

Comparative kinetics of the snowball respect to other dynamical objects

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We examine the kinetics of a snowball that is gaining mass while is rolling downhill. This dynamical system combines rotational motion with effects involving the variation of mass. In order to understand the consequences of both effects we compare its behavior with the one of some objects in which they are absent, so we compare the snowball with a ball with no mass variation and with a skier with no mass variation nor rotational motion. Environmental conditions are also included. We conclude that the comparative velocity of the snowball respect to the other objects is particularly sensitive to the hill profile and also depend on some retardation factors such as the friction, the drag force, the rotation, and the increment of mass (inertia). We emphasize that the increase of inertia could surprisingly diminish the retardation effect owing to the drag force. Additionally, when an exponential trajectory is assumed, the maximum velocity for the snowball can be reached at an intermediate step of the trip.

Keywords: Snowball dynamics; mass variable systems; dynamics of rigid bodies.

Se estudia la dinámica de una bola de nieve que gana masa mientras rueda cuesta abajo. Este sistema dinámico combina movimiento rotacional con efectos que involucran la variación de la masa. Con el fin de entender las consecuencias de combinar ambos efectos, comparamos su comportamiento con el de algunos objetos en los cuales éstos están ausentes, por lo cual comparamos a la bola de nieve con una bola sin variación de masa y con un esquiador que no varía su masa ni posee movimiento rotacional. También se incluyen los efectos de las condiciones ambientales. Concluimos que la velocidad comparativa de la bola de nieve respecto a los otros objetos es particularmente sensible al perfil del camino seguido y también depende de algunos efectos retardadores tales como la fricción, la fuerza de arrastre, la rotación y el incremento de masa (inercia). Como un efecto sorprendente, el incremento de inercia puede disminuir el efecto retardador debido a la fuerza de arrastre. Adicionalmente, cuando se asume una trayectoria exponencial, la velocidad máxima para la bola de nieve se puede alcanzar en una etapa intermedia del camino.

Descriptores: Dinámica de la bola de nieve; sistemas de masa variable; dinámica de cuerpos rígidos.

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1. Introduction

The snowball is a dynamical object that gains mass while is rolling downhill. It is a particularly interesting problem since it permits to combine several concepts of mechanics such as the translational and rotational dynamics, the rigid body, mass variable systems, normal and tangential forces, radius of curvature for a trajectory, etc [1,2]. Modeling this problem implies many input parameters and the use of numerical methods. Additionally, an ansatz should be made about the way in which the mass (or volume) is growing with time. Environmental conditions are also considered utilizing well known assumptions about friction and drag forces. The dynamical behavior of the snowball will be studied in the *velocity vs time*, and *velocity vs length* —planes.

Moreover a comparison with other dynamical objects could clarify many aspects of the very complex behavior of the snowball. Therefore, we will develop our analysis by comparing the snowball (**SB**) motion with the one obtained from a skier sliding without friction (**SNF**), a skier sliding with friction (**SF**), and a ball with constant mass and volume (**B**).

In Sec. 2., we discuss the basic assumptions and write out the equations of motion for the snowball. In Sec. 3., the comparison between the four dynamical objects mentioned above

is performed in the asymptotic regime. Section 4. describes some proves of consistency to test our results. Section 5. provides a complete analysis of the comparative kinetics of the four dynamical objects based on some environmental retardation factors. Section 6. is regarded for the conclusions.

2. The problem of the snowball rolling downhill

2.1. Basic assumptions

The complete analysis of a snowball requires many input parameters. The problem does not have any analytical solution, so it has to be solved numerically. We have listed the numerical values assumed in this paper in table I on page 4. Besides, we make the following assumptions in order to get an equation of motion.

1. The snowball is always spherical meanwhile is growing and acquiring mass. Its density is constant as well.
2. It is supposed that there is only one point of contact between the snowball and the ground, and that the snowball rolls without slipping throughout the whole motion. We shall call it the No Slipping Condition (**NSC**).

3. In order to accomplish the previous condition, the frictional static force that produces the rotation of the snowball (F_{Rs}), has to hold the condition $F_{Rs} \leq \mu_s N$ where N is the normal (contact) force to the surface, and μ_s is the coefficient of static friction. We assume that μ_s is independent on the load (Amontons' law), this statement applies for not very large loads [4].
4. The drag force owing to the wind is assumed to be of the form

$$F_v = -\frac{\rho_A C_d A}{2} v^2 \quad (1)$$

where ρ_A is the air density, C_d is the air drag coefficient, and A is the snowball's projected frontal area i.e., $A = \pi r^2$.

We assume the air drag coefficient C_d to be constant, since this assumption has given reasonable results in similar problems [3]. On the other hand, it has been established that the force F_v could be a linear or quadratic function of the speed depending on the Reynolds number (Re) [5]. For $Re > 1$ (which is our case) a quadratic dependence fits well with experimental issues [6].

5. The mass of the snowball increases but finally reaches an asymptotic value. Furthermore, a specific function of the mass (or volume) in terms of time must be supposed. In our case we assume the following functional form

$$M(t) = M_0 + K_0 (1 - e^{-\beta t}) \quad (2)$$

where M_0 is the initial mass of the snowball and clearly the final mass is $M_0 + K_0$.

6. A hill profile must be chosen in order to make the simulation, like in other similar problems [3], it has an important effect on exit speed. Specifically, we have chosen an inclined plane and an exponential trajectory.

2.2. Equations of motion

To simplify the problem we start assuming the snowball rolling downhill on a wedge whose angle is θ (see Fig. 1). In the time t the snowball has a mass M , and in the time $t + dt$ its mass is $M + dM$; let us consider a system consisting of the original snowball of mass M plus the little piece of snow with mass dM . At the time t , the momentum of such system is $\mathbf{P}(t) = M\mathbf{v}$ (bold letters represent vectors) since the little piece of snow is still on the ground and at rest. At the time $t + dt$ the ball has absorbed the piece of snow completely, so the momentum is given by $\mathbf{P}(t + dt) = (M + dM)(\mathbf{v} + d\mathbf{v})$ then the momentum change is $d\mathbf{P} = M d\mathbf{v} + \mathbf{v} dM$ (where we have neglected the differential of second order) and the total force will be

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{dM}{dt} \quad (3)$$

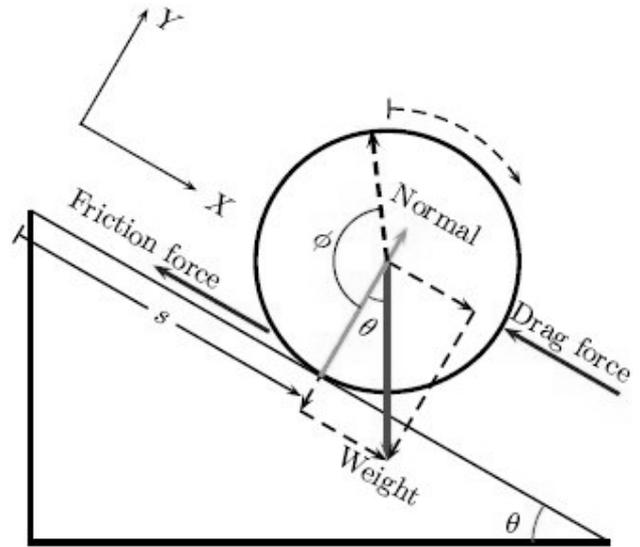


FIGURE 1. A snowball rolling downward on a wedge, the X -axis is parallel to the wedge surface and the Y -axis is perpendicular.

where \mathbf{v} corresponds to the velocity of the center of mass respect to the ground.

Further, the system is rotating too, we also suppose that such rotation is always made around an axis which passes through the center of mass (perpendicular to the sheet in Fig. 1). In this case the rotation is around a principal axis, hence the equation of motion for the angular momentum is given by

$$\mathbf{L}_C = I_C \vec{\omega}_C \quad (4)$$

where the subscript C refers to the center of mass coordinates. I_C denotes the moment of inertia of the snowball measured from an axis passing through the center of mass, and $\vec{\omega}_C$ refers to the angular velocity. According to Fig. 1, \mathbf{L}_C is directed inside the sheet, and the torque will be

$$\frac{d\mathbf{L}_C}{dt} = \vec{\tau}_C \quad (5)$$

We should remember that this equation is valid even if the center of mass is not an inertial frame [1], which is clearly our case. To calculate $d\mathbf{L}_C$ we make an analogous procedure as in the case of $d\mathbf{P}$, and the Eq. (5) is transformed into

$$I_C \frac{d\vec{\omega}_C}{dt} + \vec{\omega}_C \frac{dI_C}{dt} = \vec{\tau}_C \quad (6)$$

where $\vec{\tau}_C$ is the total torque measured from the center of mass. For the sake of simplicity, we will omit the subscript C from now on.

On the other hand, the external forces and torques exerted on the system are similar to the ones in the simple problem of a ball on a wedge [1]

$$\mathbf{F} = \mathbf{W} + \mathbf{N} + \mathbf{F}_{Rs} + \mathbf{F}_a; \quad \vec{\tau} = \mathbf{r} \times \mathbf{F}_{Rs} \quad (7)$$

where \mathbf{W} is the weight (which acts as the driving force), \mathbf{N} the normal force, \mathbf{F}_{Rs} the static friction force, and \mathbf{F}_a is any applied force which does not produce torque. If we use as a coordinate system the one indicated in Fig. 1 (with the Z - axis perpendicular to the sheet) we can convert these equations into scalar ones; then using Eqs. (3), (6) and (7) we get by supposing that \mathbf{F}_a is parallel to the X - axis.

$$\begin{aligned} N - Mg \cos \theta &= 0, \\ Mg \sin \theta - F_{Rs} + F_a &= M \frac{dv}{dt} + v \frac{dM}{dt}, \\ rF_{Rs} &= I \frac{d\omega}{dt} + \omega \frac{dI}{dt}. \end{aligned} \tag{8}$$

The first equation decouples from the others whenever the **NSC** is maintained; so we will forget it by now. We should be careful in utilizing the **NSC** since the radius is not constant. The correct **NSC** in this case consists of the relation $ds = r(d\phi)$ where ds is the differential length traveled by the center of mass of the snowball (in a certain differential time interval dt), $d\phi$ is the angle swept along the differential path, and r is the snowball radius in the time interval $[t, t + dt]$. Using the correct **NSC**, we get

$$v = \frac{ds}{dt} = r \frac{d\phi}{dt} = r\omega \Rightarrow \omega = \frac{v}{r} \tag{9}$$

and taking into account that the radius depends explicitly on the time, we obtain

$$\alpha = \frac{a}{r} - \frac{1}{r} \frac{dr}{dt} \omega \tag{10}$$

where ω is the angular velocity, α is the angular acceleration, and v, a are the translational velocity and acceleration respectively [Observe that we can start from the traditional **NSC** with $v = \omega r$. Nevertheless, the other traditional **NSC** $a = r\alpha$, is not valid anymore].

It is convenient to write everything in terms of the displacement s , taking into account the following relations

$$v = \frac{ds}{dt}; a = \frac{d^2s}{dt^2}; \omega = \frac{d\phi}{dt} = \frac{1}{r} \frac{ds}{dt} \tag{11}$$

replacing (9, 10, 11) into (8), we find

$$rF_{Rs} = I \left[\frac{1}{r} \frac{d^2s}{dt^2} - \frac{1}{r^2} \frac{dr}{dt} \frac{ds}{dt} \right] + \left(\frac{1}{r} \frac{ds}{dt} \right) \frac{dI}{dt} \tag{12}$$

Now we use the moment of inertia for the sphere $I = (2/5)Mr^2$ and the fact that the Mass “ M ” and the radius “ r ” are variable, then

$$\frac{dI}{dt} = \frac{2}{5} \left(r^2 \frac{dM}{dt} + 2Mr \frac{dr}{dt} \right), \tag{13}$$

$$M = \frac{4}{3} \pi \rho r^3; \frac{dM}{dt} = 4\pi \rho r^2 \frac{dr}{dt} \tag{14}$$

where the snowball density ρ has been taken constant. Additionally, we assume the applied force F_a to be the drag force

in Eq. (1). With all these ingredients and replacing Eq. (12) into equations (8), they become

$$\begin{aligned} \frac{d^2s}{dt^2} + \frac{15\rho_A C_d}{56\rho} \frac{1}{r} \left(\frac{ds}{dt} \right)^2 + \frac{23}{7} \frac{1}{r} \frac{dr}{dt} \frac{ds}{dt} - \frac{5}{7} g \sin \theta &= 0, \\ F_{Rs} &= \frac{8}{3} \pi \rho r^2 \left[\frac{1}{5} r \left(\frac{d^2s}{dt^2} - \frac{1}{r} \frac{dr}{dt} \frac{ds}{dt} \right) + \left(\frac{ds}{dt} \right) \frac{dr}{dt} \right]. \end{aligned} \tag{15}$$

Finally, to complete the implementation of Eqs. (15), we should propose an specific behavior for $r(t)$ or $M(t)$; we shall assume that $M(t)$ behaves like in Eq. (2). On the other hand, as a proof of consistency for Eqs. (15), we see that if we take $r = constant$, and $C_d = 0$, we find that

$$\begin{aligned} \frac{d^2s}{dt^2} &= \frac{5}{7} g \sin \theta \\ F_{Rs} &= \frac{2}{7} M_0 g \sin \theta \end{aligned} \tag{16}$$

which coincides with the typical result obtained in all common texts for the ball rolling downward on a wedge with the **NSC** [1].

2.3. A snowball downhill on an arbitrary trajectory

In this case the acceleration written above converts into the tangential one, and the Eq. (8) for the normal force becomes

$$N - Mg \cos \theta = M \frac{v^2}{R} \tag{17}$$

where R is the radius of curvature. In order to solve Eq. (17), it is convenient to use the coordinate axis plotted in Fig. 2. In cartesian coordinates, the radius of curvature is given by

$$R(x) = \frac{[1 + (y')^2]^{3/2}}{y''}; y' \equiv \frac{dy}{dx} \tag{18}$$

Moreover, the angle θ is not constant any more, and according to the Fig. 2 we see that

$$\begin{aligned} \sin \theta &= \frac{-dy}{ds} = \frac{-dy}{\sqrt{(dx)^2 + (dy)^2}} = -\frac{y'}{\sqrt{1 + (y')^2}} \\ \cos \theta &= \frac{dx}{ds} = \frac{1}{\sqrt{1 + (y')^2}} \end{aligned} \tag{19}$$

where the minus sign in the differential dy is due to the decrease of the coordinate y .

So the problem of the snowball rolling on an arbitrary trajectory can be solved by replacing (19) into the first of Eqs. (15), and making an assumption like (2). Additionally, Eq.(17) provides the solutions for the normal force by considering Eqs. (18), (19), and the solution for the velocity obtained from (15). Notwithstanding, the first of Eqs. (15) does not depend on the normal force whenever the **NSC** is maintained. So, we shall ignore it henceforth.

TABLE I. Input parameters to solve the equation of motion for the **SNF**, **SF**, **SB**, and the **B**. All measurements are in the MKS system of units. M_0 is the initial mass of all objects, K_0 defines the increment of mass for the **SB** see Eq.(2), ρ and ρ_A are the snow and air densities respectively, μ_s the static coefficient of friction between the **B** and the ground (and also between the **SB** and the ground). C_d is the air drag coefficient, θ the angle of the wedge, A_S is the skier frontal area, β is a parameter that defines the rapidity of increase of mass in the **SB**, see Eq. (2). g is the gravity acceleration, μ_D is the dynamical coefficient of friction between the **SF** and the ground, and v_0 is the initial velocity of the four objects. Further, in the exponential trajectory $y = he^{-\alpha x}$, where h is the height of the hill.

$M_0 = K_0 = 85$	$\rho = 917$	$\rho_A = 0.9$	$\mu_s = 0.03$
$C_d = 0.3$	$\theta = (4.76)^\circ$	$A_S = 0.6$	$\beta = 0.07$
$g = 9.8$	$h = 25, \alpha = 0.035$	$\mu_D = 0.03$	$v_0 = 0$

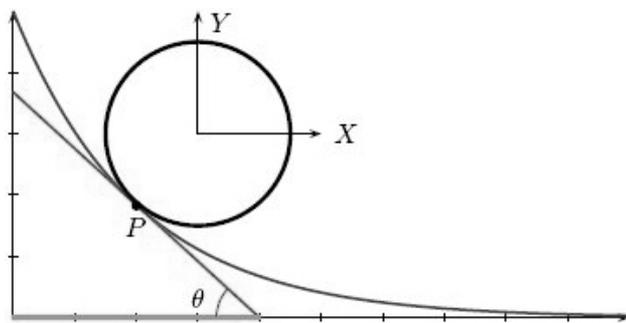


FIGURE 2. A snowball rolling downward on an exponential trajectory. To find the local value of the angle θ we define the X, Y - axis as indicated in the figure. We see that for a sufficiently large value of x we get $\theta \rightarrow 0$.

3. Comparison of the snowball with other dynamical objects

Maybe the clearest way to understand the dynamical behavior of the snowball, is by comparing it with the dynamics of other simpler objects. In our case we shall compare the dynamics of four dynamical objects

1. A skier sliding with no friction (**SNF**).
2. The same skier but sliding with friction (**SF**).
3. A ball with constant mass and volume (**B**).
4. A snowball with variable mass and volume. (**SB**)

Such comparison will be performed in the $v - t$ and $v - x$ planes. The behavior of the first two objects were reproduced from the article by Catalfamo [3], and the equations for the ball were obtained from the ones of the snowball by setting $r \rightarrow \text{constant}$. In making the comparison we use the input parameters of Table I, most of the parameters in this table were extracted from [3][However, we have changed mildly the parameters that describe the profile of the exponential trajectory, in order to keep the NSC throughout].

As it can be seen, we assume the same mass M_0 for the skier and the ball, and this is also the initial mass of the snowball ending with a mass of $2M_0$.

3.1. Asymptotic limits

Before solving all these problems, we shall study the asymptotic limits of the four objects in the inclined plane, and the exponential trajectory with and without drag force. The asymptotic regime provides useful information, and can be used to analyze the consistency of the numerical solutions. These limits depend on the drag force, the trajectory, and the object itself.

3.1.1. Inclined plane with no drag force

For each object we obtain the following limits

- For the **SF** its velocity is easily found

$$v = v_0 + gt(\sin \theta - \mu_D \cos \theta) \quad (20)$$

so there is no finite value for the velocity, and its behavior is linear respect to time. The **SNF** asymptotic limit is obtained just taking $\mu_D \rightarrow 0$.

- For the **SB** from Eq. (15), and assuming that the radius reaches an asymptotic limit *i.e.*, $dr/dt \rightarrow 0$ when $t \rightarrow \infty$ we get

$$v(t \rightarrow \infty) \equiv v_\infty = v_0 + \frac{5}{7}gt \sin \theta \quad (21)$$

getting a linear behavior respect to time. The same behavior is exhibited by the **B** when $t \rightarrow \infty$.

Observe that v_∞ in the **SB** is independent on the mass, and equal to the value for the ball **B**, it is reasonable owing to the asymptotic behavior assumed for the **SB**, Eq. (2).

3.1.2. Inclined plane with drag force

- For the **SF** its equation of motion is easily obtained

$$\frac{dv}{dt} = -\mu_D g \cos \theta - \frac{\rho_A C_d A}{2M} v^2 + g \sin \theta$$

and in the asymptotic limit $dv_\infty/dt \rightarrow 0$, so we get that

$$v_\infty^2 = \frac{2g}{\rho_A C_d} \frac{M}{A} (\sin \theta - \mu_D \cos \theta) \quad (22)$$

Now, by setting $\mu_D \rightarrow 0$, we get v_∞ for the **SNF**

- For the **SB** from Eq. (15), and setting $d^2s/dt^2 \rightarrow 0$, $dr/dt \rightarrow 0$, we obtain v_∞

$$v_\infty^2 = \frac{56\rho}{21\rho_A C_d} r_\infty g \sin \theta = \frac{2g}{\rho_A C_d} \frac{M_\infty}{A_\infty} \sin \theta \quad (23)$$

where $r_\infty \equiv r(t \rightarrow \infty)$. The second term in (23) is obtained from Eq.(2) by taking the asymptotic limit when $t \rightarrow \infty$, in this case r_∞ is given by

$$r_\infty = \left[\frac{3}{4\pi\rho} (M_0 + K_0) \right]^{\frac{1}{3}}$$

In the case of the **B**, we obtain the expression (23), but r_∞ , M_∞ , A_∞ are constant in time and equal to its initial values r_0 , M_0 , A_0 .

3.1.3. Exponential trajectory with no drag force

In this case it is easier to examine the asymptotic limits, since when the objects have traveled a long path, $\theta \rightarrow 0$, and the object is reduced to run over a horizontal plane, see Fig. 2. Therefore the limits are

- For the **SF**, $v_\infty = 0$.
- For the **SNF**, it is found easily by energy conservation

$$v_\infty^2 = 2gh + v_0^2$$

where h is the height of the hill.

- For the **B** we can find v_∞ by taking into account that energy is conserved because friction does not dissipate energy when the **NSC** is held [1]. By using energy conservation we obtain [Since what really matters for energy conservation is the height of the center of mass, there is a tiny difference that can be neglected if the radius of the ball is much smaller than the height of the hill]

$$v_\infty^2 = \frac{10}{7}gh$$

For the snowball the limit is not easy to obtain because energy is not conserved and Eq. (15) does not provide any useful information [By using $\theta \rightarrow 0$, and $d^2s/dt^2 \rightarrow 0$, the second of Eqs. (15) becomes trivial]. However, according to Figs. 5, this limit is lower than the one of the **B** owing to the increment of inertia.

3.1.4. Exponential trajectory with drag force

In this case all velocities vanish for sufficiently long time.

4. Proves of consistency

The equations of motion for each object were solved by a fourth order Runge Kutta integrator [7]. To verify the correct implementation of the program, we reproduce the results obtained by [3], and solve analytically the problem of the ball of constant radius in the inclined plane without drag force, the results were compared with the numerical solution. Additionally, all the asymptotic values discussed above were obtained consistently.

Finally, the reader could have noticed that one of our main assumptions was the **NSC**. However, this condition can only be valid if we ensure that the static friction force does not exceed the value $\mu_s N$ all over the trajectory in each case. Otherwise, the snowball would start slipping preventing us of using the fundamental relations (9) and (10). Additionally, if the snowball started sliding, the frictional force would become dynamical *i.e.* $F_{RD} = \mu_D N$ producing a coupling among the three Eqs. (8), remember that we have assumed that the first one was decoupled. As a first approach, we study the validity of the **NSC** in the asymptotic regime by calculating $F_{RS}(t \rightarrow \infty)$ and $\mu_s N(t \rightarrow \infty)$.

For the inclined plane, these limits can be estimated by using Eq. (2), the first of Eqs. (8), and the second of Eqs. (15)

$$\begin{aligned} \mu_s N(t \rightarrow \infty) &\rightarrow \mu_s M_\infty g \cos \theta \\ F_{RS}(t \rightarrow \infty) &\rightarrow \frac{2}{5} M_\infty \left(\frac{d^2 s}{dt^2} \right)_\infty \end{aligned} \quad (24)$$

it is easy to verify that the **NSC** is valid in the asymptotic limit for the wedge, since in the case of the presence of a drag force we get $F_{RS}(t \rightarrow \infty) = 0$. Additionally, in the case of absence of the drag force it is found that $F_{RS}(t \rightarrow \infty) = (2/7)(M_0 + K_0)g \sin \theta$, and the condition $F_{RS}(t \rightarrow \infty) \leq \mu_s N(t \rightarrow \infty)$ is accomplished by our input parameters (see Table I).

For the exponential trajectory the analysis is even simpler, since the path for large times becomes a horizontal straight line, the asymptotic limits for F_{RS} and $\mu_s N$ are the same as in Eqs. (24), but with $\theta \rightarrow 0$; and the **NSC** condition is held when $t \rightarrow \infty$.

However, the **NSC** in the asymptotic regime does not guarantee that it is held throughout the path. For example, in the case of the exponential trajectory, the maximum slope of the profile is found at the beginning of the trajectory, it was because of this fact that we changed mildly the profile parameters defined in Ref. [3]. Consequently, by using the first of Eqs. (8) as well as the Eqs. (15), (17), (18), and (19); we solved numerically for F_{RS} vs *time and length*, and for $\mu_s N$ vs *time and length*; utilizing the numerical input values of Table I. We then checked that $F_{RS} \leq \mu_s N$ throughout the time or length interval considered in each case.

5. Analysis

In order to make a clear comparison, we take the initial mass of all the objects to be equal, and all initial velocities are zero. In Fig. 3–6 we use the following conventions: The dashed dotted line corresponds to the **SNF**, the solid line represents the **SF**, the dashed line refers to the **B**, and finally the dotted line corresponds to the **SB**. In both, ball and snowball we only consider static friction, and neglect possible dynamic frictional effects due to sliding because we have guaranteed that the **NSC** is valid throughout the trajectory, as explained in Sec. 4.

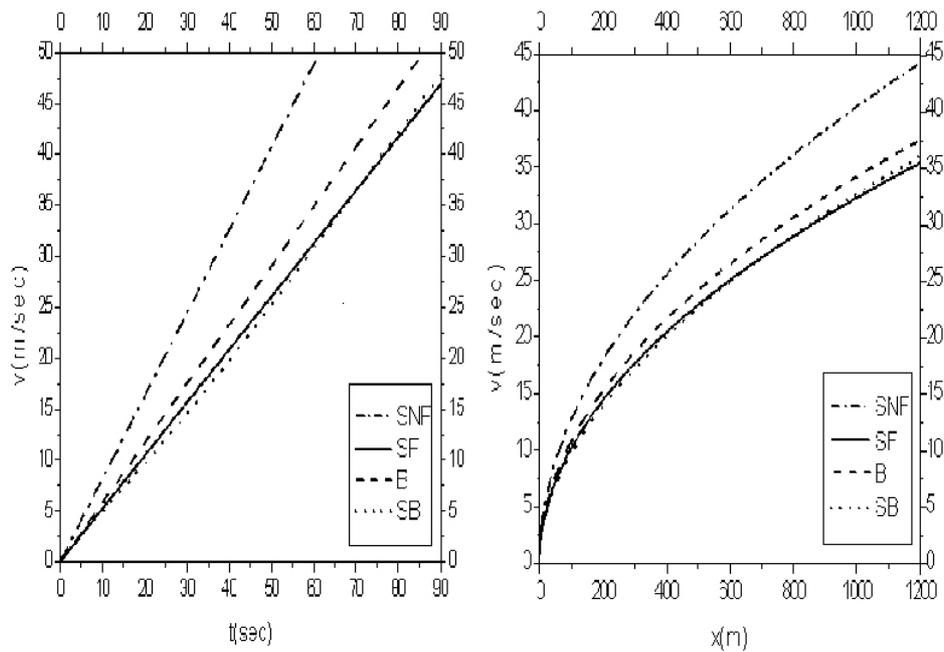


FIGURE 3. Plots in the $v - t$ plane (left) and the $v - x$ plane (right) when the objects travel in a wedge of constant slope with no drag force.

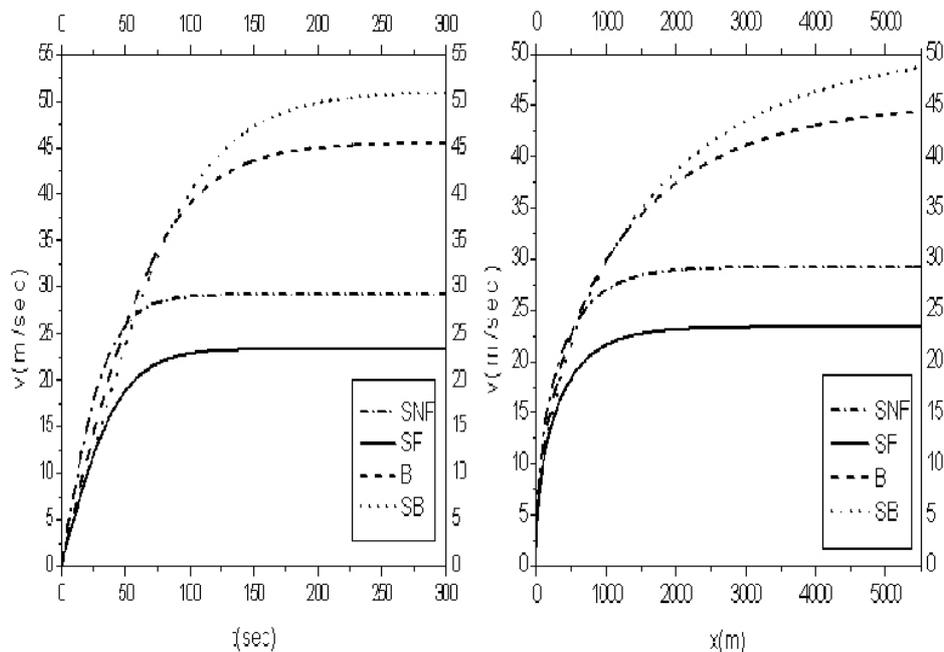


FIGURE 4. Plots in the $v - t$ plane (left), and the $v - x$ plane (right) when the objects run over a wedge of constant slope with drag force.

In Fig. 3 we plot v vs t , and v vs x for constant slope of the wedge without drag force. Of course, all graphics in the $v - t$ plane are straight lines except the one for the snowball. We can see that the **SNF** is the fastest object as expected, since no retardation factors are acting on it, next we have the **B** which possesses the rotation as a retardation factor. Additionally, the **SB** line is always below the **B** line because in the former, two retardation factors are present: the rotation and the increase of inertia. However, for sufficiently long

time (or length) the increase of inertia vanishes (according to our assumptions) so that the velocities of both **B** and **SB** coincide, in agreement with the analysis made in Sec. 3.1.1. We checked that condition, though it does not appear in Fig. 3, because of the short time and length interval displayed.

The line corresponding to the **SF** is below the line corresponding to the **SNF** as it must be, however the relation between the **SF** and **SB** lines is particularly interesting and deserves more attention. At the beginning the **SB** is slightly

slower than the **SF**, but for sufficiently long time, the **SB** becomes faster. It can be explained in the following way, at the beginning the **SB** has two retardation factors: the rotation and the increase of inertia, while the **SF** only has one retardation factor: the sliding friction. On the other hand, for sufficiently long time the increment of inertia becomes negligible in the **SB**, and only the rotation acts as a retardation factor, consequently the **SB** behaves like the **B** as shown in Sec. 3. Therefore, the combination of the two retardation

factors at the beginning makes the **SB** slower but when the increase of inertia is small enough, the **SB** becomes faster than the **SF**. Nevertheless, we should point out that this behavior depend on the value of μ_D , if it were large enough the line for the **SF** would lie below the lines for **B** and **SB** at all times, in contrast if it were small enough the **SF** line would lie above the **B** and **SB** lines. Notwithstanding, the rapidity of the **SF** must be smaller than the **SNF** speed at any time and for any value of μ_D .

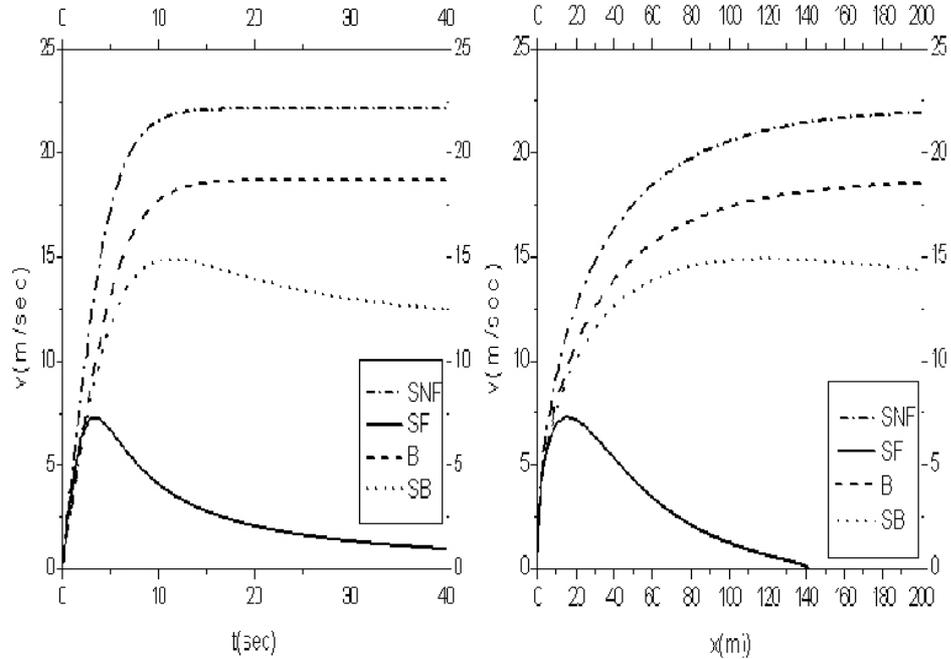


FIGURE 5. Plots in the $v-t$ plane (left), and the $v-x$ plane (right) when the objects travel on an exponential trajectory with no drag force.

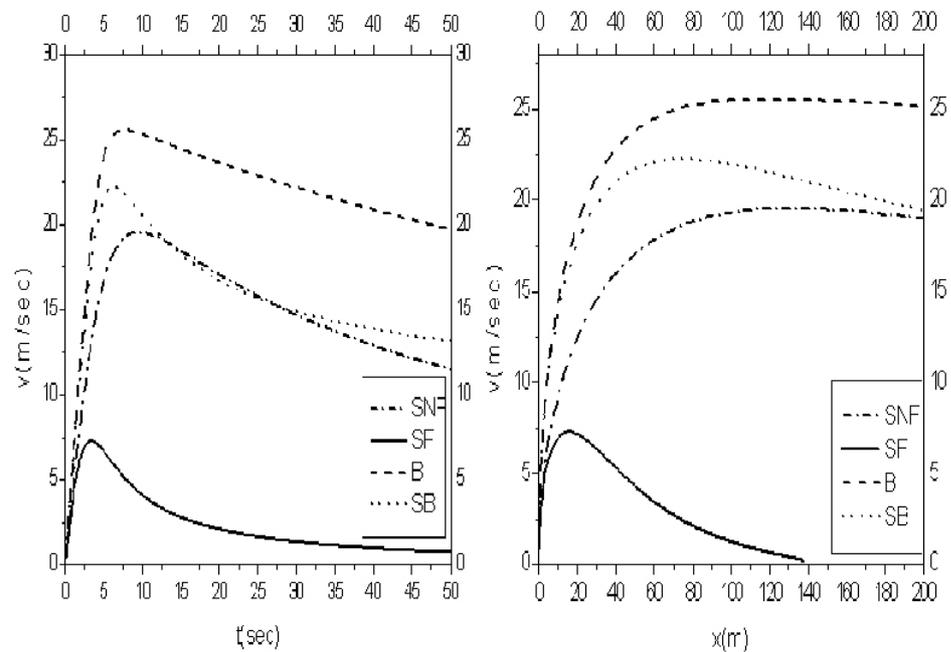


FIGURE 6. Plot in the $v-t$ plane (left), and the $v-x$ plane (right) when the objects run over an exponential trajectory with drag force.

According to this analysis, when the objects travel in a wedge with no drag force, the pattern of velocities in descending order for any set of the input parameters (as long as the initial masses and velocities are the same) is the following: the **SNF**, the **B** and the **SB**. The comparative velocity of the **SF** depend on the input parameters, but it must be always slower than the **SNF**. As a proof of consistency, it can be checked that the asymptotic limits in Eqs. (20), (21) obey this pattern.

Figure 4 correspond to a wedge with constant slope including drag force. In this case the comparative behavior among the four elements is not as simple as in Fig. 3, because in this case the lines cross out each other. However, the line describing the **SF** is always below the line describing the **SNF** as it should be. This more complex behaviour owes to the frontal area dependence of the drag force. For instance, we can realize that at short times the comparative behavior is very similar to the one in Fig. 3, since the drag force has not still acted significantly. All these elements get an asymptotic limit as we described in section 3.. We see that the largest asymptotic limit correspond to the **SB**, in opposition to the case of Fig. 3 with no drag force, in which the snowball was one of the slowest objects; the clue to explain this fact resides in the frontal area dependence of the drag force. From Eqs. (22) and (23) we can verify that for all these objects the terminal velocity behaves as $v^2 \propto M/A$, this quotient is larger for the **B** than for the **SNF** and the **SF** in our case, then the asymptotic velocity v_B is larger than v_{SNF} and v_{SF} , for both the skier and the ball this ratio is a constant. In contrast, since in the snowball the mass grows cubically with the radius while the area grows quadratically, its velocity behaves such that $v_{SB}^2 \propto r(t)$. Therefore, for sufficiently long times, its velocity grows with the radius of the **SB**, getting a higher terminal velocity (of course it depends on the asymptotic value of $r(t)$). Observe that if we had assumed a non asymptotic behavior of $r(t)$ in (15) we would have not obtained any finite terminal velocity for the snowball even in the presence of a drag force. Furthermore, we see that the terminal velocity for the **SB** is reached in a longer time than the others, it is also because of the slow growth of $r(t)$.

Figure 5 describes the four elements traveling in an exponential hill with no drag force. Two features deserve special attention: (1) The terminal velocity is achieved in a very short time specially in the cases of the **SNF** and the **B**, these limits coincides with the ones obtained in Sec. 3.. (2) For the **SB** and the **SF** there is a local maximum velocity at a rather short time, the diminution in the velocity since then on, owes to the decreasing in the slope of the path in both cases, the increment of inertia in the case of the **SB**, and the friction in the **SF**. Such local maximal velocity cannot be exhibited by the **SNF** and the **B** because conservation of energy applies for them, and as they are always descending their velocities are

always increasing, though for long times they are practically at the same height henceforth, getting the terminal velocity. In particular, we see that the terminal velocity of the **SF** is zero as it was shown in Sec. 3.1.3.

In Fig. 6 the elements travel in an exponential hill with drag force. In this case, the conservation of energy does not hold for any of the objects, consequently maximum velocities in intermediate steps of the trajectory are allowed for all of them. All terminal velocities are zero as expected. Because of the same arguments discussed above, the line of the **SF** is below to the one of the **SNF**. However, any other pattern depends on the numerical factors used.

A final comment is in order, we can realize that though the solution of the kinetics of the snowball depends on the ansatz made about the mass growth, the bulk of our results and analysis only depend on the fact that the snowball mass reaches a finite asymptotic value. So that the discussion is quite general, especially in the asymptotic regime.

6. Conclusions

We have described the behavior of a snowball acquiring mass while rolling downhill, taking into account the environmental conditions. The dynamics of the snowball is very complex because it is a rotating object, and at the same time its mass and moment of inertia are variables. In order to visualize better the effects due to the rotation and mass variation, we compare its motion with the kinetics of two objects in which the rotation and mass variational effects are absent (the Skier with Friction and the Skier with No Friction), and with one object in which the rotation is present but no the mass variation (the Ball of constant mass and radius).

The comparative behavior of these objects depend on the trajectory but also on some retardation factors: the friction, the drag force, the increase of mass (inertia), and the rotational effects. It is worth remarking that despite the increment of inertia is a retardation factor in some circumstances, it could surprisingly diminish the retardation effect due to the drag force. In addition, some local maxima of the velocities for each object appears in an exponential trajectory, showing that the maximum velocity might be achieved at an intermediate step of the path.

Finally, we point out that despite the complete solution of the snowball depends on an ansatz about the way in which its mass grows; its comparative dynamics respect to the other objects in the asymptotic regime is basically independent of the details of the growth, and only depend on the assumption that the mass reaches an asymptotic value, a very reasonable supposition. Therefore, we consider that our analysis is not very model dependent at least in the regime of large times or lengths. In addition, these asymptotic limits serves also to show the consistency of our results.

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