

A geodesic approach for the harmonic oscillator

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The harmonic oscillator (HO) is present in all contemporary physics, from elementary classical mechanics to quantum field theory. It is useful in general to exemplify certain techniques in theoretical physics. In this work, we use a method for solving classical mechanics problems by first transforming them into a free particle form. This technique has been used before for solving the one-dimensional hydrogen atom, and also for solving the motion of a particle in a one-dimensional dipolar potential. Using canonical transformations, we convert the HO Hamiltonian into a free particle form which is very easy to solve. Our approach may be helpful to exemplify how canonical transformations may be used in mechanics and how it is possible to visualize the geometry of the new phase space. Besides, we expect it will help students to grasp what is meant when a problem has been transformed into another completely different one. We also intend the paper to exhibit the power granted by the Hamiltonian approach for analyzing mechanical systems.

Keywords: Hamiltonian mechanics; geodesic motion; harmonic oscillator; coordinates transformation.

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1. Introduction

The formalism of Hamiltonian mechanics [1–3] is frequently not appreciated in full by most undergraduate and even by some graduate students. Instructors usually exemplify the use of Hamiltonian techniques by dealing with one-dimensional problems, particles interacting through the Newtonian potential or joined by springs, and the like. However, the mentioned problems may be more readily addressed using the Lagrangian formalism [1–4]. Therefore, an example is needed, which may be quickly posed using the Hamiltonian and which could be difficult to formulate within the Lagrangian formalism. This kind of problem is seldom if ever discussed in advanced undergraduate or beginning graduate classical mechanics. For the lack of this kind of examples, many students wonder if the Hamiltonian is just another technique to write down the classical equations of motion, which just happen to be more cumbersome to employ than the Lagrangian approach.

With the aforementioned difficulty in mind, in this work we purport to use the Hamiltonian approach to establish a quite different property, one that should appear at first difficult to believe to a student of advanced mechanics which is learning from standard textbooks as [3] or from more elementary books as [4], namely that motion in Kepler's prob-

lem is equivalent to geodesic (free) motion on a hypersurface in phase space. In this paper, we discuss *not* the Kepler problem *but* the harmonic oscillator, a simpler problem, showing that it may be regarded as equivalent to the motion of a non-interacting or free particle but moving on a curved space. In posing the problem, we were partially inspired by the Boya *et al.* paper [5], which establishes the equivalence between the motion under the 1D Kepler potential [6, 7] and free motion on a circle. The Boya *et al.* paper, in turn, follows in the wake of Moser proof of the equivalence between bound Kepler's motions with geodesic motions on a sphere [8].

A note of caution is in order before embarking in our analysis Moser used global techniques for giving his proof but we are instead going to discuss the matter in the standard local methods of Hamiltonian mechanics [1, 3, 4, 9]. This paper, we emphasize, intends to be a pedagogical introduction to the capabilities of the Hamiltonian formulation for an audience of physics and mathematics students or instructors.

2. Hamiltonian approach to the harmonic oscillator

The Hamiltonian of a particle of mass m in a harmonic oscillator potential is

$$H(p, q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2 \quad (1)$$

where ω is the natural frequency of the oscillator. As the system is autonomous, the energy, E , is a constant of the motion.

In Lagrangian mechanics, the Euler-Lagrange equations, the differential equations which describe the motion, are invariant under change of coordinates. Therefore, one is allowed to perform a suitable change of coordinates, $q_i \rightarrow Q_j(q_i)$, without losing physical information. The same thing does not happen in the framework of Hamiltonian mechanics. Since the coordinates and the momenta are in the same hierarchy level, we are free to carry out changes in the form

$$(p_i, q_i) \rightarrow (P_j(p_i, q_i), Q_j(p_i, q_i)). \quad (2)$$

However, unlike E-L equations, Hamilton equations do not hold for *any* transformation of the type above, but for some special category, named *canonical transformations* (CT). The principal feature of a CT is that it preserves the canonical relationship between the pairs (P_i, Q_i) , it is $\{P_i, Q_i\} = 1$, where

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} \quad (3)$$

is the Poisson bracket of the functions $f(p, q)$ and $g(p, q)$.

Let us begin mentioning that, in advanced classical mechanics, the following canonical transformation (CT) [3, 9, 15, 16],

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q, \quad p = \sqrt{2m\omega P} \cos Q. \quad (4)$$

is sometimes used as a way for solving the problem.

In the new coordinates, Q and P , introduced in (4), the Hamiltonian (1) becomes independent of Q ,

$$H'(P, Q) = \omega P. \quad (5)$$

However, it is not easy to associate any physical significance to the new momentum, P^i , besides the fact that it has dimensions of action. In any case, such an association is not necessary for any way to solve the problem. In this work, we present another canonical transformation with the same purpose but offering a menu richer in physical interpretations. But, in the end, our purpose is to find new canonical coordinates able to transform (1) into a Hamiltonian with no potential energy term. That is a free particle Hamiltonian, but at the prize of converting the Euclidean space in which the particle moves into a curved one. Earlier works in such directions may be found in [12, 13] but without the emphasis in the geometry of the transformed phase-space manifold.

3. A geodesic approach in a different phase space

The way the coordinates are used in the canonical transformations (4), and since the angular argument does not appear

explicitly in the result, suggests that to obtain an appropriate CT, we must begin by defining,

$$q = \frac{P}{m\omega} \sin \Phi, \quad p = P \cos \Phi, \quad (6)$$

where $\Phi(P, Q)$ is a function that needs to be chosen to make the transformation canonical. Notice that the change (6) removes the potential in the Hamiltonian, leaving only the kinetic energy terms in the Hamiltonian expressed in the new (P, Q) phase space coordinates,

$$\tilde{H}(P, Q) = \frac{P^2}{2m}. \quad (7)$$

As the solutions following from this Hamiltonian ought to behave as free particles in the phase space, defined by the transformations (6), it is natural to call the motion of the particle in the new coordinates *geodesic*. That is, we are substituting dynamics by geometry. A similar but more general approach has been previously taken in [12]. It should be noted also that the new coordinates have the same dimensions as the original pair (p, q) .

For P and Q to be canonical their Poisson bracket should equal 1,

$$\{q, p\}_{Q,P} = \frac{\partial q}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q} \frac{\partial q}{\partial P} = 1, \quad (8)$$

this condition gives a simple differential equation for the function $\Phi(P, Q)$,

$$\frac{P}{m\omega} \frac{\partial \Phi}{\partial Q} = 1. \quad (9)$$

After integrating we have

$$\Phi(P, Q) = \frac{m\omega Q}{P}, \quad (10)$$

where we have chosen as zero the arbitrary function of P required. Therefore, the complete transformation is

$$q = \frac{P}{m\omega} \sin \left[\frac{m\omega Q}{P} \right], \quad p = P \cos \left[\frac{m\omega Q}{P} \right]. \quad (11)$$

As this transformation is canonical, Hamilton equations hold for the new Hamiltonian, \tilde{H} . We should mention that the above transformation is the same as the given in [13].

The time evolution of the system is

$$P(t) = P_0, \quad Q(t) = (P_0/m)t + Q_0. \quad (12)$$

where P_0 and Q_0 are starting conditions. Along the process energy, E , is conserved $H = \tilde{H} = E$, so we can choose

$$P_0 = \pm \sqrt{2mE}. \quad (13)$$

Hence, the solutions are

$$\begin{aligned} q(t) &= \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \phi_0), \\ p(t) &= \sqrt{2mE} \cos(\omega t + \phi_0), \end{aligned} \quad (14)$$

where $\phi_0 = m\omega Q_0/P_0$ is a constant depending on the initial conditions. On the other hand, if our purpose were simply to find a transformation that eliminates potential-like terms in (1), we may proceed in a more general way as follows. We may separate the Hamiltonian as

$$H = \frac{p^2}{2m} \left(1 + \frac{m^2\omega^2 q^2}{p^2} \right), \quad (15)$$

this separation suggests a definition for the new momentum as

$$P(p, q) = p \sqrt{1 + \frac{m^2\omega^2 q^2}{p^2}}, \quad (16)$$

where the positive sign of the square root has been taken arbitrarily; also note that the dimensions of the momentum are not altered by the last step. In order to complete the canonical transformation, we use a generating function G such that [2, 3]

$$p = \frac{\partial G}{\partial q} \quad \text{and} \quad Q = \frac{\partial G}{\partial P}. \quad (17)$$

Solving for $p(P, q)$, from definition (16) and integrating with respect to q , we find

$$G(P, q) = \frac{q}{2} \sqrt{P^2 - m^2\omega^2 q^2} + \frac{P^2}{2m\omega} \arctan \left[\frac{m\omega q}{\sqrt{P^2 - m^2\omega^2 q^2}} \right] \quad (18)$$

up to an additive arbitrary function of P , which we choose as a vanishing one. From the second condition of (17) and writing P in terms of the original coordinates we obtain

$$Q(q, p) = \frac{\sqrt{p^2 + m^2\omega^2 q^2}}{m\omega} \arctan \left[\frac{m\omega q}{p} \right]. \quad (19)$$

Equations (16) and (19) are actually the inverse transformation of Eqs. (11).

In some cases it is useful to perform a change of coordinates to simplify the description of some physical problems and find their solutionⁱⁱ. For example, let $X = X(x, y)$ and $Y = Y(x, y)$ be two independent continuous functions of the points on the plane. If they have continuous inverse, $x = x(X, Y)$ and $y = y(X, Y)$, we can identify any point of the plane by using these new pair of coordinates (X, Y) . In order to see how the new coordinates looks geometrically, we plot the graphic of the simplest equations of the new system, *i.e.* the curves generated by setting $X = \text{constant}$ and $Y = \text{constant}$. Since we associate each of these simple curves to one single coordinate, they are named *coordinates curves* [17].

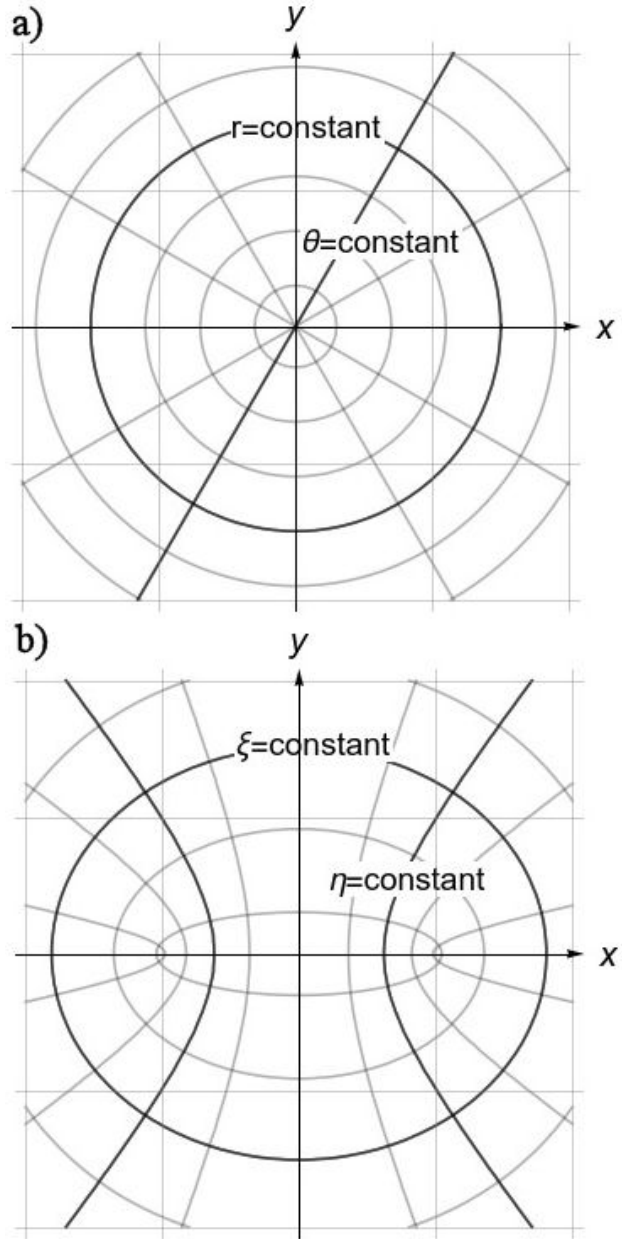


FIGURE 1. Two coordinates systems, the polar coordinate system at the left and the elliptical coordinate system at the right, described by the variables (r, θ) and (ξ, η) respectively. In both systems, there are represented a pair of coordinate curves on the original xy plane.

Now we will plot some coordinates curves $Q = \text{constant}$ and $P = \text{constant}$ in the original phase space to give some idea about how the transformation looks geometrically.

In the figure above the curves of constant, P coincide with the well-know ellipses of constant energy of the problem, while the curves associated to constant Q are simply given by (19), and do not have any special registered name. Therefore, it is easy to see that the CT presented in this work makes the system to move in the “natural” coordinate curves.

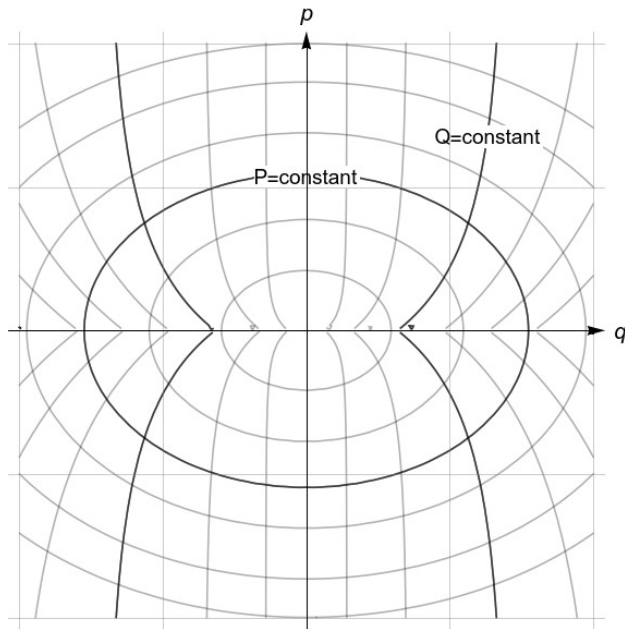


FIGURE 2. Representation of the canonical transformation in the original coordinates in arbitrary units, with $m = 1$ and frequency $\omega = 1$. It is easy to see that the CT presented in this work finds a new coordinate system for the phase space with the special feature that the P -curves coincide always with the energy constant curves on which, indeed, the system moves.

4. Conclusions

In conclusion, we have found a CT: $(p, q) \rightarrow (P, Q)$ that makes the equations of motion take the form of the simplest physical problem, namely, the free particle. That is, the system moves freely along the curves with P constant, which for the HO and in the original phase space, are ellipses representing the curves of constant energy. The way we have obtained this transformation, by defining the new momentum as in (15), can be extended easily to other problems of one degree of freedom and time-independent. This excludes the possibility of transforming in the same way, a harmonic system with a frequency mass changing in time.

We expect the analysis of an HO given in this work would help clarify the importance of the Hamiltonian formulation, which allows transformations like the one discussed here: a CT making a completely bounded system equivalent to one without any interaction whatsoever—at the prize of deforming the phase space manifold. We have shown that the classical motion under an elastic potential should be made equivalent to free motion on a certain curve in phase space. Of

course, we do not intend the discussion in this work to be interpreted as simply another way of solving the harmonic oscillator. Our point here is only to show beginning graduate students the enormous power the Hamiltonian formulation affords over the description of the motion of bodies.

Some students may still wonder whether we could extend this method to the quantum mechanics. The answer is definitively no since the quantum transformations need unitary operators \hat{U} to be produced, and not all conceivable canonical transformation may have an equivalent \hat{U} associated. More simply, think of our example at hand: the HO, if it could be possible to make a quantum transformation from this system to a free particle, then the HO energy spectrum would be continuous in the transformed system. This is impossible in quantum mechanics as, remember, unitary transformations always preserve the operator eigenvalues [13]. However, the possibility of constructing solutions of the Schrödinger equation, using canonical transformations involving non-unitary transformations, is discussed in [14].

We have one more point to mention given the free nature of the motion in the transformed phase space, the topology of phase space should be special, a sort of cylinder in which the position coordinate can grow without limit, but without the particle which it describes going too far from the point to which the spring is attached. This feature may also offer the opportunity of relating the techniques of the Hamiltonian formulation with certain aesthetic aspects of the description of motion. Such features have always played a relatively important part in the theoretical aspects and in the development of mechanics [15, 16, 18]. As we said before, if we accept the freedom granted by the Hamiltonian formulation and adopt a geometric standpoint, we may convince ourselves that the notion of interaction potential is used because we insist in describing motion using Euclidean geometry. That is, such interaction potential partially lose their importance to be substituted by phase space geometric features, as we exhibited in this work. From this standpoint, interactions may be conceived as a warping of phase space. Dynamics can be replaced by geometry.

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i. The CT proposed has a clear relationship with a transformation to action angle variables, but such matters may be distracting when we first discuss the Hamiltonian formalism pinpointing

its advantages and discussing possible uses.

ii. This occurs not only in elementary physics such as classical mechanics but also in advance topics like the path integral for-

- mulation in quantum mechanics [10, 11].
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