \mathcal{E}\Psi. \tag{12}$$

Equations (12) can be reduced to two uncoupled Schrödinger equations $\hat{H}_i\psi_i = 0$, for $i = 1, 2$, given by

$$\hat{H}_i\psi_i = \left(\frac{\partial^2}{\partial x^2} + U_i(x)\right)\psi_i(x) = 0 \tag{13}$$

where

$$U_{1,2}(x) = \left[-\left(\frac{E_x}{V_0}\right)^2 + \mathcal{E}^2 \pm \frac{1}{V_0}\frac{dE_x}{dx}\right]. \tag{14}$$

Clearly, $\hat{H}_{1,2}$ are supersymmetric partner Hamiltonians which can be factorized as $\hat{H}_1 = \hat{A}^\dagger\hat{A} - \mathcal{E}^2$ and $\hat{H}_2 = \hat{A}\hat{A}^\dagger - \mathcal{E}^2$ where $\hat{A} = (\partial_x + E_x/V_0)$ and $\hat{A}^\dagger = (-\partial_x + E_x/V_0)$. The relation between Poisson's equation and Schrödinger equation in one dimension has been pointed out before by one of the authors (GG) [20, 21].

We can easily construct the zero energy mode by setting $\mathcal{E} = 0$ in Eq. (12) and solving for the uncoupled first order differential equations for $\psi_{1,2}$, *i.e.*

$$\psi_i = C_{\mp}\exp\left[\mp\int\left(\frac{E_x}{V_0}\right)dx\right] \tag{15}$$

where $C_{\mp}$ is a normalization constant and the double sign in Eq. (15) is $-(+)$ for $i = 1(2)$. The existence of a zero energy state then depends on the asymptotic behavior of E_x.

We know from basic electrostatics that the electric field due to an infinite charged sheet with volume charge density $\rho(x) = \sigma\delta(x)$ with $\sigma > 0$ separating two dielectric materials with electric permittivity ϵ_1 and ϵ_2 is given by

$$E_x(x) = \begin{cases} \dfrac{\sigma}{2\epsilon_1}, & \text{for} \quad x > 0 \\ -\dfrac{\sigma}{2\epsilon_2}, & \text{for} \quad x < 0. \end{cases} \tag{16}$$

Interestingly, the electrostatic field given by Eq. (16) has the same form as the external scalar field given by Eq. (3) that allows the existence of the zero energy state in the JR model.

The electrostatic potential for the electric field given by Eq. (16) is

$$V(x) = -\int_0^x E_x(x)dx = \begin{cases} -\dfrac{\sigma}{2\epsilon_1}x, & \text{for} \quad x > 0 \\ \dfrac{\sigma}{2\epsilon_2}x, & \text{for} \quad x < 0. \end{cases} \tag{17}$$

Using Eq. (15) we see that we need to set $C_+ = 0$ in order to make the two-component spinor normalizable. Therefore, the normalized wave function for the zero mode is given by

$$\Psi(x) = \sqrt{\frac{\sigma}{V_0(\epsilon_1 + \epsilon_2)}}\begin{pmatrix} e^{V(x)/V_0} \\ 0 \end{pmatrix}. \tag{18}$$

In Fig. (3) we show the electrostatic zero energy wave function for the Jackiw-Rebbi model, the wave function dominantly distributes near the interface $x = 0$ and decays exponentially away. The solution given in Eq. (18) for $\epsilon_1 = \epsilon_2$ is the same as the Jackiw-Rebbi zero energy state.

Having found the Jackiw-Rebbi zero energy state through electrostatic methods it is interesting to note that we can rewrite the Dirac Hamiltonian given in Eq. (12) as

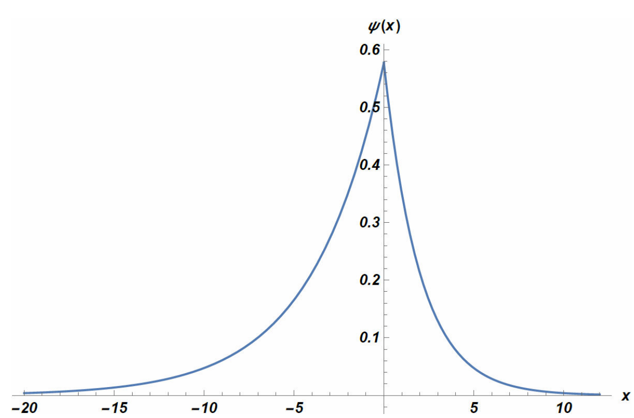

FIGURE 3. The figure shows the electrostatic Jackiw-Rebbi zero energy mode given by Eq. (18) for the following values $\sigma = V_0 = 1$, $\epsilon_1 = 1$ and $\epsilon_2 = 2$.

$$\hat{H}_D = \sigma_y \left(\hat{p} + i\sigma_z \frac{E_x}{V_0} \right), \tag{19}$$

where we have used the fact that $\sigma_x = -i\sigma_y\sigma_z$. The Dirac Hamiltonian given in Eq. (19) is obtained by performing the non-minimal substitution $\hat{p} \to \hat{p} + i\sigma_z E_x/V_0$ in the massless free particle Dirac Hamiltonian. The prescription of non-minimal substitution in the free particle Dirac Hamiltonian was used by Moshinsky and Szczepaniac for the so called Dirac Oscillator [22, 23]. For the special case of a uniform charge distribution, *i.e.* $\rho = \rho_0 = constant$, we would have $E_x = (\rho_0/\epsilon)x$ which gives rise to the one dimensional Moshinsky Dirac Oscillator [24, 25].

4. Conclusions

In conclusion, we have shown that the Poisson equation in one dimensional inhomogeneous media can be used to simulate the Jackiw-Rebbi model in one space dimension for the zero energy state. In particular, we demonstrate how the zero energy state of the Jackiw-Rebbi model can be implemented in an electrostatic set up with an infinite charged sheet that separates two different media. We have also pointed out the similarities of this system with the one dimensional Dirac Oscillator. Based on these findings, we have introduced an electrostatic platform for realizing the zero energy state of the Jackiw-Rebbi model which allows one to probe in the laboratory.

Acknowledgments

This work was supported by the program "Cátedras CONACYT". FJG would like to acknowledge support from project 32 of "Centro Mexicano de Innovación en Energía Solar" and by the National Laboratory program from CONACYT through the Terahertz Science and Technology National Lab (LANCYTT).

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