Spectral generalized function method for solving homogeneous partial differential equations with constant coefficients

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A method based on a generalized function in Fourier space gives analytical solutions to homogeneous partial differential equations with constant coefficients of any order and number of dimensions. The method exploits well-known properties of the Dirac delta, reducing the differential mathematical problem into the factorization of an algebraic expression that is finally integrated. In particular, the method was utilized to solve the most general homogeneous second-order partial differential equation in Cartesian coordinates, finding a general solution for non-parabolic partial differential equations, which can be observed as a generalization of d'Alambert solution. We found that the traditional classification, *i.e*., parabolic, hyperbolic and elliptic, is not necessary, reducing the classification to only parabolic and non-parabolic cases. We paid special attention for parabolic partial differential equations, analyzing the general 1D homogeneous solution of the photoacoustic and photothermal equations in the frequency and time domain. Finally, we also used the method to solve the Helmholtz equation in cylindrical coordinates, showing that it can be used in other coordinates systems.

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1. Introduction

In 1992 Donnelly and Ziolkowski [1] reported a powerful method for constructing solutions of partial differential equations, based on the use of the Dirac delta distribution. This method allows constructing analytical solutions to partial differential equations by Fourier transforming the partial differential equation. At first thought, this method seems to be similar to techniques found in the classical literature [2–5] however after a deeper revision, the referred technique turns out to be especially powerful as it allows constructing new solutions as shown in the referred manuscript.

In this report, we extend the method reported by Donnelly and Ziolkowski in two ways. First, we introduce spectral generalized functions (SGFs) in Fourier space and then, taking advantage of some properties of the Dirac delta, we transform the differential mathematical problem into the factorization of an algebraic expression. By doing so, we make the method more powerful allowing us to find the general solutions of partial differential equations (PDEs). The SGF method allowed to find the solution to the most general homogeneous second order PDE in Cartesian coordinates, readily,

$$(A\partial_{xx} + B\partial_{xy} + C\partial_{yy} + D\partial_x + E\partial_y + F)f(x,y) = 0, \tag{1}$$

where A, B, C, D, E, and F are constants. To the best of our knowledge, there is no report on a solution to Eq. (1) for coefficients (A, B, C, D, E, and F) non zero, which may find several applications in physics and engineering. Interestingly, the method also allowed to redefine the classification of this PDE in only parabolic and non-parabolic.

In the next section, to elucidate the method, we solve two simple ordinary differential equations. Next, the method is used to find the most general solution to Eq. (1) for the case $D = E = F = 0$. To verify the solution, well-known examples, are given. Then, by performing a change of variables, we transform Eq. (1) into a new 2D-homogeneous PDE, which is easily solved to obtain solutions to Eq. (1). Finally, the SGFs method is utilized to solve the 2D Helmholtz equation in cylindrical coordinates, showing that it can be used in other coordinate systems.

2. Description of the spectral generalized function method

To describe our method, let us begin the description by considering the following simple, but yet, illustrative example of solving,

$$f'(x) = 0, \tag{2}$$

whose solution is a constant and where $f' \equiv df/dx$ throughout the manuscript. Instead of proposing the obvious solution, let us consider the one-dimensional Fourier transform pair

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \exp(-i2\pi ux)\, dx, \tag{3}$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(u) \exp(+i2\pi ux)\, du. \tag{4}$$

By Fourier transforming equation (2) through Eq. (4),

$$f'(x) = \int_{-\infty}^{\infty} \hat{f}(u) i2\pi u \exp(i2\pi ux)\, du = 0, \tag{5}$$

by the linear independence of the functions $\exp(i2\pi ux)$, one obtains

$$u\hat{f} = 0. \tag{6}$$

To solve Eq. (6), let us recall some well-known properties of the one-dimensional Dirac delta distribution $\delta(u)$

$$\begin{aligned} \int_{-\infty}^{\infty} u^1 \delta(u) du &= u^1|_{u=0} = 0,\\ \int_{-\infty}^{\infty} u^2 \delta(u) du &= u^2|_{u=0} = 0,\\ &\vdots \\ \int_{-\infty}^{\infty} u^{n-1} \delta(u) du &= u^{n-1}|_{u=0} = 0,\\ \int_{-\infty}^{\infty} u^{n} \delta(u) du &= u^{n}|_{u=0} = 0, \end{aligned} \tag{7}$$

or

$$\begin{aligned} u^1\delta(u) &= 0,\\ u^2\delta(u) &= 0,\\ &\vdots\\ u^{n-1}\delta(u) &= 0,\\ u^n\delta(u) &= 0, \end{aligned} \tag{8}$$

n being a positive integer. We have additionally that,

$$\begin{aligned} u^2 \frac{d^1\delta(u)}{du^1} &= 0,\\ u^3 \frac{d^2\delta(u)}{du^2} &= 0,\\ &\vdots\\ u^{n-1} \frac{d^{n-2}\delta(u)}{du^{n-2}} &= 0,\\ u^{n} \frac{d^{n-1}\delta(u)}{du^{n-1}} &= 0, \end{aligned} \tag{9}$$

for integers $n > 1$.

Before proceeding further, it is worth outlining that the Dirac delta is not strictly a function. The Dirac delta is a singular distribution that assigns to each test function φ the value $\varphi(0)$. This distribution is commonly denoted as a δ distribution. As it is widely referred to as a function in electrical engineering, optical and physical sciences [6–8], in the herein report we will use this notation being aware of the limits of its use.

Applying the Dirac delta Eqs. (7) and (8), for our example given in Eq. (6), one could simple propose as a first possible solution,

$$\hat{f}(u) = \delta(u). \tag{10}$$

As $u\delta(u) = 0$, Eq. (8), one can propose a more general solution for Eq. (6) as,

$$\hat{f}(u) = \delta(u)\hat{g}(u), \tag{11}$$

being $\hat{g}(u)$ an arbitrary well-behaved function in the frequency domain. Because this function is in Fourier, space we will refer to it as our ***spectral generating function*** (**SGF**).

By using Eq. (3) to inverse transform Eq. (11), we can now see that the inclusion of the SGF allows us to obtain the general solution of Eq. (2) as,

$$f(x) = \int_{-\infty}^{\infty} \delta(u)\hat{g}(u) \exp(i2\pi ux)\, du = \hat{g}(0). \tag{12}$$

In Eq. (12), $\hat{g}(0)$ is an arbitrary constant that should be determined by the boundary conditions.

Now that a preliminary idea of our method has been described, we will proceed with our description by considering a second simple example,

$$f''(x) = 0. \tag{13}$$

The Fourier transform of Eq. (13) reads,

$$-4\pi^2 u^2 \hat{f}(u) = 0. \tag{14}$$

Again, based on the properties given in Eqs. (8) and (9), one can propose the general form of $\hat{f}$ in Eq. (14) as,

$$\hat{f}(u) = \hat{g}_1(u)\delta(u) + \hat{g}_2(u)\delta'(u). \tag{15}$$

It should be noticed that now there are two SGF's in Eq. (15).

By inverse transforming Eq. (15) one obtains,

$$f(x) = \int_{-\infty}^{\infty} [\hat{g}_1(u)\delta(u) + \hat{g}_2(u)\delta'(u)] \exp(i2\pi ux)du$$
$$= [\hat{g}_1(u) - \hat{g}_2'(u)]_{u=0} + [-i2\pi\hat{g}_2(0)]\, x, \tag{16}$$

since $[\hat{g}_1(u) - \hat{g}_2'(u)]_{u=0}$ and $-i2\pi\hat{g}_2(0)$ are constants, Eq. (16) is the expected solution.

With the simple above examples, the usefulness of the inclusion of the SGFs has been shown. In the following, we apply our method for more general examples.

3. A solution to the homogeneous second order PDE with constant coefficients and $D = E = F = 0$

First, I consider Eq. (1) with $D = E = F = 0$

$$(A\partial_{xx} + B\partial_{xy} + C\partial_{yy})\, f(x,y) = 0. \tag{17}$$

The two-dimensional Fourier transform of Eq. (17) can be written as,

$$\hat{f}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \times \exp\left[-i2\pi(ux+vy)\right] dxdy, \tag{18}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u,v) \times \exp\left[+i2\pi(ux+vy)\right] dudv. \tag{19}$$

By using Eq. (19) it follows,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u,v)\left[-4\pi^2\left(Au^2 + Buv + Cv^2\right)\right] \times \exp\left[i2\pi(ux+vy)\right] dudv = 0. \tag{20}$$

Again, from the linear independence of $\exp\left[i2\pi(ux+vy)\right]$ functions, we have

$$-4\pi^2 A\left(u^2 + \frac{B}{A}uv + \frac{C}{A}v^2\right)\hat{f}(u,v) = 0. \tag{21}$$

To apply our method, one has to look for factorization in Eq. (21) of the form,

$$\left(u^2 + \frac{B}{A}uv + \frac{C}{A}v^2\right)\hat{f}(u,v) = (u-\alpha v)(u-\beta v)\hat{f}(u,v) = 0. \tag{22}$$

In Eq. (22)

$$\alpha = -\frac{B - \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad \beta = -\frac{B + \sqrt{B^2 - 4AC}}{2A}. \tag{23}$$

By using properties from Eq. (7) and (8) to fulfill Eq. (21), the solution in the frequency domain can be written as,

$$\hat{f}(u,v) = \hat{g}_1(u,v)\delta(u-\alpha v) + \hat{g}_2(u,v)\delta(u-\beta v). \tag{24}$$

By inverse Fourier transforming Eq. (24) one obtains,

$$f(x,y) = g_1(\alpha x + y) + g_2(\beta x + y). \tag{25}$$

In Eq. (25) g_1 and g_2 represent arbitrary functions that correspond to the inverse Fourier transform of $\hat{g}_1$ and $\hat{g}_2$, respectively. By direct substitution in the differential Eq. (17) one can verify that Eq. (25) is a correct general solution.

3.1. Non-parabolic PDE

We first want to note that Eq. (25) is the most general solution of Eq. (1) with $D = E = F = 0$ for the non-parabolic PDE (hyperbolic or elliptic), which is shown in the next two examples.

3.1.1. 1D-wave equation (Hyperbolic partial differential equation)

If in Eq. (17) $A = c^2$, $B = 0$, $C = -1$ and $y = t$ we obtain the $1-$D wave equation

$$\left(c^2\partial_{xx} - \partial_{tt}\right) f(x,t) = 0. \tag{26}$$

Thus $B^2 - 4AC = 4c^2 > 0$; in Eq. (22), $\alpha = -\beta = 1/c$ and the solution (25) can be written as

$$f(x,t) = g_1\left(t + \frac{x}{c}\right) + g_2\left(t - \frac{x}{c}\right), \tag{27}$$

this is the d'Alambert solution of the 1D$-$wave equation. Since Eq. (25) includes the solution for $B \neq 0$, this result shows that Eq. (25) can be considered a generalization of the d'Alambert solution.

3.1.2. 2D-Laplace equation (Elliptic partial differential equation)

Here we consider Eq. 17 $A = C = 1$ and $B = 0$

$$(\partial_{xx} + \partial_{yy})\, f(x,t) = 0, \tag{28}$$

consequently $B^2 - 4AC = -4 < 0$; in Eq. (23) $\alpha = -\beta = i$, and the solution (25) becomes

$$f(x,y) = g_1(ix + y) + g_2(-ix + y). \tag{29}$$

To show that Eq. (29) is indeed a correct solution, let us consider the 2D$-$Laplace equation subject to the boundary conditions, (i) $f(x,0)=0$, (ii) $f(x,a)=0$, (iii) $f(0,y)=f_0(y)$ and (iv) $f(x,y)=0$ when $x\to\infty$. Since $g_1(ix+y)$ and $g_2(-ix+y)$ are both arbitrary functions, and because there are two homogeneous boundary conditions in x, harmonic solutions $g_1(x,y)=\Gamma_1 \mathrm{e}^{+ik(ix+y)}+\Gamma_2\mathrm{e}^{-ik(ix+y)}$ and $g_2(x,y)=\Gamma_3\mathrm{e}^{-ik(ix+y)}+\Gamma_4\mathrm{e}^{-ik(-ix+y)}$ are proposed; where k and Γ_j, are complex constants. Thus,

$$f(x,y)=\left(\Gamma_1\mathrm{e}^{iky}+\Gamma_4\mathrm{e}^{-iky}\right)\mathrm{e}^{-kx} + \left(\Gamma_3\mathrm{e}^{iky}+\Gamma_2\mathrm{e}^{-iky}\right)\mathrm{e}^{kx}. \quad (30)$$

Boundary condition (iv) implies that k must be a positive or negative real number, but not both. Taking $k\in\mathbb{R}^+$, Γ_2 and Γ_3 must be equal to zero, thus $f(x,y)=\left(\Gamma_1\mathrm{e}^{iky}+\Gamma_4\mathrm{e}^{-iky}\right)\mathrm{e}^{-kx}$. Applying boundary condition (i), $\Gamma_1=-\Gamma_4$, and from boundary condition (ii), it is obtained that $k=n\pi/a$, $n\in\mathbb{N}\cup\{0\}$. With these restrictions, the following family of solutions is obtained,

$$f_n(x,y)=\Gamma_n\sin\left(n\pi\frac{y}{a}\right)\exp\left(-n\pi\frac{x}{a}\right). \quad (31)$$

And the general solution

$$f(x,y)=\sum_{n=0}^{\infty}\Gamma_n\sin\left(n\pi\frac{y}{a}\right)\exp\left(-n\pi\frac{x}{a}\right). \quad (32)$$

To obtain Γ_n we apply the boundary condition (iii),

$$f(0,y)=\sum_{n=0}^{\infty}\Gamma_n\sin\left(n\pi\frac{y}{a}\right)=f_0(y). \quad (33)$$

Multiplying this equation by $\sin(m\pi y/a)$, where $m\in\mathbb{N}\cup\{0\}$ and integrating from 0 to a one obtains:

$$\Gamma_n=\frac{2}{a}\int_0^a f_0(\eta)\sin\left(n\pi\frac{\eta}{a}\right)d\eta. \quad (34)$$

Solution (32) with Γ_n given by Eq. (34), for the proposed boundary value problem can be corroborated by the separation of variables method [9].

3.2. Parabolic PDE

A special study is necessary when in Eq. (17) $B^2-4AC=0$ (i.e., parabolic PDE), in this case $\alpha=\beta=-B/2A$ and solutions (25) are degenerated. To find the second solution, we write

$$\left(\partial_x+a\partial_x\right)\left(\partial_x+a\partial_x\right)f(x,y)=0, \quad (35)$$

the Fourier transform of the differential equation (35) implies that,

$$(u+av)^2\hat{f}(u,v)=0, \quad (36)$$

with $a\equiv B/2A$. For Eq. (36) the general form of the SGF can be written as,

$$\hat{f}(u,v)=\delta\left(u+av\right)\hat{g}_1(u,v) + \frac{d\delta\left(u+av\right)}{d\left(u+av\right)}\hat{g}_2(u,v). \quad (37)$$

Thus, by inverse transforming Eq. (37) one obtains $f(x,y)=f_1(x,y)+f_2(x,y)$, where

$$f_1(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(u+av\right)\hat{g}_1(u,v) \times\exp\left[i2\pi(ux+vy)\right]dudv=g_1\left(-ax+y\right), \quad (38)$$

and

$$f_2(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\delta\left(u+av\right)}{d\left(u+av\right)}\hat{g}_2(u,v) \times\exp\left[i2\pi(ux+vy)\right]dudv. \quad (39)$$

Making the change of variable $s=u+av$, one obtains,

$$f_2(x,y)=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{d\delta(s)}{ds}\hat{g}_2\left(s-av,v\right) \times\exp\left[i2\pi\left[\left(s-av\right)x+vy\right]\right]ds\right\}dv$$
$$=\int_{-\infty}^{\infty}\left[x\hat{g}_2(v)+\hat{g}_3(v)\right]\exp\left[i2\pi\left(ax+y\right)v\right]dv$$
$$=xg_2\left(-ax+y\right)+g_3\left(-ax+y\right). \quad (40)$$

By switching the integration order and following a similar procedure as above one obtains

$$f_2(x,y)=\frac{1}{a}\left[yg_2\left(x-ay\right)+g_3\left(x-ay\right)\right]. \quad (41)$$

Thus, the general solution is,

$$f(x,y)=g_1\left(ax-y\right)+\left(ax-y\right)g_2\left(ax-y\right). \quad (42)$$

Being g_i, $i=$1, 2, arbitrary functions. By direct substitution of Eq. (42) into Eq. (17) one can confirm that the solution is correct.

4. Solutions of 2D-homogeneous PDE

The SGFs method also allows finding solutions to the general Eq. (1). By making the following change of variables in Eq. (1),

$$\gamma(x,y)=\alpha x+y \quad \text{and} \quad \eta(x,y)=\beta x+y, \quad (43)$$

one gets,

$$\begin{aligned}&[-\left(B^2-4AC\right)\partial_{\gamma\eta}+A\left(D\alpha+E\right)\partial_\gamma\\&+A\left(D\beta+E\right)\partial_\eta+AF]f(\gamma,\eta)=0.\end{aligned}\tag{44}$$

Here, α and β are given by Eq. (23).

4.1. Non-parabolics PDEs

For $B^2-4AC\neq 0$, Eq. (44) can be written as

$$\left(\partial_{\gamma\eta}-G\partial_\gamma-H\partial_\eta-I\right)f(\gamma,\eta)=0,\tag{45}$$

where

$$G\equiv A\frac{D\alpha+E}{B^2-4AC},\qquad H\equiv A\frac{D\beta+E}{B^2-4AC},\qquad\text{and}$$
$$I\equiv A\frac{F}{B^2-4AC}.\tag{46}$$

Before finding a complete solution to Eq. (45), two examples we will analyze.

4.1.1. $D=E=F=0$

If $D=E=F=0$ then $G=H=I=0$ and Eq. (45) becomes,

$$\partial_{\gamma\eta}f(\gamma,\eta)=0,\tag{47}$$

whose solutions are:

$$f(\gamma,\eta)=g_1f(\gamma)+g_2(\eta),\tag{48}$$

obtained by following Eq. (24).

4.1.2. F=0

If $F=0$, then

$$\left(\partial_{\gamma\eta}-G\partial_\gamma-H\partial_\eta\right)f(\gamma,\eta)=0.\tag{49}$$

Equation (49) can be rewritten as,

$$\begin{aligned}&\partial_\gamma\left[\left(\partial_\eta-G\right)f(\gamma,\eta)\right]=\partial_\eta\left[Hf(\gamma,\eta)\right]\qquad\text{or}\\&\partial_\eta\left[\left(\partial_\gamma-H\right)f(\gamma,\eta)\right]=\partial_\gamma\left[Gf(\gamma,\eta)\right].\end{aligned}\tag{50}$$

Equations (50) suggest that a family of solutions can be obtained if one chooses,

$$f(\gamma,\eta)=g(K\gamma+J\eta),\tag{51}$$

here J and K are non-zero complex constants. By defining $\varphi(\gamma,\eta)\equiv K\gamma+J\eta$ an ordinary differential equation is obtained, namely,

$$g''(\varphi)-\left(\frac{GK+HJ}{KJ}\right)g'(\varphi)=0,\tag{52}$$

and their solution are given by

$$\begin{aligned}f(\gamma,\eta)=\Gamma_1\exp&\left[+\sqrt{\frac{GK+HJ}{KJ}}(K\gamma+J\eta)\right]\\&+\Gamma_2\exp\left[-\sqrt{\frac{GK+HJ}{KJ}}(K\gamma+J\eta)\right].\end{aligned}\tag{53}$$

Where Γ_1 and Γ_2 are constants that must be determined by the particular boundary conditions.

Since $K\gamma+J\eta=(\alpha K+\beta J)x+(K+J)y$, the solutions in the (x,y) variables are,

$$\begin{aligned}f(x,y)&=\Gamma_1\exp\left[+(Lx+My)\right]\\&+\Gamma_2\exp\left[-(Lx+My)\right],\end{aligned}\tag{54}$$

where $L\equiv(\alpha K+\beta J)\sqrt{(GK+HJ)/KJ}$ and $M\equiv(K+J)\sqrt{(GK+HJ)/KJ}$.

By direct substitution into differential Eqs. (1) or (49) one can verify that Eq. (54) or (53), respectively, are correct solutions.

4.1.3. All coefficients non-zero

Equation (52) suggests the same form of solutions for Eq. (45), thus,

$$g''(\varphi)-\left(\frac{GK+HJ}{KJ}\right)g'(\varphi)-\frac{I}{KJ}g(\varphi)=0,\tag{55}$$

By proposing exponential solutions, $g(\varphi)=\exp\left(N\varphi\right)$, in Eq. (55) ones gets

$$\left[N^2-\left(\frac{GK+HJ}{KJ}\right)N-\frac{I}{KJ}\right]g(\varphi)=0,\tag{56}$$

here N is a constant. From (56) the non-trivial solution is obtained when,

$$N^2-\left(\frac{GK+HJ}{KJ}\right)N-\frac{I}{KJ}=0,\tag{57}$$

whose roots are

$$N_1=\frac{GK+HJ}{2KJ}\left[1+\sqrt{1-\frac{4IKJ}{(GK+HJ)^2}}\right],$$

and

$$N_2=\frac{GK+HJ}{2KJ}\left[1-\sqrt{1-\frac{4IKJ}{(GK+HJ)^2}}\right].$$

Thus, solutions to Eq. (55) are,

$$\begin{aligned}f(\gamma,\eta)&=\Gamma_1\exp\left[N_1(K\gamma+J\eta)\right]\\&+\Gamma_2\exp\left[N_2(K\gamma+J\eta)\right],\end{aligned}\tag{58}$$

and in the (x,y) variables,

$$f(x,y)=\Gamma_1\exp\left(Ox+Py\right)+\Gamma_2\exp\left(Qx+Ry\right).\tag{59}$$

Where $O \equiv (\alpha K + \beta J)N_1$, $P \equiv (K + J)N_1$ $Q \equiv (\alpha K + \beta J)N_2$ and $R \equiv (K + J)N_2$.

Again, by direct substitution in the differential equations (1) or (45) one can verify that Eq. (58) or (59), respectively, are correct solutions.

4.2. Parabolic 2D-PDE

If $B^2 - 4AC = 0$ or $C = B^2/4A$, (1) becomes

$$\left(A\partial_{xx} + B\partial_{xy} + \frac{B^2}{4A}\partial_{yy} + D\partial_x + E\partial_y + F\right) f(x,y) = 0, \tag{60}$$

and the solutions are degenerated, as shown in Sec. 2.3. Also, above it was found that solutions for the 2D-parabolic PDEs in the (x,y) variables are of the form,

$$f(x,y) = g\left(\gamma(x,y)\right) \quad \text{with} \quad \gamma(x,y) = -\frac{B}{2A}x + y. \tag{61}$$

Substituting Eq. (61) into Eq. (60),

$$\frac{2AE - BD}{2A} g'(\gamma) + Fg(\gamma) = 0. \tag{62}$$

Non-trivial solution occurs when $BD - 2AE \neq 0$, thus

$$g'(\gamma) = -\left(\frac{2AF}{2AE - BD}\right) g(\gamma). \tag{63}$$

Whose solutions are:

$$g(\gamma) = \exp\left(-\frac{2AF}{2AE - BD}\gamma\right). \tag{64}$$

Then

$$f(x,y) = \exp\left[\frac{2AF}{2AE - BD}\left(\frac{B}{2A}x - y\right)\right]. \tag{65}$$

Similar to the case in section 3, the second solution can be found,

$$f(x,y) = \left(x - \frac{D}{E}y\right) \times \exp\left[\frac{2AF}{2AE - BD}\left(\frac{B}{2A}x - y\right)\right], \tag{66}$$

which can be verified by direct derivation.

Since Eq. (65) becomes either one-variable function for $B = 0$ or a constant for $F = 0$, general solutions are not possible until one proposes specific expressions for the SGFs.

4.2.1. *1D-diffusion-like differential equation*

As an illustrative example, let us consider a 1D-diffusion-like differential equation. In this case $B = C = D = F = 0$, and Eq. (1) becomes

$$\left(A\partial_{xx} + E\partial_y\right) f(x,y) = 0. \tag{67}$$

The Fourier transform of Eq. (67) reads,

$$\left(-4\pi^2 u^2 + i2\pi\frac{E}{A}v\right)\hat{f}(u,v) = 0. \tag{68}$$

Following the above description, it is possible to factorize Eq. (68) as,

$$\left(u - \sqrt{\frac{iE}{2\pi A}v}\right)\left(u + \sqrt{\frac{iE}{2\pi A}v}\right)\hat{f}(u,v) = 0. \tag{69}$$

Following now the method as described in the above examples we have the two solutions,

$$\hat{f}(u,v) = \delta\left(u - \sqrt{\frac{iE}{2\pi A}v}\right)\hat{g}_1(u,v) + \delta\left(u + \sqrt{\frac{iE}{2\pi A}v}\right)\hat{g}_2(u,v). \tag{70}$$

By inverse Fourier transforming Eq. (70) in one spectral variable one obtains,

$$\hat{f}_v(x,y) = \left[\hat{g}_1(v)\exp\left(+i2\pi\sqrt{\frac{iE}{2\pi A}}vx\right) + \hat{g}_2(v)\exp\left(-i2\pi\sqrt{\frac{iE}{2\pi A}}vx\right)\right] \times \exp\left(i2\pi vy\right). \tag{71}$$

An application of Eq. (71) is obtained by taking $y = t$ (time), $v = f$ (frequency), $E/A = -1/D$ (D thermal diffusivity) and $\hat{f}_v = \hat{T}$ (Fourier transform of temperature). In this case, Eq. (71) becomes the general homogeneous solution of the Photoacoustic and Photothermal boundary value problem [10],

$$\hat{T}_\omega(x,t) = \left\{\Gamma_1(\omega)\exp\left[-i\sigma(\omega)x\right] + \Gamma_2(\omega)\exp\left[+i\sigma(\omega)x\right]\right\}\exp\left(i\omega t\right). \tag{72}$$

Where $\omega = 2\pi f$ and $\sigma(\omega) = (1+i)\sqrt{\omega/2D}$. $\Gamma_1(\omega)$ and $\Gamma_2(\omega)$ are functions that should be determined by the boundary conditions in the frequency domain. The inverse Fourier transform of Eq. (72) reads,

$$T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_1(\omega)\exp\left\{-i[\sigma(\omega)x + \omega t]\right\}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_2(\omega)\exp\left\{+i[\sigma(\omega)x + \omega t]\right\}d\omega. \tag{73}$$

Since photothermal measurements are performed utilizing a monochromatic source, with a blocking sinusoidal chopper operating at a frequency ω_0, [10] $I(\omega_0) = I_0[1 + \cos(\omega_0 t)]/2$ and the Fourier transform of $\cos(\omega_0 t)$ is proportional to $\delta(\omega \pm \omega_0)$, a direct proposal is $\Gamma_1 = 2\pi S_1(\omega)\delta(\omega + \omega_0)$ and $\Gamma_2 = 2\pi S_2(\omega)\delta(\omega - \omega_0)$, which gives the solution,

$$T(x,t) = S_1(\omega_0)\exp\{-i[\sigma(\omega_0)x + \omega_0 t]\} + S_2(\omega_0)\exp\{+i[\sigma(\omega_0)x + \omega_0 t]\}. \qquad (74)$$

The above approach shows how the SGFs method can be chosen for constructing different solutions.

5. SGF method in other coordinate systems: The Helmholtz differential equation in Cylindrical coordinates

Helmholtz equation with axial symmetry occurs when $A = C = 1$, $B = D = E = 0$, and $F = 0$ in Eq. (1)

$$(\partial_{xx} + \partial_{yy} + F)\, f(x,y) = 0. \qquad (75)$$

To obtain a solution to Eq. (75) in cylindrical coordinates, first we Fourier transform the function f from the (x,y) space to the frequency domain,

$$\left[-4\pi^2\left(u^2+v^2\right) + F\right]\hat{f}(u,v) = 0, \qquad (76)$$

being (u,v) the corresponding spectral variables.

It is straightforward to obtain solutions for this equation in Cartesian coordinates with one of the approaches described above. Instead, let,

$$2\pi u = k_x = k\cos(\alpha) \quad \text{and} \quad 2\pi v = k_y = k\sin(\alpha), \qquad (77)$$

and additionally

$$x = \rho\cos(\varphi) \quad \text{and} \quad x = \rho\sin(\varphi). \qquad (78)$$

Substituting (77) in (76) one obtains, $\left(k^2 - F\right)\hat{f}\left(k_x, k_y\right)$; or

$$\left(k - \sqrt{F}\right)\left(k + \sqrt{F}\right)\hat{f}(k_x,k_y) = 0, \qquad (79)$$

applying our method to Eq. (79) implies that,

$$\hat{f}(k_x,k_y) = \delta\left(k - \sqrt{F}\right)\hat{g}_1\left(k_x,k_y\right) + \delta\left(k + \sqrt{F}\right)\hat{g}_2\left(k_x,k_y\right). \qquad (80)$$

The inverse transform of Eq. (80) now reads,

$$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}\left(k_x,k_y\right) \times \exp\left[i(k_x x + k_y y)\right]dk_x dk_y. \qquad (81)$$

By substituting (79) and (80) in (81) one obtains,

$$f(\rho,\varphi) = \frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{2\pi}\left[\delta\left(k-\sqrt{F}\right)\hat{g}_1(k,\alpha) + \delta\left(k+\sqrt{F}\right)\hat{g}_2(k,\alpha)\right] \times \exp\left[ik\rho\cos(\alpha-\varphi)\right]kdkd\alpha. \qquad (82)$$

After performing the integrals involving the Dirac function, one obtains,

$$f(\rho,\varphi) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\hat{g}_1\left(\sqrt{F},\alpha\right)\exp\left[i\sqrt{F}\rho\cos(\alpha-\varphi)\right] \times \sqrt{F}d\alpha - \frac{1}{(2\pi)^2}\int_0^{2\pi}\hat{g}_2\left(-\sqrt{F},\alpha\right) \times \exp\left[-i\sqrt{F}\rho\cos(\alpha-\varphi)\right]\sqrt{F}d\alpha.$$

At this point, the generalized functions $\hat{g}_{1,2}(\pm\sqrt{F},\alpha) = \hat{g}_{1,2}(\alpha)$ can be used to obtain different solutions. A natural choice can be $\hat{g}_1(\alpha) = A\exp(+in\alpha)$ and $\hat{g}_2(\alpha) = B\exp(-in\alpha)$, being A and B arbitrary constants, and n an integer. With this choice, one obtains,

$$f(\rho,\varphi) = \frac{A\sqrt{F}}{(2\pi)^2}\int_0^{2\pi}\exp(-in\alpha) \times \exp\left[i\sqrt{F}\rho\cos(\alpha-\varphi)\right]d\alpha - \frac{A\sqrt{F}}{(2\pi)^2} \times \int_0^{2\pi}\exp(+in\alpha)\exp\left[-i\sqrt{F}\rho\cos(\alpha-\varphi)\right]d\alpha. \qquad (83)$$

Making the variable change $\Psi = \alpha - \varphi$, in the integrals of Eq. (83), we have

$$f(\rho,\varphi) = A'\sqrt{F}\exp(+in\varphi)\frac{1}{2\pi}\int_{-\varphi}^{2\pi-\varphi}\exp(+in\Psi) \times \exp\left[+i\sqrt{F}\rho\cos(\Psi)\right]d\Psi - B'\sqrt{F}\exp(-in\varphi) \times \frac{1}{2\pi}\int_{-\varphi}^{2\pi-\varphi}\exp(-in\Psi)\exp\left[-i\sqrt{F}\rho\cos(\Psi)\right]d\Psi, \qquad (84)$$

where $A' = A/2\pi$ and $B' = B/2\pi$. Since the functions in arguments are $2\pi-$periodic, we can write

$$f(\rho,\varphi)=A'\sqrt{F}\exp\left(+in\varphi\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(+in\Psi\right)\times\exp\left[+i\sqrt{F}\rho\cos(\Psi)\right]d\Psi-B'\sqrt{F}\exp\left(-in\varphi\right)\times\frac{1}{2\pi}\int_0^{2\pi}\exp\left(-in\Psi\right)\exp\left[-i\sqrt{F}\rho\cos(\Psi)\right]d\Psi.$$

By using the next integral form of the Bessel functions of the first kind order n,

$$J_n\left(\sqrt{F}\rho\right)=\exp\left(-i\frac{n\pi}{2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(in\Psi\right)\times\exp\left[i\sqrt{F}\rho\cos(\Psi)\right]d\Psi$$

and the fact that

$$J_n(-\sqrt{F}\rho)=(-1)^nJ_n(\sqrt{F}\rho)=\exp\left(-in\pi\right)J_n(\sqrt{F}\rho),$$

we have

$$\exp\left(-in\pi\right)J_n\left(\sqrt{F}\rho\right)=\exp\left(-i\frac{n\pi}{2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(-in\Psi\right)\times\exp\left[-i\sqrt{F}\rho\cos(\Psi)\right]d\Psi,$$

then

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(+in\Psi\right)\exp\left[+i\sqrt{F}\rho\cos(\Psi)\right]d\Psi=\exp\left(+i\frac{n\pi}{2}\right)J_n\left(\sqrt{F}\rho\right),$$

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(-in\Psi\right)\exp\left[-i\sqrt{F}\rho\cos(\Psi)\right]d\Psi=\exp\left(-i\frac{n\pi}{2}\right)J_n\left(\sqrt{F}\rho\right).$$

by substituting these functions into Eq. (84), ones obtain the family of solutions,

$$f_n(\rho,\varphi)=\sqrt{F}\left\{A_n\exp\left[+in\left(\varphi+\frac{n\pi}{2}\right)\right]+B_n\exp\left[-in\left(\varphi+\frac{n\pi}{2}\right)\right]\right\}J_n\left(\sqrt{F}\rho\right). \quad (85)$$

If in Eq. (83) we set $\hat{g}_2(\alpha)=0$, we obtain the well know solution to the Helmholtz differential equation in cylindrical coordinates

$$f_n(\rho,\varphi)=A'_n\exp\left[+in\left(\varphi+\frac{n\pi}{2}\right)\right]J_n\left(\sqrt{F}\rho\right)$$

where $A'_n=A_n\sqrt{F}$. Finally, the obtained general solution of Helmholtz differential equation in cylindrical coordinates with axial symmetry is,

$$f(\rho,\varphi)=\sqrt{F}\sum_{n=0}^{\infty}\left\{A_n\exp\left[+in\left(\varphi+\frac{n\pi}{2}\right)\right]+B_n\exp\left[-in\left(\varphi+\frac{n\pi}{2}\right)\right]\right\}J_n\left(\sqrt{F}\rho\right). \quad (86)$$

Before finishing the report, we summarize the proposed method for solving homogeneous partial differential equations with constant coefficients. First, factorization of the resulting transformed equation (when possible) is performed, and then, the properties of the Dirac delta function listed in Eq. (7)-(9) are used. When the method of factorization does not allow obtaining solutions, there is still the possibility of constructing solutions. It could be done particular SGFs or by using the properties of the product of two or more Dirac deltas.

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