Scalar field radiation emitted by an accelerated scalar point source: A classical field theory approach

D.P. Meira Filho, G. dos Santos de Sousa, L. Helena Silva de Sousa, and R. da Silva Sales
Federal Institute of Pará, Campus Santarém, Santarém, Brazil.
e-mail: damiao.meira@ifpa.edu.br

J. Kysnney Santos Kamassury
Graduate Program in Electrical Engineering, Technological Center
Federal University of Santa Catarina, Florianópolis, Brazil.
e-mail: jorge.kamassury@posgrad.ufsc.br

J. Ferreira Lopes
Geophysics Graduate Program, Federal University of Pará, Belém, Brazil.

Received 19 June 2018; accepted 26 January 2019

In this paper we will use classical field theory to address the interaction of an accelerated point source with a non-massive Klein-Gordon-Fock field in Minkowski spacetime. For this, initially, we obtain the equation for the non-massive scalar field via lagrangian formalism and the scalar potential through Green's function formalism. Finally, we reach the expression of the power radiated by a point scalar source under the influence of this field and its covariant generalization.

Keywords: Classical field theory; non-massive classical scalar field; calculation of emitted power; Green's function formalism; scalar radiation.

PACS: 03.50.De; 03.50.Kk; 03.65.Pm

DOI: https://doi.org/10.31349/RevMexFisE.65.105

1. Introduction

It is known from the history of science that the first essentially scientific investigations of the phenomenon of radiation dates from the first half of the 19th century. Although the first studies focused on the radiation emitted by the incandescence of the bodies, in the course of the 19th century, many other phenomena began to gain prominence, as is the case with the X-ray.

The work related to electromagnetic theory, especially those resulting from the studies of the Scottish physicist and mathematician James Clerk Maxwell (1831-1879) enabled the radiation to be treated with important mathematical consistency. Under the influence of the works of the British physicist and chemist Michael Faraday (1791-1867) and the Irish physicist and mathematician Sir William Thomson (1824-1907), Maxwell presented his contributions to electromagnetic theory through three important works [1–5]. However, it is through his historical work entitled *A Treatise on Eletricity and Magnetism* [6] that he publishes a coherent synthesis of electricity, magnetism and optics (with the results of his earlier articles), unifying such domains.

Also during this period, we cite the important contributions of the German mathematician Bernhard Riemann (1826-1866) on the retarded potentials [7], by the Danish physicist Ludwig Lorenz (1829-1891) with the use of Lorenz gauge [8,9] and French physicist Alfred-Marie Liénard (1869-1958) and German geophysicist Emil Wiechert (1861-1928) with solutions of the homogeneous wave equation for specific problems [10].

In 1897 the English physicist and mathematician Sir Joseph Larmor (1857-1942) was able to demonstrate, although only for non-relativistic regimes, that an accelerated electric charge emitted radiation [11]. Soon after, Liénard reached a generalized form for the emitted power that was valid for any electronic velocity [12]. These results were decisive, for example, for the calculation of the intensity of the spectral lines of hydrogen, for the evolution of the atomic model and even as a basic aspect for the development of the Matrix Mechanics [13, 14].

Under the light of the classical radiation theory, several experimental results could be treated coherently, as is the case of Thomson scattering, resonance scattering and termal radiation. With the advent of quantum mechanics and relativity, many other aspects of radiation have been better understood.

In this paper, we will discuss a quantum scalar field described by the Klein-Gordon equation, proposed in 1926 by the Swedish physicist Oskar Klein (1894-1977) and the German physicist Walter Gordon (1893-1939) to describe relativistic electrons [15, 16]. It should be mentioned that sometimes this equation is also known as the Klein-Gordon-Fock (KGF) equation due to the Soviet physicist Vladimir Fock (1898-1974), who also obtained the same expression when presenting a relativistic treatment of the kleperian motion of bodies according to the Wave Mechanics [17].

Despite its failure to treat electrons in relativistic conditions, the KGF equation, based on the Feynman-Stueckelberg interpretation, allows us to describe the behavior of particles with spin 0 [18], as the mesons (π^+, π^- and π^0) and, consequently, to approach certain bosonic fields. It is mentioned that although the KGF field has no classical analogue because it is strictly quantum, it can be treated as a classical field [19].

In fact, just as the classical electromagnetic field, considered to be the high photon density limit of the quantized field, in a wide variety of physical applications, we study the classical scalar field as a possible approximation to a meson field [20, 21].

Scalar fields and, consequently, scalar radiation are common ingredients of models in particle physics, cosmology and gravitation, in particular, semiclassical gravity [22]. The semiclassical gravity also known as Quantum Field Theory in Curved Spacetimes [23,24], is devoted to investigating the consequences of defining a quantum theory of fields for matter and their interactions on a classical curved space-time underlying [25]. Indeed, this theory has been responsible for the prediction of important effects, such as the creation of particles in expanding universes, evaporation of black holes by virtue of quantum effects [26,27] (Hawking radiation) and thermal radiation obtained by accelerated observers (Fulling-Davies-Unruh effect) [28–30].

Recently, the framework of semiclassical gravity has been used at tree level to compute the (massless) scalar radiation of a point source in circular orbit around Reissner-Nordström [31], Schwarzschild [32–34] and Kerr black holes [35].

In this sense, throughout the text we will use the classical field theory to treat the non-massive scalar field, as well as analyze the interaction of accelerated point (scalar) classical currents with this field in Minkowski spacetime, thus observing relevant aspects of classical theory of radiation which, in turn, serves as introductory topics in the study of Quantum Field Theory.

It is emphasized that throughout the text, we will be using the Heaviside-Lorentz system of units, assuming $c = 1$ [36, 37].

2. Obtaining the field equations through lagrangian formalism

First, consider a massless scalar field $\phi(x)$ by interacting with an accelerated point source $J(x)$. In this case, the lagrangian density that describes this system is given by:

$$\pounds = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + J(x)\phi(x)\,. \tag{1}$$

In this work, we assume that physical phenomena happen in Minkowski spacetime whose metric is [38]:

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Indeed, using the variational principle for continuous systems [39], that is,

$$S = \int \pounds\, d^4x \tag{2}$$

we obtain the Euler-Lagrange equations, namely

$$\frac{\partial\pounds}{\partial\phi} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial\pounds}{\partial\left(\partial^\mu\phi\right)}\right] = 0. \tag{3}$$

In focus, we can use the Euler-Lagrange equations to obtain the KGF equation. Thus, by calculating each of the terms separately, we obtain:

$$\frac{\partial\pounds}{\partial\phi} = J(x) \tag{4}$$

$$\frac{\partial\pounds}{\partial\left(\partial^\mu\phi\right)} = \partial_\mu\phi(x)\,. \tag{5}$$

In addition, by properly replacing the Eq. (4) and Eq. (5) in the Eq. (3), we get the following expression:

$$\partial^\mu\partial_\mu\phi = J(x). \tag{6}$$

This last expression is known as the inhomogeneous KGF equation for the non-massive classical scalar field generated by a scalar source $J(x)$.

3. Solutions of field equations by the formalism of Green invariant functions

The solution of the Eq.(6) can be obtained by employing the invariant Green function. In this case, we will use the expression

$$(\partial^\mu\partial_\mu)_x D(x,x^{'}) = \delta^{(4)}(x-x^{'}) \tag{7}$$

where, $D(x,x^{'})$ is the Green function, $(\partial^\mu\partial_\mu)_x$ denotes the d'Alembertian operator acting at the coordinate x where $x^\mu = (x^0,x^1,x^2,x^3)$, and $\delta^{(4)}(x-x^{'})$ is the fourth-dimensional Dirac delta function.

The Green function of the d'Alembertian operator in the absence of boundary surfaces, depends solely on the difference between the spacetime position vectors $x-x^{'}=\zeta$ [40], where we define the spatial range $\zeta^i = x^i - x^{i'}$ such as R^i. It is worth emphasizing that in this case, we must perform a variable transformation for the operator $\partial_\mu\partial^\mu$ to act on the variable ζ. In this way, we will have:

$$\left[\partial_\mu\partial^\mu\right]_x = \left[\partial_\mu\partial^\mu\right]_\zeta\,.$$

By doing $D(x,x^{'}) = D(x-x^{'}) = D(\zeta)$, we can rewrite the Eq. (7) as:

$$\left[\partial_\mu\partial^\mu\right]_\zeta\left(D(x,x^{'})\right) = \delta\left(\zeta\right)\,. \tag{8}$$

To transform from the coordinate space to the space of the wave number, we will use the Fourier integral, whose transformation $\tilde{D}(k)$ is defined by

$$D(\zeta) = \frac{1}{16\pi^4}\int \tilde{D}(k)e^{-ik_\nu\zeta^\nu}d^4k \tag{9}$$

in which the exponential argument is given by:

$$k_\mu\zeta^\mu = k^0\zeta^0 - |\vec{k}|R. \tag{10}$$

The delta function, in turn, can be represented by:

$$\delta(\zeta) = \frac{1}{16\pi^4}\int e^{-ik_\nu\zeta^\nu}d^4k\,.$$

By properly replacing the expression (9) in (8) and applying the operator $(\partial^\mu\partial_\mu)_\zeta$ we get:

$$(\partial^\alpha\partial_\alpha)_\zeta D(\zeta) = \frac{1}{16\pi^4}\int \tilde{D}(k)(\partial^\alpha\partial_\alpha)_\zeta e^{-ik_\nu\zeta^\nu}d^4k\,. \tag{11}$$

Analyzing the application of the operator in detail, we obtain the expression

$$(\partial^\mu\partial_\mu)_\zeta\, e^{-i\eta_{\alpha\nu}k^\alpha\zeta^\nu} = -k^\alpha k_\alpha\, e^{-ik^\alpha\zeta_\alpha} \tag{12}$$

that when applied in the Eq. (11), results in the following equation:

$$\tilde{D}(k) = \frac{i^2}{k^\alpha k_\alpha} \tag{13}$$

Substituting this last result into the Fourier transform of the Green function, we have:

$$D(\zeta) = \frac{i^2}{16\pi^4}\int \frac{e^{-ik_\nu\zeta^\nu}}{k^\alpha k_\alpha}d^4k\,. \tag{14}$$

Using the following expression

$$k^\alpha k_\alpha = (k^0)^2 - |\vec{k}|^2 \tag{15}$$

we can rewrite the invariant Green function

$$D(\zeta) = -\frac{1}{16\pi^4}\int e^{i\vec{k}\cdot\vec{R}}d^3k \int_{-\infty}^{+\infty}\frac{e^{-ik^0\zeta^0}}{\left[(k^0)^2 - |\vec{k}|^2\right]}dk^0 \tag{16}$$

where the integrand in dk^0 is not set to $k^0 = \pm|\vec{k}|$.

In this context, we will consider the solution of the integral in k^0 , this is:

$$\int_{-\infty}^{+\infty}\frac{e^{-ik^0\zeta^0}}{[(k^0)^2 - k^ik^i]}\,dk^0 \tag{17}$$

In this case, the 4-vector k^μ is represented in terms of its components, $k = (\omega, \vec{k})$, where ω is the frequency and $\vec{k}$ is the wave vector.

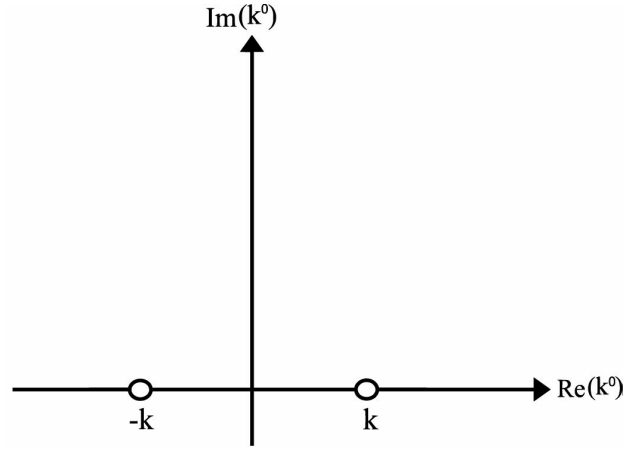

FIGURE 1. The singularity points of the integrand of the expression (17) arranged on the real axis k^0.

Since the integrand has singularity points (Fig. 1), we will use the residuals and poles theory to solve the integral in dk^0. In this case, we consider the integral in k^0 as a complex variable and solve the resulting integral as a contour integral in the complex plane k^0. As shown in Fig. 1, the integrand has two simple poles, which are:

$$k^0 = \pm(k^ik^i)^{\frac{1}{2}} = \pm|\vec{k}|\,. \tag{18}$$

The different solutions of the Green functions can be obtained by taking the closed contours r and a and shifting the poles on the imaginary axis by an amount $-\varepsilon$ for the contour r or $+\varepsilon$ for the contour a as shown in Fig. 2 and Fig. 3. Lastly, we take the limit $\varepsilon \to 0$.

In this context, the contour r is characterized by the boundary of a half-circle of radius R defined in the lower half-plane and containing the poles $k^0 = \pm|\vec{k}| - i\varepsilon$ should be selected when $\zeta^0 > 0$, since the term $e^{-ik^0\zeta^0}$ diverges in the upper half-plane when $R \to \infty$. On the other hand, the contour a, characterized by the boundary of a semicircle of radius R traced in the upper half-plane and enclosing the

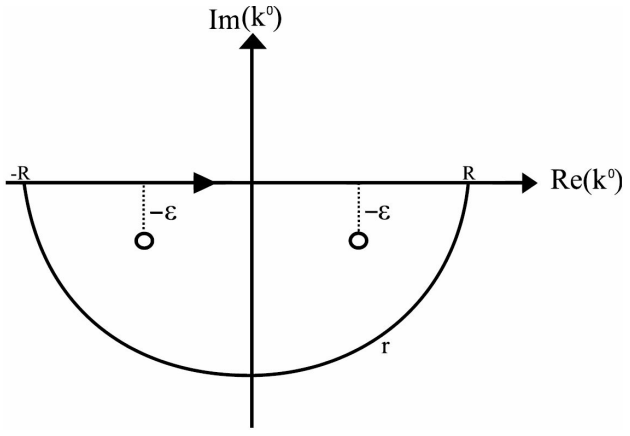

FIGURE 2. The poles displaced on the imaginary axis of a quantity $-\epsilon$.

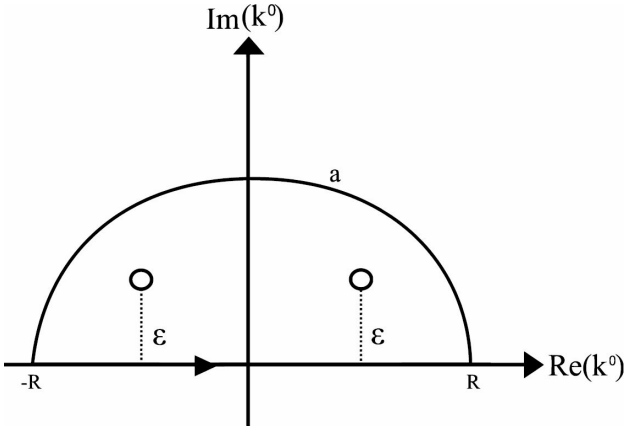

FIGURE 3. The poles displaced on the imaginary axis of a quantity $-\epsilon$.

poles $k^0 = \pm k + i\varepsilon$, should be selected when $\zeta^0 < 0$, because $e^{-ik^0\zeta^0}$ diverges in the lower half-plane when $R \to \infty$.

Using the residue and poles theory [41] in the Eq. (18) and assuming the closed contour r, we have:

$$\int_{-\infty}^{+\infty} \frac{e^{-ik^0\zeta^0}}{(k^0)^2 - |\vec{k}|^2} dk^0 = \frac{\pi i}{|\vec{k}|}\left[e^{i|\vec{k}|\zeta^0} - e^{-i|\vec{k}|\zeta^0}\right]. \quad (19)$$

Substituting the Eq. (19) into Eq. (16), we reach the delayed or causal Green function:

$$D_{\text{ret}}(\zeta) = \frac{\theta(\zeta^0)}{8\pi^3}\int \frac{\sin(|\vec{k}|\zeta^0)e^{i\vec{k}\cdot\vec{R}}}{|\vec{k}|}\, d^3\vec{k}\,. \quad (20)$$

The Heaviside function $\theta(\zeta^0)$ arises because of the fact that we adopt the closed contour r in which $\zeta^0 > 0$.

Let the heaviside function be defined as

$$\theta(\zeta^0) = \begin{cases} 1, & \text{if} \quad \zeta^0 > 0, \\ 0, & \text{if} \quad \zeta^0 < 0 \end{cases}$$

where, in spherical coordinates

$$d^3\vec{k} = |\vec{k}|^2 \sin(\theta)\, d\theta\, d\phi\, d|\vec{k}|\,,$$

so:

$$D_{\text{res}}(\zeta) = \frac{\theta(\zeta^0)}{2\pi^2 R}\int_0^{+\infty} \sin(|\vec{k}|R)\sin(|\vec{k}|\zeta^0) d|\vec{k}|\,. \quad (21)$$

In Eq. (21), R refers to the spatial distance between x^μ and $x^{'\mu}$. Rewriting trigonometric functions in terms of complex exponents, this is integrating the exponentials

$$\int_0^{+\infty} e^{\pm i|\vec{k}|\left(R \mp \zeta^0\right)}\, d|\vec{k}| = \int_{-\infty}^{0} e^{i|\vec{k}|\left(\zeta^0 \mp R\right)}\, d|\vec{k}| \quad (22)$$

and appropriately replacing Eq. (22) in the Eq. (21), we reach:

$$D_{\text{ret}}(\zeta) = \frac{\theta(\zeta^0)}{8\pi^2 R}\int_{-\infty}^{+\infty}\Big[e^{i|\vec{k}|(\zeta^0 - R)} - e^{i|\vec{k}|(\zeta^0 + R)}\Big] d|\vec{k}|. \quad (23)$$

Writing the Dirac delta in terms of the wave vector, this is

$$\delta\left(\zeta^0 \pm R\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i|\vec{k}|\left(\zeta^0 \pm R\right)}\, d|\vec{k}|\,, \quad (24)$$

the Eq. (23), becomes:

$$D_{\text{ret}}(\zeta) = \frac{1}{4\pi R}\theta\left(x^0 - x^{'0}\right)\delta\left(x^0 - x^{'0} - R\right). \quad (25)$$

In fact, the Green functions can be arranged in a covariant form from the following Dirac delta property:

$$\delta\left[f(x)\right] = \sum_i \delta(x - x_i)\left|\frac{df(x_i)}{dx}\right|^{-1}. \quad (26)$$

In this case:

$$\delta\left[\left(x^\mu - x^{'\mu}\right)^2\right] = \delta\left[\zeta^\mu\zeta_\mu\right] = \delta\left[\left(\zeta^0\right)^2 - R^2\right]. \quad (27)$$

Finally, by differentiating $\delta[\left(\zeta^0\right)^2 - R^2]$ with respect to z^0 and developing some algebraic manipulations, we reach:

$$\delta\left[\left(\zeta^0\right)^2 - R^2\right] = \frac{1}{2R} \times \left[\delta(x^0 - x^{'0} - R) + \delta(x^0 - x^{'0} + R)\right] \quad (28)$$

Knowing that the theta functions select one or the other between the two terms of the Eq. (28), then the delayed Green function will be given by:

$$D_{\text{ret}}(\zeta) = \frac{1}{2\pi}\,\theta\left(x^0 - x^{'0}\right)\delta\left[\left(x^\nu - x^{'\nu}\right)^2\right]. \quad (29)$$

It is emphasized that the theta function, apparently non-invariant, when subjected to the constraints of the delta function, becomes invariant under its own Lorentz transformations [42]. A more detailed discussion of the related physical aspects of the Eq. (29) can be seen in [43].

4. Obtaining the scalar potential through the formalism of Green's functions

At once, the field equations with source are expressed by

$$\left(\partial^\mu\partial_\mu\right)\,\phi(x) = J(x) \quad (30)$$

and the delayed Green function set to:

$$\left(\partial^\mu\partial_\mu\right)_x D_{\text{ret}}(x,y) = \delta^{(4)}(x-y)\,. \quad (31)$$

Thus, in terms of the Green function, we can write the scalar field, solution of the Eq. (30), as:

$$\phi(x) = \int D_{\text{ret}}(x,y)J(y)d^4y. \tag{32}$$

In detail, to verify the previous statement, we simply replace the Eq. (31) in the inhomogeneous KGF equation. In this way, we have:

$$\begin{aligned}(\partial^\mu\partial_\mu)_x\ \phi(x) &= \int J(y)\,(\partial^\mu\partial_\mu)_x\ D_{\text{ret}}(x,y)\,d^4y \\ &= \int \delta^4(x-y)\,J(y)\,d^4y = J(x)\,.\end{aligned}$$

This development evidences that the proposed scalar field is a solution of the inhomogeneous KGF equation.

At this time, let us consider the scalar current associated with a point source following a world line $z^\mu(\tau)$, with 4-velocity $U^\mu\left(z(\tau)\right) \equiv (dz^\mu/d\tau)$ given by

$$J(y) \;=\; \frac{q}{U^0(y)}\,\delta^3\left[\vec{y}-\vec{z}(\tau)\right]\,, \tag{33}$$

whose term q corresponds to the scalar charge [25]. By replacing the current expression in the scalar field equation and knowing that $d^4y = dy^0d^3y$, it follows that:

$$\phi(x) \;=\; q\int \frac{1}{U^0(y)}\,\delta^3\left[\vec{y}-\vec{z}(\tau)\right]D_{\text{ret}}(x,y)dy^0d^3y\,. \tag{34}$$

Since $U^0(y) = dy^0/d\tau$, we get:

$$\begin{aligned}\phi(x) &= q\int \delta^3\left[\vec{y}-\vec{z}(\tau)\right]D_{\text{ret}}(x,y)\,d\tau d^3y \\ &= q\int D_{\text{ret}}(x,z)\,d\tau\,.\end{aligned} \tag{35}$$

It is a delayed Green function (analogous to the Eq. (29)) represented by

$$D_{\text{ret}}\left(x,z(\tau)\right) \;=\; \frac{1}{2\pi}\,\theta(x^0-z^0)\,\delta\left[(x-z(\tau))^2\right]$$

is that

$$\delta\left[(x-z(\tau))^2\right] \;=\; \left[\frac{\delta\left(\tau-\tau_p\right)+\delta\left(\tau-\tau_f\right)}{2\left|U_\mu(x-z)^\mu\right|}\right]\,.$$

Then we can rewrite $D_{\text{ret}}\ (x,z(\tau))$ as:

$$D_{\text{ret}}\left(x,z(\tau)\right)=\frac{\theta(x^0-z^0)}{4\pi}\left[\frac{\delta(\tau-\tau_p)+\delta(\tau-\tau_f)}{|U_\mu(x-z)^\mu|}\right]. \tag{36}$$

Substituting the Eq. (36) into the Eq. (35), we obtain

$$\phi(x) = \frac{q}{4\pi}\int\theta\left(x^0-z^0(\tau)\right)\left[\frac{\delta\left(\tau-\tau_p\right)+\delta\left(\tau-\tau_f\right)}{|U_\mu(x-z)^\mu|}\right]d\tau$$

$$\begin{aligned}\phi(x) = {}& \frac{q}{4\pi}\theta\left(x^0-z^0(\tau_p)\right)\left[|U_\mu(\tau_p)\,(x-r(\tau_p))^\mu|\right]^{-1} \\ &+ \frac{q}{4\pi}\theta\left(x^0-z^0(\tau_f)\right)\left[|U_\mu(\tau_f)\,(x-z(\tau_f))^\mu|\right]^{-1}\end{aligned}$$

where to ensure the causality $x^0 - z^0(\tau_0 = \tau_p) > 0$. Based on this condition, we have:

$$\phi(x) \;=\; \frac{q}{4\pi\left|U_\mu(\tau_0)\,(x-z(\tau_0))^\mu\right|}. \tag{37}$$

In the meantime, to simplify the equation (37) we can refer to the property of the Dirac delta given by the Eq. (38), where $f(\tau) = [x-r(\tau)]^\mu\ [x-r(\tau)]_\mu$.

$$\delta\left[f(\tau)\right] \;=\; \sum_i \delta(\tau-\tau_i)\left|\left(\frac{df}{d\tau}\right)_{\tau=\tau_i}\right|^{-1} \tag{38}$$

Differentiating the Eq. (38), we obtain the following equation

$$\frac{df}{d\tau} \;=\; -2U_\alpha(x-r)^\alpha \tag{39}$$

where, $U_\mu(\tau) = \gamma(1,\vec{v})$.

Substituting the Eq. (39) into Eq. (38) and developing this last equation, we have:

$$\delta\left[f(\tau)\right] \;=\; \frac{\delta(\tau-\tau_p)+\delta(\tau-\tau_f)}{2\left|U_\mu(x-r)^\mu\right|}\,. \tag{40}$$

Considering $\vec{x}-\vec{r}(\tau_0) \;=\; \vec{R}$, we have the cone of light that $x^0 - r^0(\tau_0) = R$ (note that τ_0 is defined by the condition of the light cone as shown in Fig. 4). In addition, by defining the relations $\vec{R} = R\hat{R}$, $\vec{\beta} = \vec{v}$ e $\gamma^{-2} = 1-\vec{\beta}$, we can rewrite the scalar product $U_\mu(x-\mu)^\mu$ as:

$$U_\mu(x-r)^\mu \;=\; \eta_{\nu\mu}\,U^\nu(x-r)^\mu \;=\; \gamma R\left(1-\left|\vec{\beta}\right|\right). \tag{41}$$

It should be noted that the expression (42) is positive, since $\left|\vec{\beta}\right| < 1$. In this case, we can write the Eq. (40) as follows:

$$\delta\left[f(\tau)\right] \;=\; \frac{\delta(\tau-\tau_p)+\delta(\tau-\tau_f)}{2\,U_\mu(x-r)^\mu}\,. \tag{42}$$

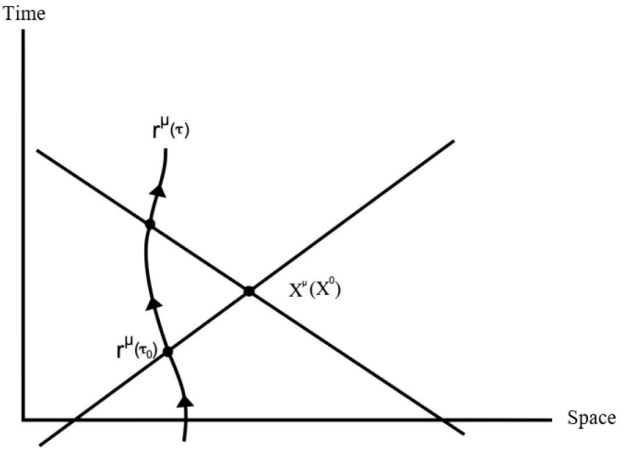

FIGURE 4. The point of intersection between the source world line and the cone of light past the observation point.

In fact, in possession of the relation (42), we can use it to rewrite the Eq. (37) without the absolute value operator, that is:

$$\phi(x) \;=\; \frac{q}{4\pi\, U_\mu(\tau_0)\,(x - z(\tau_0))^\mu}\,. \tag{43}$$

In this last expression (43), the term τ_0 represents the proper time of the source at the time of emission of the radiation, event associated with the position $z^\mu(\tau_0)$ of the charge in the Minkowski spacetime.

In this case, the non-massive classical scalar field is represented in the covariant form. However, it is convenient to write in the most usual form

$$\phi(x) \;=\; \frac{q}{4\pi\left[\gamma R\left(1 - \left|\vec{\beta}\right|\right)\right]} \tag{44}$$

where $\gamma = U^0 = dz^0/d\tau$ and $\vec{\beta} = \vec{v} = d\vec{z}/d\tau$.

5. Scalar field radiation emitted by an accelerated scalar point source

It is known that the emitted power is given by the flow of the Poynting vector $\vec{S}$ through the spherical surface (in the inertial coordinate system) with the origin in the source $J\,(x^\mu)$ and radius R, whose area element is denoted by $d\vec{A}$. The emitted power is thus equated by:

$$P = \oiint \vec{S}\cdot d\vec{A}\,. \tag{45}$$

To obtain the Poynting vector, in this particular case, we will manipulate the canonical energy-momentum tensor. This tensor for the classical scalar field is given by:

$$T_{\mu\nu} = \frac{\partial \pounds}{\partial\,(\partial^\mu\,\phi)}\partial_\nu\phi - \eta_{\mu\nu}\pounds\,. \tag{46}$$

In this case, the lagrangian is expressed by:

$$\pounds = \frac{1}{2}\eta_{\mu\nu}\,(\partial^\mu\,\phi)\,(\partial^\nu\,\phi) + J\phi\,. \tag{47}$$

Calculating the partial derivative of $\pounds$ with respect to $\partial^\lambda\phi$

$$\begin{aligned}\frac{\partial \pounds}{\partial\,(\partial^\lambda\phi)} &= \frac{1}{2}\eta_{\mu\nu}\frac{\partial\,(\partial^\mu\phi)}{\partial\,(\partial^{\,\lambda}\phi)}\partial^\nu\phi + \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\frac{\partial\,(\partial^\nu\phi)}{\partial\,(\partial^\lambda\phi)}\\ &= \frac{1}{2}\eta_{\mu\nu}\,\delta^\mu_\lambda\partial^\nu\phi + \frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi\,\delta^\nu_\lambda\\ &= \frac{1}{2}\eta_{\lambda\nu}\partial^\nu\,\phi + \frac{1}{2}\eta_{\mu\lambda}\,\partial^\mu\,\phi\end{aligned}$$

we will have

$$\frac{\partial \pounds}{\partial\,(\partial^\lambda\phi)} = \partial_\lambda\,\phi$$

where $\eta_{\mu\nu} \;=\; \eta_{\nu\mu}$. Thus, we will have:

$$\frac{\partial \pounds}{\partial\,(\partial^\mu\,\phi)} = \partial_\mu\phi\,.$$

In addition, the energy-momentum tensor becomes:

$$T_{\mu\nu} = (\partial_\mu\phi)\,(\partial_\nu\,\phi) - \eta_{\mu\nu}\,\pounds\,. \tag{48}$$

To obtain the countervariant components of the tensor $T_{\mu\nu}$, initially we will raise the indices using the metric tensor, namely:

$$\begin{aligned}T^{\mu\nu} &= \eta^{\alpha\mu}T_{\alpha\beta}\eta^{\beta\nu}\\ T^{\mu\nu} &= (\partial^\mu\phi)\,(\partial^\nu\,\phi) - \eta^{\mu\nu}\,\pounds\,.\end{aligned} \tag{49}$$

Knowing also that the components T^{oi} of the energy-momentum tensor represent the components S^i of the Poynting vector $\vec{S}$, it follows that:

$$\begin{aligned}S^i = T^{oi} \;&=\; \left(\partial^0\phi\right)\left(\partial^i\phi\right) - \eta^{oi}\pounds\\ S^i &= \eta^{00}\,(\partial_0\phi)\,\eta^{ij}\,(\partial_j\phi) = -\,(\partial_0\phi)\;\delta^{ij}\,(\partial_j\phi)\\ S^i &= -\,(\partial_0\phi)\;(\partial_i\phi)\end{aligned}$$

where we use $\eta^{00} \;=\; +1$, $\eta^{ii} \;=\; -1$ ($i \;=\; 1,2,3$) e $\eta^{oi} \;=\; \eta^{io} = 0$.

In fact, we can express $\partial_\mu\,\phi$ in terms of velocity and acceleration of the moving source. Using the expression (48) we reach:

$$\partial_\mu\,\phi(x) \;=\; \int J(y)\,[\partial_\mu D_{\text{ret}}(x,y)]\;d^4y\,. \tag{50}$$

Replacing the Eq. (49) in the Eq. (50), we have:

$$\partial_\mu\phi(x) = q\int \frac{\delta^3\,[\vec{y} - \vec{z}(\tau)]\,[\partial_\mu\,D_{\text{ret}}(x,y)]}{U^0(y)}\,dy^0d^3y\,.$$

Since $U^0(y) = dy^0/d\tau$, then:

$$\partial_\mu\phi(x) \;=\; q\int \partial_\mu\,D_{\text{ret}}\,(x, z(\tau))\;d\tau. \tag{51}$$

In this case, $\partial_\mu\,D_{\text{ret}}\,(x, z(\tau))$ it will be:

$$\begin{aligned}\partial_\mu\,D_{\text{ret}}(x,z) &= \frac{\partial\,[D_{\text{ret}}\,(x, z(\tau))]}{\partial\left[(x-z)^2\right]}\,\frac{\partial\left[(x-z)^2\right]}{\partial x^\mu}\\ &= \frac{d\,[D_{\text{ret}}\,(x, z(\tau))]}{d\tau}\,\frac{d\tau}{d\left[(x-z)^2\right]}\\ &\times\,\frac{\partial\left[(x - z(\tau))^2\right]}{\partial x^\mu}\,.\end{aligned}$$

In this last expression,

$$\frac{d\tau}{d\left[(x-z)^2\right]} = \left\{\frac{d\left[(x-z)^2\right]}{d\tau}\right\}^{-1} = \left[-2\,(x^\nu - z^\nu)\,U_\nu\right]^{-1}$$

and

$$\frac{\partial\left[(x-z)^2\right]}{\partial x^\mu}=\delta^\alpha_\mu(x_\alpha-z_\alpha)+\delta_{\mu\alpha}(x^\alpha-z^\alpha)=2(x_\mu-z_\mu).$$

Knowing that $U^\mu = dz^\mu/d\tau$ and $dx^\mu/d\tau = 0$, the expression $\partial_\mu D_{\text{ret}}$ can be rewritten as:

$$\partial_\mu D_{\text{ret}}(x,z)=\frac{-d\left[D_{\text{ret}}\left(x,z(\tau)\right)\right]}{d\tau}\frac{(x_\mu-z_\mu)}{(x^\nu-z^\nu)\,U_\nu}. \quad (52)$$

By adequately replacing the Eq. (52) in the Eq. (51), we have:

$$\partial_\mu\,\phi(x)=-\,q\int_{-\infty}^{+\infty}\frac{(x-z)_\mu}{(x-z)^\nu\,U_\nu}\left\{\frac{d\left[D_{\text{ret}}\left(x,z(\tau)\right)\right]}{d\tau}\right\}d\tau. \quad (53)$$

Integrating in parts the second member of the expression (53), we reach:

$$\partial_\mu\phi(x)=-q\left[\frac{(x-z)_\mu}{(x-z)^\nu\,U_\nu}D_{\text{ret}}\left(x,z(\tau)\right)\right]_{-\infty}^{+\infty}+q\int_{-\infty}^{+\infty}D_{\text{ret}}(x,z(\tau))\frac{d}{d\tau}\left(\Upsilon_\mu\right)d\tau \quad (54)$$

where $\Upsilon_\mu = ((x-z)_\mu/(x-z)^\nu U_\nu)$.

Using the fact that

$$D_{\text{ret}}\left(x,z(\tau)\right)=\frac{1}{2\pi}\,\theta(x^0-z^0)\,\delta\left[(x-z(\tau))^2\right]$$

and where $x^0 - z^0(\tau) > 0$, we develop the Eq. (54) as evidenced below.

$$\partial_\mu\phi(x)=\frac{-q}{2\pi}\left\{\Upsilon_\mu\theta\left[x^0-z^0(\tau)\right]\delta\left[(x-z(\tau))^2\right]\right\}_{\tau\to-\infty}^{\tau\to+\infty}+\frac{q}{2\pi}\int_{-\infty}^{+\infty}\theta\left[x^0-z^0(\tau)\right]\delta[(x-z(\tau))^2]\Lambda_\mu d\tau \quad (55)$$

where $\Lambda_\mu = (d/d\tau)((x-z)_\mu/[(x-z)^\nu U_\nu])$.

Assuming further that the radiation emitted at an infinite instant implies an infinite distance to be traveled by the radiation, then in this case the asymptotic terms obtained for $\tau \to \pm\infty$ will not contribute, *i.e.* $\delta[(x-z(\tau))^2] = 0$. Using these observations in the Eq. (55), we can obtain:

$$\partial_\mu\phi(x)=\frac{q}{2\pi}\int_{-\infty}^{+\infty}\theta\left[x^0-z^0(\tau)\right]\delta[(x-z(\tau))^2]\Lambda_\mu d\tau \quad (56)$$

Substituting the expression (42) into the last result (56) and admitting the condition of the last light cone, we observe that

$$\partial_\mu\phi(x)=\frac{q}{4\pi}\frac{1}{\left[(x-z(\tau_0))_\nu\,U^\nu(\tau_0)\right]^2}\frac{d}{d\tau}\times\left[\frac{(x-z)_\mu}{(x-z)^\nu U_\nu}\right]_{\tau_0}. \quad (57)$$

We also have:

$$\frac{d}{d\tau}\left[\frac{(x-z)_\mu}{(x-z)_\nu U^\nu}\right]=-\frac{(x-z)_\nu U^\nu U_\mu}{\left[(x-z)_\nu U^\nu\right]^2}+\frac{(x-z)_\mu U_\nu U^\nu}{\left[(x-z)_\nu U^\nu\right]^2}-\frac{(x-z)_\mu(x-z)_\nu a^\nu}{\left[(x-z)_\nu U^\nu\right]^2} \quad (58)$$

Using this last equation, we can rewrite the Eq. (57) as:

$$\partial_\mu\phi(x)=\frac{-q}{4\pi}\left\{\frac{(x-z)_\nu\left(U_\mu U^\nu-\delta^\nu_\mu\right)}{\left[(x-z)_\alpha U^\alpha\right]^3}\right\}_{\tau=\tau_0}+-\frac{q}{4\pi}\left\{\frac{(x-z)_\mu\,(x-z)_\nu}{\left[(x-z)_\alpha U^\alpha\right]^3}\frac{dU^\nu}{d\tau}\right\}_{\tau=\tau_0}. \quad (59)$$

It is worth noting that the first term of the second member of the Eq. (59) decays with R^{-2}, while the second term decays with R^{-1}. This fact indicates that for points sufficiently distant from the J source world line, the term proportional to the 4-velocity is negligible in relation to the term proportional to the 4-acceleration. Under these conditions, the Eq. (59) becomes:

$$\partial_\mu\phi(x)\cong-\left[\frac{q}{4\pi}\frac{(x-z)_\mu(x-z)_\nu}{\left[(x-z)_\alpha U^\alpha\right]^3}\frac{dU^\nu}{d\tau}\right]_{\tau=\tau_0}. \quad (60)$$

Developing this last expression, we will have:

$$\partial_\mu\phi(x)\cong-\frac{q}{4\pi}\frac{(x-z)_\mu}{\left[(x-z)_\alpha U^\alpha\right]^3}\left[(x-z)^0\frac{dU^0}{d\tau}\right]_{\tau=\tau_0}+\frac{q}{4\pi}\frac{(x-z)_\mu}{\left[(x-z)_\alpha U^\alpha\right]^3}\left[(x-z)^i\frac{dU^i}{d\tau}\right]_{\tau=\tau_0}. \quad (61)$$

Given the following relations

$$(x-z)_0=(x-z)^0=R\,,$$
$$(x-z)_i=-(x-z)^i=-R(\hat{R})^i,$$
$$\left[x^\nu-z^\nu(\tau_0)\right]^2=0\,,$$
$$\frac{dU^0}{d\tau}=\frac{dz^0}{d\tau}\frac{dU^0}{dz^0}=\frac{dt}{d\tau}\frac{dU^0}{dt}=\gamma\frac{dU^0}{dt}=\gamma\frac{d\gamma}{dt}=\gamma^4\left(\vec{v}.\vec{a}\right),$$
$$\frac{dU^i}{d\tau}=\frac{d\left(\gamma\vec{v}\right)}{d\tau}=\gamma^4\frac{d\gamma}{d\tau}+\gamma\frac{d\vec{v}}{d\tau}=\left[\gamma^4v^i\left(\vec{v}.\vec{a}\right)+\gamma^2a^i\right].$$

and using them in the Eq. (61), we obtain:

$$\partial_\mu\phi\cong\frac{-q}{4\pi}\frac{(x-z)_\mu R\gamma^2}{\left[(x-z)_\alpha U^\alpha\right]^3}\times\left[\gamma^2\left(\vec{v}.\vec{a}\right)\left(1-\vec{v}.\hat{R}\right)-\left(\vec{a}.\hat{R}\right)\right]_{\tau=\tau_0} \quad (62)$$

In a frame of reference where the accelerated source moves with a smaller velocity than the speed of light, we have $\gamma = 1$ and $(x-z)_\alpha U^\alpha = R$. In this case, the Eq. (62) becomes:

$$\partial_\mu \phi \cong \frac{q}{4\pi}\frac{1}{R^2}\left[(x-z)_\mu \left(\hat{R}.\vec{a}\right)\right]_{\tau=\tau_0}. \tag{63}$$

Explaining the temporal and spatial components of $\partial_\mu\phi$, we get the relations (64) and (65), respectively

$$\partial_0 \phi = \frac{\partial \phi}{\partial t} = \frac{q}{4\pi}\frac{1}{R}\left(\hat{R}.\vec{a}\right)_{\tau=\tau_0} \tag{64}$$

$$\partial_i \phi \cong \frac{q}{4\pi}\frac{(x-z)_i}{R^2}(\hat{R}.\vec{a})_{\tau=\tau_0} = \frac{-q}{4\pi}\frac{R^i}{R^2}\left(\hat{R}.\vec{a}\right)_{\tau=\tau_0} \tag{65}$$

Remembering that $S^i = -\left(\partial_0\phi\right)\left(\partial_i\phi\right)$, we get

$$\vec{S} = \left[\frac{q^2}{16\pi^2}\frac{\left(\hat{R}.\vec{a}\right)}{R^2}\hat{R}\right]_{\tau=\tau_0}, \tag{66}$$

and knowing that $d\vec{A} = dA.\hat{R} = \hat{R}R^2\sin(\theta)d\theta d\phi$ we will reach the expression:

$$P = \oiint \vec{S}\cdot d\vec{A} = \frac{q^2a^2}{12\pi} \tag{67}$$

The Eq. (67) is equal to half the classical value of the radiated power for the electromagnetic field [43] given by Eq. (68). In fact, this was already an expected result, since the electromagnetic field as a non-massive vector field has two degrees of freedom associated with the two degrees of polarization of the same [44]. For more detailed discussions see [43].

$$P = \frac{q^2a^2}{6\pi} \tag{68}$$

As $P = dE/dz^0 = \gamma dE'/\gamma dz^{0'} = dE'/dz^{0'}$ is a Lorentz invariant, it is possible to propose a Lorentz invariant that is reduced to the expression (67) for a frame of reference where the accelerated source moves with a velocity small compare with the speed of light [42]. In addition, based on the expression (60), it is observed that the invariant sought will depend only on U^μ and $dU^\mu/d\tau$.

Indeed, a convenient form is given by

$$P = \frac{-q^2}{12\pi}\frac{dU^\mu}{d\tau}\frac{dU_\mu}{d\tau} \tag{69}$$

where

$$\frac{dU^\mu}{d\tau}\frac{dU_\mu}{d\tau} = \gamma^8\left(\vec{v}.\vec{a}\right)^2 - \left[\gamma^4 v^i\left(\vec{v}.\vec{a}\right) + \gamma^2 a^i\right]^2.$$

In the case of assuming an inertial frame instantaneously at rest with the source, the Eq. (69) is reduced to Eq. (67).

Conclusion

We discuss throughout the present text the interaction of an accelerated point source with the non-massive scalar field making use of the formalism of the Green functions, in addition to determining the expression of the total power radiated in the covariant form. It was verified that the power obtained in the case of the scalar source is half the value of that referring to the electric charge, which is due to the two degrees of freedom (of polarization) of the electromagnetic wave (photon) in contrast with a single degree of freedom of the scalar field.

Although the approach employed essentially focused on the study of the phenomenon of accelerated (in this case, scalar) radiation through classical field theory, the presented development can be used as an initial step to investigate different fields and their interactions with the subject, as well as a preparatory didactic text for the study of quantum field theory.

Acknowledgments

We thank the distinguished Professor Luis Carlos Bassalo Crispino for the relevant guide lines on the topic addressed in this text. We also endorse our thanks to the fruitful partnership between the Federal Institute of Pará (IFPA), Federal University of Santa Catarina (UFSC) and Federal University of Pará (UFPA). Finally, the author J. K. S. Kamassury would like to thank the financial support provided by the National Council for Scientific and Technological Development (CNPq).

1. J. C. Maxwell, *Trans. Cambridge Philos. Soc.*, **10** (1855) 27-51.
2. J. C. Maxwell, *Trans. Cambridge Philos. Soc.*, **10** (1856) 51-83.
3. J. C. Maxwell, *Philo. Mag.*, **21** (1861) 161-175.
4. J. C. Maxwell, *Philo. Mag.*, **21** (1862) 281-291.
5. J. C. Maxwell, *Phil. Trans.*, **155** (1865) 459-512.
6. J. C. Maxwell, *A Treatise on Eletricity and Magnetism* (Clarendon Press, Oxford, 1873).
7. B. Riemann, *Philo. Mag.*, **34** (1867) 368-372.
8. L. Lorenz, *J. reine angew. Math*, **58** (1861) 329.
9. L. Lorenz, *Philo. Mag.*, **34** (1867) 287-301.
10. E. Wiechert, *Arch. Néerl.*, **5** (1900) 549.
11. J. Larmor, *Philo. Mag.*, **44** (1897) 503-512.
12. A. Liénard, *L'Éclairage Électrique*, **16** (1898) 106.
13. J. M. F. Bassalo, *Crônicas da Física: Tomo I* (Editora da Universidade Federal do Pará, Belém, 1987).
14. F. Caruso and V. Oguri, *Física Moderna: Origens Clássicas e Fundamentos*, 2nd Edition, (LTC, Rio de Janeiro, 2016).

15. O. Klein, *Z. Phys.*, **37** (1926) 895.
16. W. Gordon, *Z. Phys.*, **40** (1926) 117.
17. V. Fock, *Z. Phys.*, **38** (1926) 242.
18. A. Molgado, O. Morales and J. A. Vallejo, *Rev. Mex. Fis. E*, **64** (2018) 1-6.
19. L. H. Ryder, *Quantum Field Theory*, 2nd Edition, (Cambridge University Press, Cambridge, 1996).
20. D. Iwanenko and A. Sokolow, *Klassische Feldtheorie* (Akademie-Verlag, Berlin, 1953).
21. R. G. Cawley and E. Marx, *Int. J. Theor. Phys.* **1** (1968) 153.
22. G. E. A. Matsas, *Rev. Bras. Ensino Fís.*, **27** (2005) 137-145.
23. L. Parker and D. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, Cambridge, 2009).
24. N. D. Birrell and P. C. W. Davies, *Quantum fields in Curved Space* (Cambridge University Press, Cambridge, 2013).
25. L. C. B. Crispino, *Quantização a Baixas Frequências de Campos Bosônicos no Espaço-tempo de Schwarzschild e Aplicações* (Doctoral Thesis). Institute for Theoretical Physics. (São Paulo State University, 2001)
26. S. W. Hawking, *Nature* **248** (1974) 30.
27. S. W. Hawking, *Commun. Math. Phys.* **43** (1975) 199.
28. S. A. Fulling, *Phys. Rev. D* **7** (1973) 2850.
29. P. C. W. Davies, *J. Phys. A* **8** (1975) 609.
30. W. C. Unruh, *Phys. Rev. D* **14** (1976) 870.
31. L. C. B. Crispino, A. R. R. da Silva, G. E. A. Matsas, *Phys. Rev. D* **79** (2009) 024004.
32. L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, *Class. Quantum Grav.* **17** (2000) 19.
33. L. C. B. Crispino, *Phys. Rev. D* **77** (2008) 047503.
34. R. P. Bernar, L. C. B. Crispino, A. Higuchi, *Phys. Rev. D* **95** (2017) 064042.
35. C. F. B. Macedo, L. C. B. Crispino, V. Cardoso, *Phys. Rev. D* **86** (2012) 024002.
36. F. B. Silsbee, *J Res Natl Bur Stand*, **66C:2** (1962).
37. R. G. Littlejohn, *Gaussian, SI and Other Systems of Units in Electromagnetic Theory*, unpublished manuscript (2017). URL: http://bohr.physics.berkeley.edu/classes/221/1112/notes/emunits.pdf
38. R. D. Inverno, *Introducing Einstein's Relativity*, 1st Edition, (Clarendon Press, Oxford, 1992).
39. W. Greiner and D. Reinhardt, *Field Quantization*, 2nd Edition, (Springer, 1997).
40. J. D. Jackson, *Classical Electrodynamics*, 3rd Edition, (John Wiley & Sons, New York, 1998).
41. J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 9th Edition, (McGraw-Hill, New York, 2013).
42. D. P. Meira Filho, *Cálculo da Potência Irradiada por Correntes Clássicas Uniformemente Aceleradas no Espaço-Tempo de Minkoswki com Campos Bosônicos sem Massa via Teoria Clássica de Campos* (Monograph). Institute of Exact and Natural Sciences. (Federal University of Pará, 2003)
43. D. P. Meira Filho and J. K. S. Kamassury, *Rev. Bras. Ensino Fís.*, **40** (2018) e3316.
44. L. Landau and E. Lifshitz, *Curso de Física: Teoria do Campo*, 1st Edition, (Hemus, 2004).