An introduction to the Kaluza-Klein formulation

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We give an elementary introduction to the Kaluza–Klein formulation, in which the gravitational and the electromagnetic fields are represented in the geometry of a five-dimensional space. We show that, in the framework of general relativity, the interaction of a point particle, or of a charged spin-zero field, with a gravitational and an electromagnetic field can be obtained through the metric of a five-dimensional space. We also show that the symmetries of the metric of this five-dimensional space lead to constants of motion for the point particles, or to operators that commute with the Klein–Gordon operator. A common misunderstanding related to the unification of gravitation and electromagnetism given by the Kaluza–Klein formulation is discussed.

Keywords: Kaluza–Klein theory; geodesics; symmetries; Klein–Gordon equation.

Damos una introducción elemental a la formulación de Kaluza–Klein, en la cual los campos gravitacional y electromagnético están representados en la geometría de un espacio de dimensión cinco. Mostramos que, en el marco de la relatividad general, la interacción de una partícula puntual, o de un campo cargado de espín cero, con un campo gravitacional y uno electromagnético puede obtenerse a través de la métrica de un espacio de dimensión cinco. Mostramos también que las simetrías de la métrica de este espacio de dimensión cinco llevan a constantes de movimiento para las partículas puntuales, o a operadores que conmutan con el operador de Klein–Gordon. Se discute un malentendido común relacionado con la unificación de la gravitación y el electromagnetismo dada por la formulación de Kaluza–Klein.

Descriptores: Teoría de Kaluza–Klein; geodésicas; simetrías; ecuación de Klein–Gordon.

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1. Introduction

In the general theory of relativity the gravitational field is associated with the geometry of the space-time and the concept of a gravitational force is no longer needed. The motion of a test particle under the sole influence of a gravitational field is obtained by assuming that its world-line is a geodesic of the space-time metric.

It is naturally interesting to see if other interactions can be incorporated in a geometrical structure. In this sense, the so-called Kaluza–Klein theory can be regarded as a partial success; in this formulation one tries to represent geometrically the effects of a gravitational and an electromagnetic field by considering a five-dimensional space with a metric made out of the space-time metric and the four-potential of the electromagnetic field. By contrast with the general relativity theory, which involves a four-dimensional space-time whose physical relevance is already established in the special relativity theory, the Kaluza–Klein theory makes use of a five-dimensional space, obtained by adding one coordinate to the four coordinates of the space-time, but this fifth dimension arises as a mathematical convenience, without experimental basis.

Even though the Kaluza–Klein theory is almost as old as the general relativity theory (see, *e.g.*, Ref. [1] and the references cited therein), it is not commonly treated in the books on general relativity (some few exceptions are Refs. [2–4]). In the usual approach to the Kaluza–Klein theory, a specific form of the metric of the five-dimensional space is proposed without any motivation. One of the aims of this paper is to show that the form of the metric employed in the Kaluza–Klein theory can be obtained in a natural manner, trying to see the world-line of a charged particle in a gravitational and an electromagnetic field as the projection on the space-time of a geodesic of a five-dimensional space. One of the advantages of relating the world-lines of charged particles with geodesics is that the continuous symmetries of a metric lead directly to constants of motion (without making use of the Noether theorem).

In Sec. 2, making use of the Hamiltonian formalism, we show that, in the framework of general relativity, the world-lines of charged particles in a gravitational and an electromagnetic field are the projections on the space-time of the geodesics of the metric of a five-dimensional space, obtained by adding a coordinate to the space-time. In Sec. 3 we show that with each continuous symmetry of the space-time metric and the electromagnetic field there is an associated constant of motion, even if the four-potential is not invariant. In Sec. 4, we show that the Klein–Gordon equation for a charged field can be expressed in a compact way, with the interactions with a gravitational and an electromagnetic field given through the five-dimensional metric proposed in Sec. 2 and that the continuous symmetries of the five-dimensional metric lead to operators that commute with the Klein–Gordon operator. In Sec. 5, we show that if one imposes the analog of the Einstein vacuum field equations on the five-dimensional metric one obtains the Einstein–Maxwell equations for a certain class of electromagnetic fields.

Throughout this paper it is assumed that the reader is acquainted with the basic formalism of the general relativity theory as presented, *e.g.*, in Refs. [5–7], and with the Hamil-

tonian formalism of classical mechanics as presented, *e.g.*, in Refs. [8–10].

2. Motion of particles in gravitational and electromagnetic fields

In the general relativity theory, the world-line of a particle (with a nonzero rest mass) in a gravitational field represented by the space-time metric $g_{\alpha\beta}$ is determined by the geodesic equations

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}+\Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}=0, \tag{1}$$

where τ is the proper time of the particle, the Greek indices run from 0 to 3 and there is summation over repeated indices. The Christoffel symbols, $\Gamma^\alpha_{\beta\gamma}$, are determined by the metric tensor according to the standard formula

$$\Gamma^\alpha_{\beta\gamma}=\frac{1}{2}g^{\alpha\rho}\left(\frac{\partial g_{\beta\rho}}{\partial x^\gamma}+\frac{\partial g_{\gamma\rho}}{\partial x^\beta}-\frac{\partial g_{\beta\gamma}}{\partial x^\rho}\right), \tag{2}$$

where $(g^{\alpha\beta})$ is the inverse of the matrix $(g_{\alpha\beta})$. The fact that τ is the proper time of the particle amounts to the condition

$$g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=-c^2 \tag{3}$$

(assuming that the signature of the metric is $(-+++)$).

By means of a straightforward computation, one can verify that the Hamiltonian

$$H=\frac{1}{2m_0}g^{\alpha\beta}p_\alpha p_\beta \tag{4}$$

leads to the geodesic equations (1), where m_0 is the rest mass of the particle and the p_α are the canonical momenta conjugated to the coordinates x^α; that is, the Hamilton equations

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}=\frac{\partial H}{\partial p_\alpha},\qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau}=-\frac{\partial H}{\partial x^\alpha}, \tag{5}$$

reproduce the geodesic equations (1), using the fact that $\partial_\alpha g^{\beta\gamma}=-g^{\beta\rho}(\partial_\alpha g_{\rho\sigma})g^{\sigma\gamma}$. (Note that p_α is the conjugate momentum to x^α, but $p^\alpha\equiv g^{\alpha\beta}p_\beta$ is not.) Similarly, a Hamiltonian for a charged particle in a gravitational field (represented by $g_{\alpha\beta}$) and an electromagnetic field (represented by the four-potential A_α) is given by

$$\begin{aligned}H&=\frac{1}{2m_0}g^{\alpha\beta}\left(p_\alpha-\frac{q}{c}A_\alpha\right)\left(p_\beta-\frac{q}{c}A_\beta\right)\\&=\frac{1}{2m_0}\left[g^{\alpha\beta}p_\alpha p_\beta-2\frac{q}{c}A^\alpha p_\alpha+\left(\frac{q}{c}\right)^2A^\alpha A_\alpha\right],\end{aligned} \tag{6}$$

where q is the charge of the particle (in cgs units) and $A^\alpha\equiv g^{\alpha\beta}A_\beta$.

Equation (6) is of the form of the Hamiltonian (4) for the geodesics of a five-dimensional space,

$$H=\frac{1}{2m_0}G^{AB}p_Ap_B, \tag{7}$$

where the upper case Latin indices, $A, B, \ldots$, run from 0 to 4, and G_{AB} is the metric tensor of the five-dimensional space. Indeed, splitting the double sum in (7) in the form

$$H=\frac{1}{2m_0}\left(G^{\alpha\beta}p_\alpha p_\beta+2G^{\alpha4}p_\alpha p_4+G^{44}p_4p_4\right)$$

(assuming, as usual, that the metric tensor is symmetric), comparison with Eq. (6) yields

$$G^{\alpha\beta}=g^{\alpha\beta},\qquad G^{\alpha4}p_4=-\frac{q}{c}A^\alpha,$$
$$G^{44}p_4p_4=\left(\frac{q}{c}\right)^2A^\alpha A_\alpha.$$

Since the components $g^{\alpha\beta}$ and A^α, appearing in (6), are functions of x^α only, we can assume that the G_{AB} (and the G^{AB}) are functions of x^α only; this implies that the new coordinate x^4 does not appear in the Hamiltonian (7) and, therefore, its conjugate momentum, p_4, is a constant of motion. Guided by the expressions above, we shall assume that the metric of the five-dimensional space is

$$G^{\alpha\beta}=g^{\alpha\beta},\quad G^{\alpha4}=-\kappa A^\alpha,\quad G^{44}=1+\kappa^2A^\alpha A_\alpha, \tag{8}$$

where κ is a constant such that $\kappa^2A^\alpha A_\alpha$ is adimensional (identifying the constant of motion p_4 with $q/\kappa c$). Note that we have arbitrarily added the constant term 1 to the expression for G^{44}; in this way the matrix (G^{AB}) is nonsingular if $A_\alpha=0$. The inclusion of this term does affect the expression of $\mathrm{d}x^4/\mathrm{d}\tau$, but, as we shall demonstrate below, it does not modify the equations of motion of the particle in the space-time.

Then, one readily verifies that the entries of the inverse of the matrix (G^{AB}) must be given by

$$G_{\alpha\beta}=g_{\alpha\beta}+\kappa^2A_\alpha A_\beta,\quad G_{\alpha4}=\kappa A_\alpha,\quad G_{44}=1 \tag{9}$$

and, therefore, the metric of the five-dimensional space is

$$\begin{aligned}G_{AB}\mathrm{d}x^A\mathrm{d}x^B&=G_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta+2G_{\alpha4}\mathrm{d}x^\alpha\mathrm{d}x^4\\&\quad+G_{44}\mathrm{d}x^4\mathrm{d}x^4=g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta\\&\quad+(\kappa A_\alpha\mathrm{d}x^\alpha+\mathrm{d}x^4)^2.\end{aligned} \tag{10}$$

This last equation shows that the added coordinate, x^4, has dimensions of length and that the metric tensor G_{AB} has signature $(-++++)$ (because the last term on the right-hand side of (10) is always positive).

Another interesting feature of the Kaluza–Klein theory is that the gauge transformations can be associated with a coordinate transformation. As is well known, for a given electromagnetic field tensor, $F_{\alpha\beta}$, the four-potential is not uniquely determined. If ξ is an arbitrary differentiable function of the x^α, then A_α and $A_\alpha+\partial_\alpha\xi$ give rise to the same electromagnetic field tensor. Equation (10) shows that the gauge transformation $A_\alpha\mapsto A_\alpha+\partial_\alpha\xi$ produces the same effect as the coordinate transformation $x^4\mapsto x^4+\kappa\xi$.

It may be pointed out that instead of choosing G_{44} as a constant, as in Eq. (9), one may assume that G_{44} is a scalar field (see, *e.g.*, Refs. [1,3,4,11]), but, as we have shown, such a field is not present in the interactions considered above. Note also that the coordinate x^4 was introduced just as a mathematical trick, to view the motion of a charged particle in a gravitational and an electromagnetic field as the projection on the space-time of a geodesic in a five-dimensional space, without having a physical or geometrical meaning for this coordinate (by contrast with the ordinary space-time coordinates). We do not have to specify the range of values of x^4; at this point we do not know whether the set of admissible values of x^4 is all of $\mathbb{R}$ or some subset of $\mathbb{R}$. (This is not strange in the general relativity theory; as an example, in the standard derivation of the Schwarzschild solution one does not know in advance the admissible values of the coordinates or if they are globally defined.) Note that the possible values of x^4 do not restrict those of p_4 (which is related to the electric charge of the particle). For instance, in the case of a simple pendulum in classical mechanics, the angular coordinate may be limited to take values in an interval of length 2π, but its conjugate momentum can take arbitrarily large values.

We shall verify directly that the projection of the geodesics of the metric (9) on the space-time are the worldlines of charged particles in the gravitational field corresponding to $g_{\alpha\beta}$ and the electromagnetic field corresponding to A_α. To this end, we calculate the Christoffel symbols corresponding to the metric tensor G_{AB}, which, in order to avoid confusion with those corresponding to the four-dimensional metric $g_{\alpha\beta}$, will be denoted by $\mathbb{\Gamma}^A_{BC}$. That is,

$$\mathbb{\Gamma}^A_{BC} = \frac{1}{2}G^{AR}\left(\frac{\partial G_{BR}}{\partial x^C} + \frac{\partial G_{CR}}{\partial x^B} - \frac{\partial G_{BC}}{\partial x^R}\right).$$

Making use of Eqs. (8) and (9) one readily finds that

$$\begin{aligned}
&\mathbb{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \tfrac{1}{2}\kappa^2(A_\beta F^\alpha{}_\gamma + A_\gamma F^\alpha{}_\beta),\\
&\mathbb{\Gamma}^4_{\beta\gamma} = \tfrac{1}{2}\kappa(\nabla_\beta A_\gamma + \nabla_\gamma A_\beta) + \tfrac{1}{2}\kappa^3 A^\mu(F_{\mu\beta}A_\gamma + F_{\mu\gamma}A_\beta),\\
&\mathbb{\Gamma}^\alpha_{\gamma 4} = -\tfrac{1}{2}\kappa F^\alpha{}_\gamma,\\
&\mathbb{\Gamma}^4_{\gamma 4} = \tfrac{1}{2}\kappa^2 A^\mu F_{\mu\gamma},\\
&\mathbb{\Gamma}^\alpha_{44} = 0 = \mathbb{\Gamma}^4_{44},
\end{aligned} \tag{11}$$

where ∇_β denotes the covariant derivative (*e.g.*, $\nabla_\beta A_\gamma = \partial_\beta A_\gamma - \Gamma^\rho_{\beta\gamma}A_\rho$, with $\partial_\alpha = \partial/\partial x^\alpha$) and

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

is the electromagnetic field tensor.

Assuming that $x^A = x^A(\tau)$ is a geodesic of the five-dimensional space (parameterized by the proper time of the particle), we have, for $\alpha = 0, 1, 2, 3$, making use of (11) [*cf.* Eq. (1)],

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \mathbb{\Gamma}^\alpha_{BC}\frac{\mathrm{d}x^B}{\mathrm{d}\tau}\frac{\mathrm{d}x^C}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \mathbb{\Gamma}^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\\
&\quad + 2\mathbb{\Gamma}^\alpha_{\beta 4}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^4}{\mathrm{d}\tau} + \mathbb{\Gamma}^\alpha_{44}\frac{\mathrm{d}x^4}{\mathrm{d}\tau}\frac{\mathrm{d}x^4}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}\\
&\quad + \left(\Gamma^\alpha_{\beta\gamma} - \kappa^2 A_\beta F^\alpha{}_\gamma\right)\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} - \kappa F^\alpha{}_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^4}{\mathrm{d}\tau},
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\\
&\qquad = \kappa\left(\frac{\mathrm{d}x^4}{\mathrm{d}\tau} + \kappa A_\gamma\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\right)F^\alpha{}_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}.
\end{aligned} \tag{12}$$

On the other hand, using again the geodesic equations for $x^A = x^A(\tau)$ and (11) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x^4}{\mathrm{d}\tau} + \kappa A_\gamma\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\right) &= -\mathbb{\Gamma}^4_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\\
&\quad - \kappa A_\alpha\mathbb{\Gamma}^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} + \kappa(\partial_\beta A_\gamma)\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0,
\end{aligned}$$

which means that $\mathrm{d}x^4/\mathrm{d}\tau + \kappa A_\gamma \mathrm{d}x^\gamma/\mathrm{d}\tau$ is a constant of motion. Then, identifying this constant with $q/\kappa m_0 c$, Eqs. (12) coincide with the standard equations of motion of a charged particle.

Note that the metric G_{AB} only involves the gravitational and the electromagnetic fields (but not the properties of the particle to be considered); the mass and the charge of the particle are part of the initial conditions that determine the specific geodesic to be followed. Actually, the mass and the charge only enter through the charge to mass ratio.

3. Constants of motion associated with symmetries of the five-dimensional metric

As pointed out in the Introduction, one advantage of relating the equations of motion of a mechanical system with the geodesic equations is that one has a constant of motion associated with each continuous symmetry of the metric.

We start by looking for constants of motion for the geodesic equations (1) associated with a vector field K^α on the space-time. More precisely, we want to find the conditions on the functions K^α for $g_{\alpha\beta}K^\alpha \mathrm{d}x^\beta/\mathrm{d}\tau$ to be conserved as a consequence of the geodesic equations (1). Making use of the chain rule and Eqs. (1) and (2) we have (replacing the summation indices where convenient)

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\beta}K^\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right) = (\partial_\gamma g_{\alpha\beta})\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}K^\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}$$
$$+ g_{\alpha\beta}(\partial_\gamma K^\alpha)\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} - g_{\alpha\beta}K^\alpha\Gamma^\beta_{\gamma\delta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau}$$
$$= \big[K^\alpha\partial_\gamma g_{\alpha\beta} + g_{\alpha\beta}\partial_\gamma K^\alpha$$
$$- \tfrac{1}{2}K^\alpha(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma})\big]\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}$$
$$= \big(\tfrac{1}{2}K^\alpha\partial_\gamma g_{\alpha\beta} - \tfrac{1}{2}K^\alpha\partial_\beta g_{\alpha\gamma} + g_{\alpha\beta}\partial_\gamma K^\alpha$$
$$+ \tfrac{1}{2}K^\alpha\partial_\alpha g_{\beta\gamma}\big)\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}.$$

The contributions of the first two terms inside the parenthesis of the last line cancel because they are antisymmetric in the indices β and γ, while the factor $(\mathrm{d}x^\beta/\mathrm{d}\tau)(\mathrm{d}x^\gamma/\mathrm{d}\tau)$ is symmetric in these two indices. Similarly, only the symmetric part on the indices β and γ of the third term gives a nonzero contribution (the fourth term is already symmetric). Thus, one obtains the conditions

$$g_{\alpha\beta}\partial_\gamma K^\alpha + g_{\alpha\gamma}\partial_\beta K^\alpha + K^\alpha\partial_\alpha g_{\beta\gamma} = 0, \qquad (13)$$

which are known as the Killing equations. (The vector fields satisfying Eqs. (13) are called Killing vectors or Killing vector fields.) The solutions of these equations determine the symmetries of the metric $g_{\alpha\beta}$ (see, *e.g.*, Ref. [4], Sec. 6.11, Ref. [5], p. 107, or Ref. [6], Sec. 33.2).

In an entirely similar way, one finds that the geodesic equations for the five-dimensional metric G_{AB} possess constants of motion of the form

$$\varphi = G_{AB}K^A\frac{\mathrm{d}x^B}{\mathrm{d}\tau} \qquad (14)$$

if the functions K^A satisfy

$$G_{AB}\partial_C K^A + G_{AC}\partial_B K^A + K^A\partial_A G_{BC} = 0, \qquad (15)$$

$B, C = 0, 1, 2, 3, 4$. In terms of the space-time metric and the four-potential, the constant of motion (14) is given by

$$\varphi = g_{\alpha\beta}K^\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + (K^4 + \kappa A_\alpha K^\alpha)\left(\frac{\mathrm{d}x^4}{\mathrm{d}\tau} + \kappa A_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right)$$
$$= g_{\alpha\beta}K^\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + (K^4 + \kappa A_\alpha K^\alpha)\frac{q}{\kappa m_0 c}. \qquad (16)$$

A special class of solutions of (15) is defined by the condition that the K^A be functions of the x^α only. In that case, making use of (9), Eqs. (15) reduce to Eqs. (13) and

$$\kappa(A_\alpha\partial_\beta K^\alpha + K^\alpha\partial_\alpha A_\beta) = -\partial_\beta K^4. \qquad (17)$$

Equation (17) means that if $\partial_\beta K^4 \neq 0$, then the four-potential is not invariant under the transformations generated on the space-time by K^α. However, one can readily verify (starting from $\partial_\beta\partial\gamma K^4 = \partial_\gamma\partial_\beta K^4$) that Eqs. (17) imply that

$$K^\gamma\partial_\gamma F_{\alpha\beta} + F_{\alpha\gamma}\partial_\beta K^\gamma + F_{\gamma\beta}\partial_\alpha K^\gamma = 0, \qquad (18)$$

which means that the electromagnetic field is invariant under the transformations generated by the Killing vector K^α. (*Cf.* Ref. [6], Eq. (33.61).)

Thus, a symmetry of the gravitational field [represented by a solution of (13)] which, at the same time, is a symmetry of the electromagnetic field [Eq. (18)], leads to a constant of motion of the form (16). That is, once we have found a Killing vector K^α of the space-time, which also leaves the electromagnetic field invariant [in the gauge-invariant way defined by (18)], the function K^4 is determined (up to an additive constant) by Eq. (17), and with all these functions we can calculate the constant of motion (16). A very simple example is given by $K^\alpha = 0$, which trivially satisfies Eqs. (13) and (18). Then, from (17) we find that K^4 must be a trivial constant (that is, a real number) and Eq. (14) yields

$$\varphi = K^4\left(\frac{\mathrm{d}x^4}{\mathrm{d}\tau} + \kappa A_\beta\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right),$$

which, apart from the arbitrary constant factor K^4, is the constant of motion obtained in Sec. 2, identified with $q/\kappa m_0 c$.

3.1. Example. A uniform electromagnetic field in the Minkowski space-time

As a simple example we take $(g_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1)$ (the metric of the flat Minkowski space-time in Cartesian coordinates). Its Killing vectors are the generators of the Poincaré group. We consider the electromagnetic field defined by $A_0 = ax^1$, $A_3 = bx^2$, where a, b are constants, and $A_1 = A_2 = 0$. The non-vanishing components of the electromagnetic field tensor are given by $F_{10} = a$ and $F_{23} = b$, corresponding to uniform electric and magnetic fields in the x^1-direction. This field is invariant under all translations (i.e., $K^\alpha = \mathrm{const.}$) and also under the transformations generated by the Killing vector $K^0 = x^1$, $K^1 = x^0$, $K^2 = K^3 = 0$ (which generates boosts in the x^1-direction). Then, from Eq. (17) we find that, up to a constant term, $K^4 = -\frac{1}{2}\kappa a[(x^0)^2 + (x^1)^2]$. Substituting into Eq. (16) we obtain the constant of motion

$$\varphi = -x^1\frac{\mathrm{d}x^0}{\mathrm{d}\tau} + x^0\frac{\mathrm{d}x^1}{\mathrm{d}\tau} + \frac{aq}{2m_0 c}[(x^1)^2 - (x^0)^2].$$

It should be remarked that the metric tensor $g_{\alpha\beta}$ and the electromagnetic field $F_{\alpha\beta}$ need not satisfy the Einstein equations and the Maxwell equations, respectively.

Of course, the existence of the constants of motion (16) must follow directly from the equations of motion of a charged particle. However, the relevant fact in this geometrical formulation is that these special constants of motion are directly related to symmetries of the five-dimensional metric defined above. (See also Sec. 4.)

4. The Klein–Gordon equation

As we shall show now, the Klein–Gordon equation,

$$\left(\nabla_\alpha - \frac{\mathrm{i}q}{\hbar c}A_\alpha\right)\left(\nabla^\alpha - \frac{\mathrm{i}q}{\hbar c}A^\alpha\right)\psi = \left(\frac{m_0 c}{\hbar}\right)^2\psi, \quad (19)$$

which would correspond to charged spin-zero particles, can also be conveniently written making use of the metric (9) (a similar result, in the restricted case of a flat space-time, is given in Ref. [4]). Using the fact that, for an arbitrary vector field V^α,

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{|g|}}\partial_\alpha\big(\sqrt{|g|}\,V^\alpha\big), \quad (20)$$

where $g \equiv \det(g_{\alpha\beta})$ (see, *e.g.*, Ref. [5], p. 66, Ref. [6], Eq. (20.7), or Ref. [7], p. 225), with a similar formula for any dimension, we find that the left-hand side of Eq. (19) is equivalent to

$$\begin{aligned}\nabla_\alpha\nabla^\alpha\psi - \frac{\mathrm{i}q}{\hbar c}\nabla_\alpha(A^\alpha\psi) - \frac{\mathrm{i}q}{\hbar c}A_\alpha\partial^\alpha\psi - \left(\frac{q}{\hbar c}\right)^2 A^\alpha A_\alpha\psi \\ = \frac{1}{\sqrt{|g|}}\partial_\alpha\big(\sqrt{|g|}\,g^{\alpha\beta}\partial_\beta\psi\big) - \frac{\mathrm{i}q}{\hbar c}\frac{1}{\sqrt{|g|}}\partial_\alpha\big(\sqrt{|g|}\,A^\alpha\psi\big) \\ - \frac{\mathrm{i}q}{\hbar c}A^\alpha\partial_\alpha\psi - \left(\frac{q}{\hbar c}\right)^2 A^\alpha A_\alpha\psi.\end{aligned}$$

Then, defining

$$\Psi(x^A) = \psi(x^\alpha)\exp(\mathrm{i}qx^4/\kappa\hbar c), \quad (21)$$

using again Eq. (20) and the fact that $\det(G_{AB}) = \det(g_{\alpha\beta})$, one finds that Eq. (19) is equivalent to

$$\nabla_A\nabla^A\Psi = \left[\left(\frac{m_0 c}{\hbar}\right)^2 - \left(\frac{q}{\kappa\hbar c}\right)^2\right]\Psi, \quad (22)$$

where the covariant derivatives on the left-hand side correspond to the five-dimensional metric G_{AB}.

Writing the Klein–Gordon equation in the form (22) allows us to make use of the symmetries of the metric (10) to find operators that commute with the Klein–Gordon operator. A straightforward but somewhat lengthy computation shows that if K^A is a Killing vector [that is, a solution of (15)] then

$$\nabla_A\nabla^A(K^B\partial_B\Psi) = K^B\partial_B(\nabla_A\nabla^A\Psi)$$

(again, a similar result holds in any dimension). The operators that commute with the Klein–Gordon operator represent conserved quantities and their knowledge simplifies the solution of the Klein–Gordon equation.

An interesting consequence of this approach is that if one supposes that the coordinate x^4 corresponds to a circle of length l, in the sense that x^4 and $x^4 + l$ represent the same point, then, assuming that the function Ψ is single-valued, from (21) one finds that the charge must be quantized

$$q = \frac{2\pi\kappa\hbar c}{l}n, \qquad n = 0, \pm1, \pm2, \ldots.$$

5. Curvature of the Kaluza–Klein metric

A curious fact is that if one assumes that the five-dimensional metric (9) satisfies the *analog* of the Einstein vacuum field equations, then (by suitably selecting the value of the constant κ) one obtains the Einstein field equations with the electromagnetic field as source, the Maxwell equations without charges or currents, and the algebraic condition $F^{\alpha\beta}F_{\alpha\beta} = 0$. In fact, if the Riemann curvature tensor is defined by

$$\mathbb{R}^A{}_{BCD} = \frac{\partial\mathbb{\Gamma}^A_{BD}}{\partial x^C} - \frac{\partial\mathbb{\Gamma}^A_{BC}}{\partial x^D} + \mathbb{\Gamma}^A_{RC}\mathbb{\Gamma}^R_{BD} - \mathbb{\Gamma}^A_{RD}\mathbb{\Gamma}^R_{BC}$$

and the Ricci tensor by $\mathbb{R}_{AB} = \mathbb{R}^C{}_{ACB}$, with similar definitions for other dimensions, with the aid of Eqs. (11) one gets

$$\begin{aligned}\mathbb{R}_{\alpha\beta} &= R_{\alpha\beta} + \tfrac{1}{4}\kappa^4(F^{\mu\nu}F_{\mu\nu})A_\alpha A_\beta \\ &\quad + \tfrac{1}{2}\kappa^2(A_\alpha\nabla_\rho F_\beta{}^\rho + A_\beta\nabla_\rho F_\alpha{}^\rho) - \tfrac{1}{2}\kappa^2 F_\alpha{}^\rho F_{\beta\rho}, \\ \mathbb{R}_{\alpha4} &= \tfrac{1}{4}\kappa\nabla_\beta F_\alpha{}^\beta + \tfrac{1}{4}\kappa^3 F^{\rho\sigma}F_{\rho\sigma}A_\alpha, \\ \mathbb{R}_{44} &= \tfrac{1}{4}\kappa^2 F^{\alpha\beta}F_{\alpha\beta}, \end{aligned} \quad (23)$$

where $R_{\alpha\beta}$ is the Ricci tensor of the space-time metric $g_{\alpha\beta}$. Hence, the fifteen equations $\mathbb{R}_{AB} = 0$ (which are the analog of the Einstein vacuum field equations in a five-dimensional space) imply that the curvature of the space-time and the electromagnetic field tensor satisfy

$$R_{\alpha\beta} = \tfrac{1}{2}\kappa^2 F_\alpha{}^\rho F_{\beta\rho}, \quad (24)$$

$$\nabla_\beta F_\alpha{}^\beta = 0, \quad (25)$$

$$F^{\alpha\beta}F_{\alpha\beta} = 0. \quad (26)$$

Equations (25) are the source-free Maxwell equations and Eqs. (24) are the Einstein field equations with an electromagnetic field satisfying (26) as its source, if $\kappa = \pm2\sqrt{G}/c^2$, where G is the Newton gravitational constant.

It has to be emphasized that there is no reason to assume that the metric G_{AB} should satisfy the analog of the Einstein field equations in vacuum, and the fact that the equations $\mathbb{R}_{AB} = 0$ lead to the Einstein–Maxwell equations for the electromagnetic fields satisfying the condition (26) (provided that we give the appropriate value to κ) can only be regarded as a curious coincidence.

A more subtle point that leads to some misunderstandings in the literature is the following. Making use of Eqs. (8) and (23) one finds that the scalar curvature of the five-dimensional metric (9), $\mathbb{R} = G^{AB}\mathbb{R}_{AB}$, is given by

$$\mathbb{R} = R - \tfrac{1}{4}\kappa^2 F^{\alpha\beta}F_{\alpha\beta}, \quad (27)$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature of the space-time. Furthermore, as pointed out above, $\det(G_{AB}) = \det(g_{\alpha\beta})$, hence,

$$\sqrt{|\det(G_{AB})|}\,\mathbb{R} = \sqrt{|g|}\big(R - \tfrac{1}{4}\kappa^2 F^{\alpha\beta}F_{\alpha\beta}\big). \quad (28)$$

Note that Eqs. (27) and (28) hold for any space-time metric $g_{\alpha\beta}$ and any electromagnetic field $F_{\alpha\beta}$, and that the left-hand sides of (27) and (28) correspond to the specific five-dimensional metric (9).

As is well known, the Lagrangian density $\mathcal{L}_{\rm EM} = \sqrt{|g|}\big(R-(1/4)\kappa^2 F^{\alpha\beta}F_{\alpha\beta}\big)$, with the appropriate value of κ, considered as a function of $g_{\alpha\beta}, A_\alpha$, and their partial derivatives, leads to the Einstein–Maxwell equations [without algebraic restrictions of the form (26)] and, similarly, the Lagrangian density $\sqrt{|\det(G_{AB})|}\,\mathbb{R}$, considered as a function of G_{AB} and their partial derivatives, leads to $\mathbb{R}_{AB} = 0$. However, Eq. (28) does not mean that the equations $\mathbb{R}_{AB} = 0$ are equivalent to the Einstein–Maxwell equations (as shown above). If we substitute the right-hand side of (28) into the Euler–Lagrange equations, treating $g_{\alpha\beta}$ and A_α as the field variables, one must obtain the Einstein–Maxwell equations; by virtue of the equality (28) these equations must also follow by substituting the left-hand side of (28) into the Euler–Lagrange equations, treating again $g_{\alpha\beta}$ and A_α as the field variables (fourteen variables in total), and the result is not $\mathbb{R}_{AB} = 0$ because, as pointed out above, $\mathbb{R}_{AB} = 0$ follow from the Lagrangian density $\sqrt{|\det(G_{AB})|}\,\mathbb{R}$ if the G_{AB} (fifteen variables in total) are treated as the field variables.

In spite of these facts, it is frequently claimed that (27) represents a unification of Einstein's theory and electromagnetism (see, *e.g.*, Ref. [1], Sec. IV, Ref. [3], or Ref. [4], Sec. 18.2).

6. Discussion

As we have shown, in the context of the general relativity theory, the effects of combined gravitational and electromagnetic fields on point particles, or on spin-zero quantum fields, can be reproduced by the appropriate definition of a metric in a five-dimensional space. One can take a conservative point of view, regarding the fifth dimension just as a useful mathematical trick, but, at the other extreme, it can be assumed that the fifth dimension is as real as the usual three spatial dimensions with the difference, it is argued, that x^4 is the coordinate on a circle of an extremely small radius (see, *e.g.*, Refs. [1, 3, 4, 11]). In that case, at the quantum level, the Kaluza–Klein theory makes predictions that have not been experimentally confirmed; by means of elementary arguments, it follows that, if the fifth dimension corresponds to a circle of radius a, there must exist infinite families of massive particles, which form what is called Kaluza–Klein towers, with masses proportional to $1/a$, comparable to the Planck mass.

At the classical level, a good reason to doubt about the physical relevance of the metric (10) is the presence of the arbitrary constant κ; recalling that $\mathrm{d}x^4/\mathrm{d}\tau + \kappa A_\gamma \mathrm{d}x^\gamma/\mathrm{d}\tau$ is identified with $q/\kappa m_0 c$, from Eq. (10) we see that (if $q \neq 0$), by choosing appropriately the value of κ, the *constant* value of

$$G_{AB}\frac{\mathrm{d}x^A}{\mathrm{d}\tau}\frac{\mathrm{d}x^B}{\mathrm{d}\tau} = -c^2 + \left(\frac{q}{\kappa m_0 c}\right)^2 \tag{29}$$

can be made positive, negative, or zero (by contrast, in general relativity there is a profound difference between spacelike, timelike, and null curves). It may be noticed that the constant (29) is essentially the factor appearing on the right-hand side of (22).

Even if one does not assign a physical reality to the fifth dimension, the Kaluza–Klein formalism can be conveniently applied, *e.g.*, in the study of the of properties of gravitational fields coupled to electromagnetic fields (see, *e.g.*, Ref. [12]).

In the recent decades, various attempts have been made trying to find unified theories that include all known fundamental interactions, where the total number of dimensions is greater than five, inspired in the Kaluza–Klein theory.

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