Dynamics for an electric pendulum

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The Lagrangian formulation is an extensive tool for the analysis of physical systems. In particular, we have applied the Lagrangian procedure to deduce the dynamics and stability for an electric pendulum system. We have considered two cases; repulsive and attractive electric interactions which modify the dynamics of the classical simple pendulum model. We contrast both scenarios studying their restrictions, phase trajectories and stability points for this purpose.

Keywords: Lagrangian dynamics; phase space; simple pendulum.

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1. Introduction

Classical mechanics for a system of particles is beautiful explained in Landau's textbook [1]. The approach employed by Landau corresponds to the Lagrangian formulation, where the energy and degrees of freedom play an important role. Since the Galileo Galilei's era, a simple pendulum has been widely studied [2]. This model corresponds to the description of harmonic oscillations for a convenient limit, this is a basic mechanical system, which is applied to diverse branches of knowledge [3]. The simple pendulum model consists of a mass tied to a string that is attached to a fixed point and then from a certain angle is released to analyze its movement [4].

The electric pendulum studied in this letter is composed by a metallic ball of mass m and charge q, this ball is tied to a nylon isolator string of length l, which is released from angle ϕ. Another metallic ball with charge q_0 and mass m_0 is placed in a fixed position of vertical axis, as can be observed in Fig. 1. Charges and masses of both balls are chosen to be different. Interaction of the hanging mass with gravitational field of the Earth is important for the developed model, the mass of the fixed ball in vertical axis does not represent a considerable contribution to the system because both masses are much smaller than the mass of the Earth $M_e \ggg m, m_0$ and 1 kg $\geq m, m_0 > 0$; thus, gravitational interaction between the two balls can be neglected compared with that of the Earth. Besides, the length of the string l is much bigger than the dimensions of the balls, which can be considered as punctual particles but mass of the string can be neglected in this model since it is lower than mass of the suspended ball [5,6].

Our main objective in this work is to formulate the equations of motion for the electric pendulum system. The study of the orbits and the nature of the motion in each region of the system evolution are also analyzed.

The remainder of this work is organized as follows. In Sec. 2, we present the physics of the problem and provide the motion equations for the system. We analyse the regions and restrictions for phase trajectories, which synthesizes the dynamics of the system, in Sec. 3. The next section is devoted to identify the range of evolution as well as the stability points. Finally, in Sec. 5 we provide the conclusions and some closing remarks.

2. Electric Pendulum

The laws of movement for any mechanical system can be deduced from the universal principle of least action or the Hamilton's principle. This statement establishes that all mechanical systems can be described by a function for all possible continuous trajectories. This is called the Lagrangian and gives rise to the so called Euler-Lagrange equations of motion [1,4].

We analyze a simple pendulum restricted to a one-degree of freedom with variable ϕ, which is depicted in Fig. 1. Here, l_0 corresponds to a fixed length from the lower position of the pendulum to the second vertical fixed particle. There are two balls with masses m, m_0 and charges q, q_0 respectively. For the system developed at this work and for the mass m, which is being considered as the dynamic body, we define the Lagrangian function as

$$L_e = \frac{1}{2} m\dot{x}^2 - U(x), \tag{1}$$

where the first element $m\dot{x}^2/2$ represents the kinetic energy and $U(x)$ is the potential energy of the system. Note

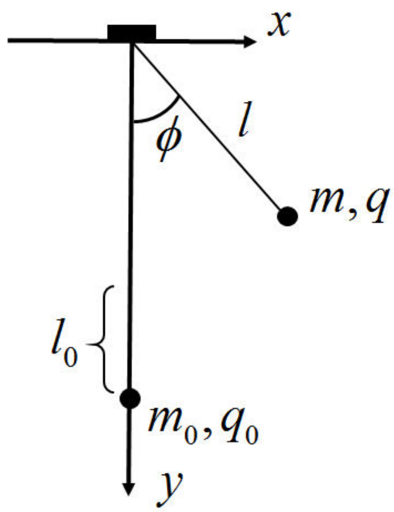


FIGURE 1. Model for an electric pendulum.

that even though motion of the system takes place in a bi-dimensional frame, actually the motion is one-dimensional according to the degrees of freedom.

The term $U(x)$ is expressed as

$$U(x) = U_g(x) + U_{qi}(x), \tag{2}$$

where $U_g = -mgl\cos\phi$ means the gravitational potential energy with $g = 9.8$ m/s^2 the acceleration due to Earth's gravity, we have taken the y axis vertical downwards. The other term $U_{qi}(x) = (-1)^i qq_0/4\pi\varepsilon_0 s$ corresponds to the electric potential energy, ε_0 denotes the electric permittivity in vacuum, and s is the positive rectilinear distance between charges. Observe that $i = \{0, 1\}$ refers to a repulsive or attractive interaction respectively [5,6]. Also, we must mentioned that nature of electric interaction depends on the sign of charges qq_0, but we have taken them as positive in magnitude and leave the repulsive or attractive interaction dependent exclusively on the sign.

We can set s in terms of ϕ by means of cosine's law

$$s^2 = (l + l_0)^2 + l^2 - 2ll_0\cos\phi.$$

We replace above expressions in Eq. (2) to obtain the total potential for the electric pendulum

$$U(\phi) = -mgl\cos\phi + \frac{(-1)^i qq_0}{4\pi\varepsilon_0\sqrt{(l+l_0)^2 + l^2 - 2ll_0\cos\phi}}, \tag{3}$$

notice that the potential function holds $U(\phi) = U(-\phi)$, then the system is governed by an even symmetry as it is known for the simple pendulum.

In Fig. 2 is shown $U(\phi)$ for specific constant values, as can be seen for $i = 0$ the repulsive electric potential creates a shape with three extreme points. Meanwhile, for $i = 1$ we observe an attractive electric potential, which is very similar to an ordinary simple pendulum with stable point around

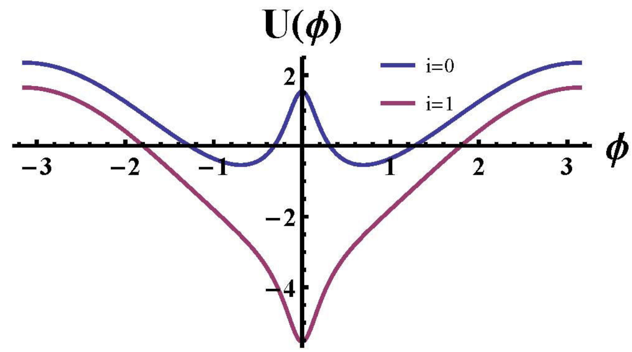


FIGURE 2. Potential for the electric simple pendulum with values, $mgl = 2$, $qq_0/4\pi\epsilon_0\sqrt{2ll_0} = 0.5$, $((l + l_0)^2 + l^2)/2ll_0 = 1.02$.

$\phi = 0$. It is worth noting that for the case $i = 0$ within a range of values, as we shall see, gravitational potential can be much bigger than the electric repulsive interaction, thus the shape of the potential is quite similar to an ordinary simple pendulum.

For the considered system, the Lagrangian function $L(\phi, \dot{\phi})$, corresponds to

$$L_e = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi - \frac{(-1)^i qq_0}{4\pi\varepsilon_0\sqrt{(l+l_0)^2 + l^2 - 2ll_0\cos\phi}}, \tag{4}$$

here we apply the formal theory of Euler-Lagrange equations, which are given by

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right) = \frac{dL}{d\phi}$$

for the one-dimensional case, thus

$$\frac{dL_e}{d\phi} = -mgl\sin\phi + \frac{(-1)^i qq_0 2ll_0\sin\phi}{4\pi\varepsilon_0\left((l+l_0)^2 + l^2 - 2ll_0\cos\phi\right)^{\frac{3}{2}}} \tag{5}$$

and

$$\frac{dL_e}{d\dot{\phi}} = ml^2\dot{\phi} \tag{6}$$

so the equation of motion for the system is given by

$$ml^2\frac{d^2\phi}{dt^2} = -mgl\sin\phi + \frac{(-1)^i qq_0 2ll_0\sin\phi}{4\pi\varepsilon_0\left((l+l_0)^2 + l^2 - 2ll_0\cos\phi\right)^{\frac{3}{2}}} \tag{7}$$

observe that all dynamic information for the electric pendulum is contained in the equation above. It is worth noting that if one of the charges $q, q_0 = 0$ in Eq. (7), we recover the typical simple pendulum motion equation.

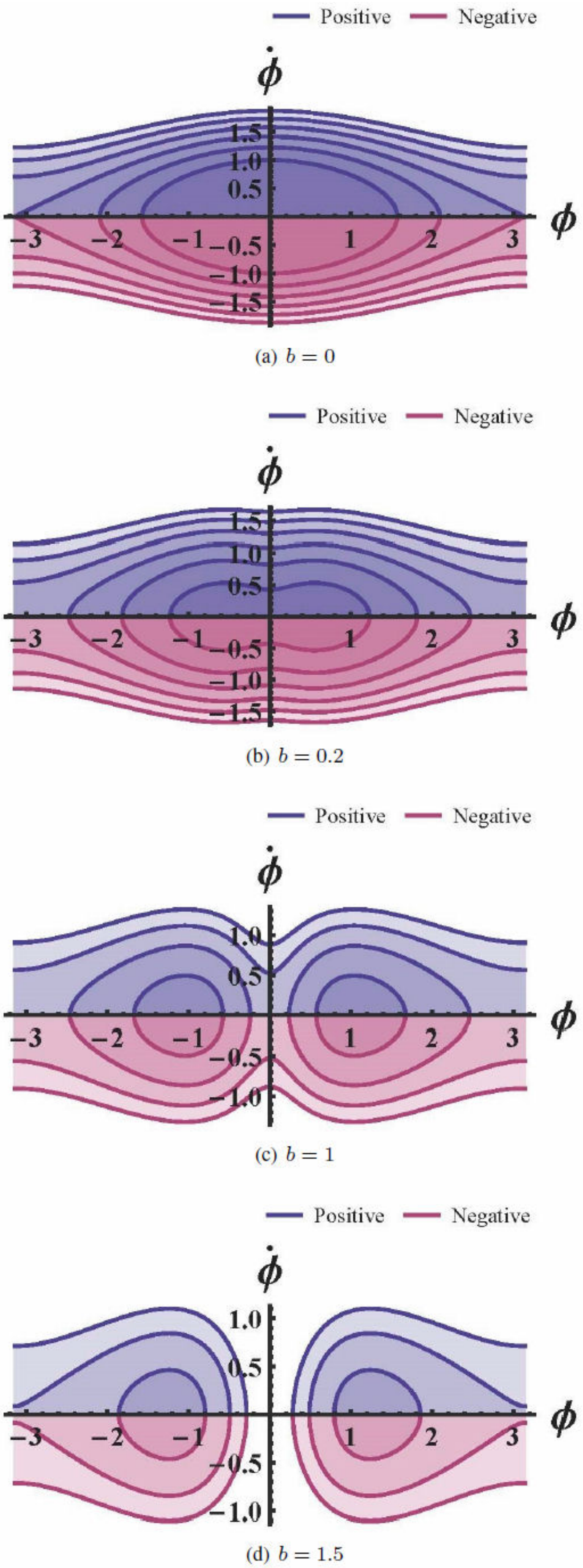


FIGURE 3. Trajectories on phase space for potential $U(\phi)$ with $a = 1$, $c = 1.135$ and $-\pi < \phi < \pi$.

3. Orbits in Phase Space

Instead of working with the equation of motion (7), we analyze the dynamics of the system by using the phase space formulation. In this context, the graph in x-y axes where positions and momenta are represented, is called phase space, every point depicted in it is called phase point, and the drawn curves correspond to phase curves; usually they are known as orbits or trajectories in phase space. Those which correspond to single points, are called stable or unstable equilibrium points [7,8].

The phase space gives a complete description about the dynamic behaviour of such system. Every phase point represents a dyad with coordinates of position and momentum of the particle at a given instant t_0. We begin by analyzing such orbits with $i = 0$, which are plotted in Fig. 3. The case $i = 1$, corresponds to rigid oscillations of the simple pendulum type, which are not considered in this work. From equations (5, 6) the one-dimensional Euler-Lagrange equation is

$$\frac{d}{dt}\left(\frac{dL_e}{d\dot{\phi}}\right) = \dot{\phi}\frac{d}{d\phi}\left(\frac{dL_e}{d\dot{\phi}}\right),$$

from where is obtained

$$\dot{\phi}^2 - a\cos\phi + \frac{b}{\sqrt{c - \cos\phi}} = E_0. \tag{8}$$

Observe that coordinates $\left(\dot{\phi}, \phi\right)$ are given for an specific value of energy E_0, where $a = 2g/l$, $b = qq_0/ml^2\pi\varepsilon_0\sqrt{2ll_0}$, $c = ((l + l_0)^2 + l^2)/2ll_0$. According to this, trajectories in the upper half plane are depicted for positive values of velocities, meanwhile negative momenta are drawn in down half plane.

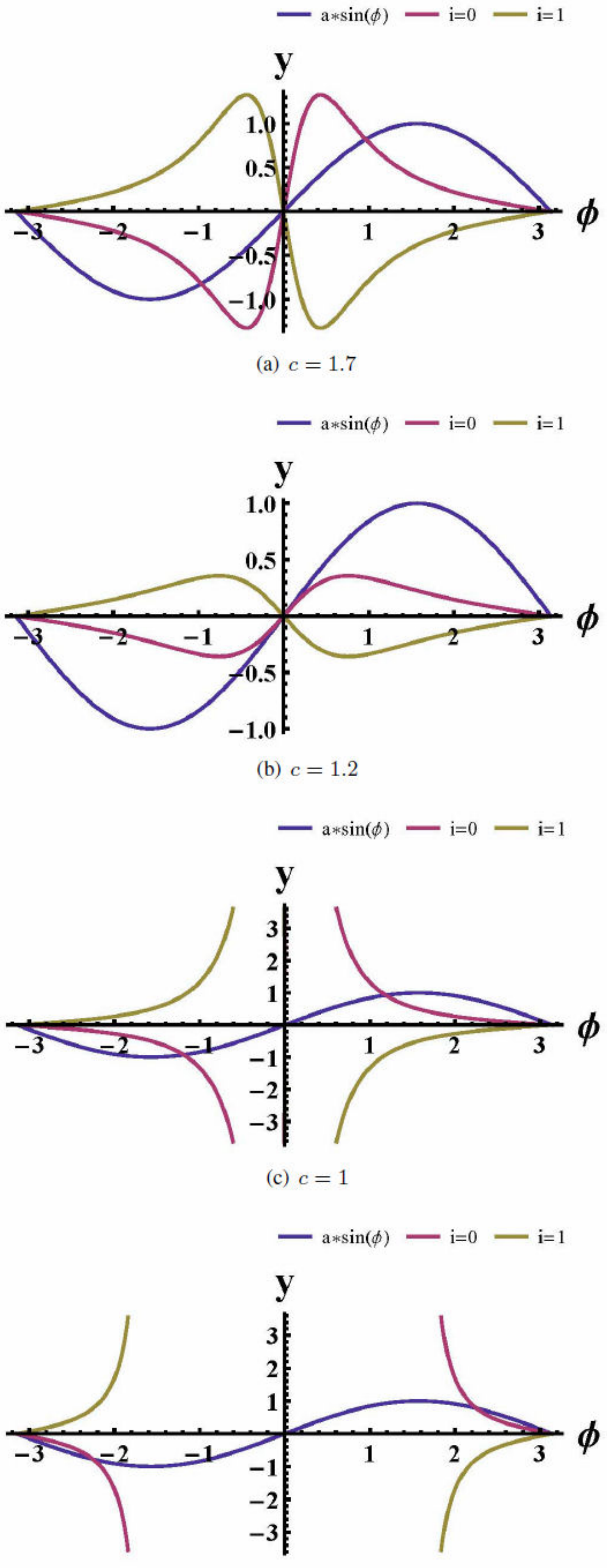
In Fig. 3(a) we observe that trajectories correspond to an stable equilibrium point $\phi = 0$, which is quite similar to simple pendulum. In Fig. 3(b) and 3(c), we note a splitting of two more stable points meanwhile the point $\phi = 0$ becomes an unstable equilibrium point given as a separatrix between both stable points. For the case 3(d) we observe a restricted region of motion around the point $\phi = 0$, and there are also two orbits around stable points at symmetric angles ϕ_0. Physically this means that the electric interaction is stronger than the gravitational for the system.

Orbits in phase space occur around stable and unstable equilibrium points of the system. This is illustrated in Fig. 4 where we have plotted both cases $i = \{0, 1\}$ in order to contrast equilibrium points. If we maximize the function $U(\phi)$ in expression (3), we must calculate $dU(\phi)/d\phi = 0$, from this relation we find a constriction for critical points according to

$$a\sin\phi + \frac{(-1)^i\, b\sin\phi}{(c - \cos\phi)^{\frac{3}{2}}} = 0,$$

with $a = mgl$, $b = qq_0/8\pi\varepsilon_0\sqrt{2ll_0}$, $c = ((l + l_0)^2 + l^2)/2ll_0$. Equilibrium points are given as the intersection of functions

$$\left\{a\sin\phi, -\frac{(-1)^i\, b\sin\phi}{(c - \cos\phi)^{\frac{3}{2}}}\right\}.$$



(a) $c = 1.7$

(b) $c = 1.2$

(c) $c = 1$

(d) $c = 0$

FIGURE 4. Stable and unstable equilibrium points for potential $U(\phi)$ with $a = 1, b = 1$ and $-\pi < \phi < \pi$.

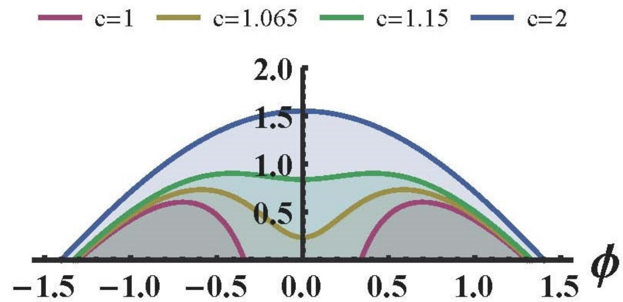


FIGURE 5. Range for evolution of ϕ with $a = 2$, $b = 0.45$ and $-\pi/2 < \phi < \pi/2$.

Note that in Fig. 4(a), there are three intersections $\{-\phi_0, 0, \phi_0\}$ for the case $i = 0$, meanwhile for $i = 1$ there is just one critical point in $\phi = 0$ which coincides with a typical simple pendulum system. In Figs. 4(c) and 4(d) for $i = 1$ a pair of stable equilibrium points appears $\{-\phi_0, +\phi_0\}$ and for $i = 0$, there are no stable or unstable points. We note a restricted region in the neighbourhood of $\phi = 0$ for both cases $i = \{0, 1\}$.

The case illustrated in Fig. 4(b) corresponds to a unique stable point $\phi = 0$ for both cases $i = \{0, 1\}$. They behave similar to simple pendulum systems, in other words, the electric interaction does not give a significative contribution.

4. Range of Evolution

Let us examine the regions in which the motion of the suspended particle occurs. Observe that the regions for motion are governed by constraint imposed in Eq. (8). This can be deduced from the positive argument in the square root radical with respect to $\dot{\phi}$. If we look for the evolution of the particle, the angle must evolve restricted to

$$a \cos\phi > \frac{b}{\sqrt{c - \cos\phi}},$$

which is plotted for several cases in Fig. 5. The intervals of oscillations are shown as bounded by shaded areas displayed on vertical axis. As parameter $c \to 1$, the range of motion goes from a real interval to imaginary values. The case $c = 0.5$ is not shown in Fig. 5 because interval of motion now turns into imaginary values.

5. Conclusions

In this work, we have deduced the equation of motion that describes the dynamics of the electric pendulum. The study of the orbits and the nature of the motion in each region of the system are also studied. We analyze the regions and restrictions for phase trajectories, which are different from those typically observed in a simple pendulum model, physically this is because bodies involved have electric charge as well as mass adding a new electric potential energy term to the Lagrangian which modifies considerably the original system.

As a consequence the ranges of stability are restricted depending on the values of parameters given in the potential function. Through this work we have contrasted the repulsive and attractive electric interaction which depends on the nature of the charge.

1. L. Landau Lifshitz. *Course of theoretical physics.* Elsevier, (2013).
2. H. Goldstein *et al*., *Classical mechanics*. Addison-Wesley, (2002).
3. D. G Zill, and M. R. Cullen *A first course in differential equations with modeling applications*. Thomson Education, (2006).
4. Dare A Wells, *Schaum's outline of theory and problems of lagrangian dynamics*. McGraw-Hill, (1967).
5. J. D. Jackson, *Classical electrodynamics*. John Wiley & Sons, (2012).
6. W. Greiner, *Classical electrodynamics*. Springer Science & Business Media, (2012).
7. V. I. Arnol'd, *Mathematical methods of classical mechanics*, Vol. **60**. Springer Science & Business Media, (2013).
8. C. Lanczos, *The Variational Principles of Mechanics*. Dover Publ., (1986).