Electrostatic, magnetostatic and electromagnetic fields for harmonically distributed sources on infinite planes

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The static uniform electric charge distribution, the stationary and time-harmonically-varying uniform electric current distributions, on infinite planes, are known to produce uniform electrostatic, uniform magnetostatic and plane wave electromagnetic fields around the respective sources, since the introductory course of electricity and magnetism. This paper presents some natural extensions of these familiar systems for harmonically distributed sources on the planes, which can be assimilated in the electromagnetic theory course.

Keywords: Harmonic sources in planes; electrostatic; magnetostatic and electromagnetic radiation fields.

Desde el curso introductorio de electricidad y magnetismo se enseña que las distribuciones uniformes en planos infinitos, de carga eléctrica estática y de corrientes eléctricas estacionarias y variables armónicamente en el tiempo, producen campos electrostáticos uniformes, campos magnetostáticos uniformes y campos electromagnéticos en ondas planas en la vecindad de las respectivas fuentes. Este trabajo presenta algunas extensiones naturales de esos sistemas familiares para fuentes distribuidas armónicamente sobre los planos, las cuales pueden ser asimiladas en el curso de teoría electromagnética.

Descriptores: Fuentes armónicas en planos; campos electrostáticos; magnetostáticos y de radiación electromagnética.

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1. Introduction

Some of the simplest electrostatic, magnetostatic and electromagnetic field configurations and their sources, covered in the electricity and magnetism introductory course [1], are briefly discussed in this paragraph in order to motivate the study of some of their natural extensions to be developed in the present work. The uniform electric intensity field associated with a static uniformly distributed electric charge on an infinite plane can be evaluated from Coulomb's law and the superposition principle, or by a direct application of Gauss's law. The uniform magnetic induction field associated with a stationary uniformly distributed electric current on an infinite plane can be evaluated from the field of a line of current and the superposition principle, or by a direct application of Ampère's law. The plane wave electromagnetic field associated with a time-harmonically-varying uniformly distributed electric current in an infinite plane can be evaluated from Maxwell's equations in the differential and boundary condition forms. The common feature of uniform distributions of the sources on the respective planes is translated into the uniform nature of the electrostatic and magnetostatic fields, and into the propagation in the directions perpendicular to the plane of the plane wave electromagnetic fields.

The geometry and the distribution of the sources in the systems discussed in the previous paragraph make the respective problems become one-dimensional, and thereby among the simplest and most didactic. This article presents some natural extensions of these familiar situations, keeping the sources on infinite planes, but changing their distributions from uniform to harmonic, leading to exact and easy to construct electrostatic, magnetostatic and electromagnetic fields.

The rest of the article is organized as follows. Section 2 covers the electrostatic case for circular and hyperbolic cosine charge distributions in one of the cartesian coordinates in the source plane. Section 3 covers the magnetostatic case for stationary transverse currents with circular and hyperbolic cosine distributions in one of the cartesian coordinates in the source plane. Section 4 covers the electromagnetic case for time-harmonically-varying, transverse and longitudinal currents with a cosine distribution in one of the cartesian coordinates in the source plane. Section 5 consists of a discussion of the results for each specific situation, the connections among them, and some points of didactic interest. The Appendix contains some results on the Laplace and Helmholtz equations and their solutions, which are relevant for the construction of the fields in Secs. 2-4.

2. Electrostatic fields from harmonically distributed charges on an infinite plane

Gauss's law in its differential and boundary conditions forms,

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{1}$$

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma \tag{2}$$

establishes the connections between the volume charge density ρ and the electric intensity field $\vec{E}$ at any point in space, and between the surface charge density σ on a boundary surface and the components of the field normal to such a surface, respectively.

The conservative nature of the electrostatic field is described by the corresponding differential and boundary conditions forms:

$$\nabla \times \vec{E} = 0 \tag{3}$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0 \tag{4}$$

expressing the curlless character of the electric intensity field, and the continuity of its tangential components at the boundary surface, respectively.

For the static electric charges distributed harmonically on an infinite plane, $\rho(\vec{r}) = 0$ for all points outside the plane. Then the electric intensity field can be constructed as a divergenceless, Eq. (1), and curlless, Eq. (3), field, with discontinuous normal components, Eq. (2), and continuous tangential components, Eq. (4), at the source plane. As shown in the Appendix, Eq. (1) and (3), are equivalent to the Laplace equation in the situation under discussion, and therefore the electric intensity field must be a harmonic function determined by the harmonicity of the source.

2.1. Circular cosine distribution of charge in the x-y plane

The static charge on the $z = 0$ plane is assumed to have the distribution defined by its surface charge density

$$\sigma(x, y, z = 0) = \sigma_0 \cos \frac{\pi x}{L} \tag{5}$$

which is cosenoidal with a period $2L$ in the x direction, and independent of the y coordinate. The latter makes the problem become two dimensional. The electrostatic field depends only on the x and z coordinates, and its components must be harmonic functions of the types of Eq. (A.4). The harmonicity of the source, Eq. (5), selects the value $k = \pi/L$ in Eq. (A.4). It is necessary to write the electrostatic field $\vec{E}(x, z)$ distinguishing between its forms above and below the source plane:

$$\vec{E}(x, z \geq 0) = \left(\hat{\imath} E^a_{0x} \sin \frac{\pi x}{L} + \hat{k} E^a_{0z} \cos \frac{\pi x}{L}\right) e^{-\frac{\pi z}{L}} \tag{6}$$

$$\vec{E}(x, z \leq 0) = \left(\hat{\imath} E^b_{0x} \sin \frac{\pi x}{L} + \hat{k} E^b_{0z} \cos \frac{\pi x}{L}\right) e^{\frac{\pi z}{L}}, \tag{7}$$

ensuring their correct asymptotic behavior for $z \to \infty$ and $z \to -\infty$, respectively. The divergenceless condition of Eq. (1) translates into the restrictions on the amplitudes in Eqs. (6) and (7):

$$E^a_{0x} = E^a_{0z} \tag{8}$$

and

$$E^b_{0x} = -E^b_{0z}. \tag{9}$$

The reader can check that the curlless condition, Eq. (3), is also satisfied. Then the continuity of the tangential components, Eq. (4), requires that

$$E^a_{0x} = E^b_{0x} \tag{10}$$

$$\therefore E^b_{0z} = -E^a_{0z}. \tag{11}$$

Finally, the discontinuity of the normal components, Eq. (3), defines their value in terms of the surface charge density amplitude,

$$E^a_{0z} = 2\pi\sigma_0. \tag{12}$$

In conclusion, the harmonic electrostatic field produced by the source of Eq. (5) takes the final forms:

$$\vec{E}(x, z \geq 0) = 2\pi\sigma_0 \left(\hat{\imath} \sin \frac{\pi x}{L} + \hat{k} \cos \frac{\pi x}{L}\right) e^{-\frac{\pi z}{L}} \tag{13}$$

$$\vec{E}(x, z \leq 0) = 2\pi\sigma_0 \left(\hat{\imath} \sin \frac{\pi x}{L} - \hat{k} \cos \frac{\pi x}{L}\right) e^{\frac{\pi z}{L}}. \tag{14}$$

2.2. Hyperbolic cosine distribution of charge in the y-z plane

The static charge on the $x = 0$ plane is chosen to have a hyperbolic cosine distribution

$$\sigma(x = 0, y, z) = \sigma_0 \cosh kz \tag{15}$$

along the z-axis, and independent of the y coordinate. The reader can recognize that such a distribution corresponds to the choice $C = D$ in Eq. (A.4). The same arguments used in Sect. 2.1 lead to propose the expressions for the field in front and in back of the source plane:

$$\vec{E}(x \geq 0, z) = \hat{\imath} E^f_{0x} \cos kx \cosh\, kz + \hat{k} E^f_{0z} \sin kx \sinh\, kz \tag{16}$$

$$\vec{E}(x \leq 0, z) = \hat{\imath} E^b_{0x} \cos kx \cosh\, kz + \hat{k} E^b_{0z} \sin kx \sinh\, kz \tag{17}$$

Again, the divergenceless condition on the electric field leads to the restrictions on the amplitudes

$$E^f_{0x} = E^f_{0z}, \tag{18}$$

$$E^b_{0x} = E^b_{0z}. \tag{19}$$

These conditions, in turn, ensure that the field is curlless. The tangential components of the field vanish at the source plane, guaranteeing that Eq. (4) is satisfied. The discontinuity of the normal components, Eq. (2), leads to

$$E^f_{0x} = -E^b_{0x} = 2\pi\sigma_0 \tag{20}$$

Therefore, the electrostatic field is given by the final expression:

$$\vec{E}(x \geq 0, z) = 2\pi\sigma_0 \times \left(\hat{\imath} \cos kx \cosh kz + \hat{k} \sin kx \sinh\, kz\right) \tag{21}$$

$$\vec{E}(x \leq 0, z) = 2\pi\sigma_0 \times \left(-\hat{\imath} \cos kx \cosh kz - \hat{k} \sin kx \sinh kz\right). \tag{22}$$

This section can be concluded by noticing that the limit situations of both alternative harmonic sources, Eq.(5) for

$L \to \infty$, and Eq. (15) for $k \to \infty$, correspond to the familiar situation of the uniformly charged plane, for which the electrostatic field is also uniform as expressed by the limits of Eqs. (13)-(14) and (21)-(22), respectively.

3. Magnetostatic fields from stationary currents harmonically distributed on an infinite plane

Ampére's law in its differential and boundary condition forms,

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}, \tag{23}$$

$$(\vec{B}_2 - \vec{B}_1) \times \hat{n} = \frac{4\pi}{c}\vec{K}, \tag{24}$$

give the connections between the surface current density $\vec{J}$ and the magnetic induction field $\vec{B}$ at any point in space, and between the linear current density $\vec{K}$ on a boundary surface and the components of the field tangential to the surface, respectively.

The non-existence of magnetic monopoles is expressed by Gauss's law for the magnetic induction field in its differential and boundary condition forms,

$$\nabla \cdot \vec{B} = 0 \tag{25}$$

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0, \tag{26}$$

reflecting the solenoidal character of the field, and the continuity of its normal components at the boundary surface, respectively.

For the stationary electric currents distributed harmonically on an infinite plane, $\vec{J}(\vec{r}) = 0$ for all points outside the plane. Therefore the magnetic induction field must be curlless, Eq. (23), and solenoidal, Eq. (25), just as it happened with the electrostatic field in the previous section. However, Eq. (24) shows that the magnetic induction field must have discontinuous tangential components, while its normal components are continuous, Eq. (26). Nevertheless, the magnetic induction field must also be a harmonic function determined by the harmonicity of the source.

3.1. Circular distribution of current in the $x - y$ plane

A stationary current in the y direction and cosenoidally distributed in the x coordinate on the $z = 0$ plane,

$$\hat{K}(x, y, z = 0) = \hat{j}K_0 \cos\frac{\pi x}{L}, \tag{27}$$

has a period $2L$ and is independent of the y coordinate. The magnetic induction field associated with such a source depends on the x and z coordinates, is divergenceless and curlless, and consequently its components must be harmonic functions of the type of Eq. (A.4) with $k = \pi/L$. The expressions of the field above and below the source plane are chosen as

$$\vec{B}(x, y, z \geq 0) = \left(\hat{i}B^a_{0x} \cos\frac{\pi x}{L} + \hat{k}B^a_{0z} \sin\frac{\pi x}{L}\right)e^{-\frac{\pi z}{L}} \tag{28}$$

$$\vec{B}(x, y, z \leq 0) = \left(\hat{i}B^b_{0x} \cos\frac{\pi x}{L} + \hat{k}B^b_{0z} \sin\frac{\pi x}{L}\right)e^{\frac{\pi z}{L}} \tag{29}$$

so that the first one vanishes asymptotically as $z \to \infty$, and the second one does likewise for $z \to -\infty$. The vanishing of the respective divergences requires that

$$B^a_{0x} = -B^a_{0z} \tag{30}$$

and

$$B^b_{0x} = B^b_{0z} \tag{31}$$

These conditions also guarantee that the curls of Eqs. (28) and (29) vanish. The continuity of the normal components of the field at the source plane, Eq. (26), requires that

$$B^a_{0z} = B^b_{0z} \tag{32}$$

and

$$\therefore B^b_{0x} = -B^a_{0x} \tag{33}$$

Then the discontinuity of the tangential components of the field at the source plane determines the value of the unknown amplitude in terms of the current amplitude:

$$B^a_{0x} = \frac{2\pi}{c}K_0. \tag{34}$$

Consequently, Eqs. (28) and (29) take the final forms

$$\vec{B}(x, z \geq 0) = \frac{2\pi}{c}K_0\left(\hat{i}\cos\frac{\pi x}{L} - \hat{k}\sin\frac{\pi x}{L}\right)e^{-\frac{\pi z}{L}}, \tag{35}$$

$$\vec{B}(x, z \leq 0) = \frac{2\pi}{c}K_0\left(-\hat{i}\cos\frac{\pi x}{L} - \hat{k}\sin\frac{\pi x}{L}\right)e^{\frac{\pi z}{L}}, \tag{36}$$

describing the magnetostatic field produced by the harmonic current of Eq. (27)

3.2. Hyperbolic cosine distribution of current in the $y - z$ plane

The stationary current on the $x = 0$ plane is chosen in the y direction and distributed as a hyperbolic cosine in the z coordinate, so that its linear current density is written as

$$\hat{K}(x = 0, z) = \hat{j}K_0 \cosh kz. \tag{37}$$

The associated magnetic induction field must satisfy the condition stated and applied in the previous subsections.

Therefore, its forms in the front and back of the source plane are

$$\vec{B}(x \geq 0, z) = \Big(\hat{i} B^f_{0x} \sin kx \sinh\, kz + \hat{k} B^f_{0z} \cos kx \cosh\, kz\Big), \tag{38}$$

$$\vec{B}(x \leq 0, z) = \Big(\hat{i} B^b_{0x} \sin kx \sinh\, kz + \hat{k} B^b_{0z} \cos kx \cosh\, kz\Big). \tag{39}$$

The vanishing of their divergences, Eq. (25), demands that

$$B^f_{0x} = -B^f_{0z} \tag{40}$$

and

$$B^b_{0x} = -B^b_{0x}. \tag{41}$$

These conditions also ensure that the curls of Eqs. (38) and (39) vanish. The normal components of this field vanish at the source plane, so that Eq. (26) is automatically satisfied. The discontinuity of the tangential components of the field at the source plane, Eq. (24), determines the connection between the field unknown amplitude and the linear current amplitude,

$$B^f_{0z} = -\frac{2\pi}{c} K_0. \tag{42}$$

Therefore, the final explicit forms of the magnetostatic field become

$$\vec{B}(x \geq 0, z) = \frac{2\pi}{c} K_0 \Big(\hat{i} \sin kx \sinh\, kz - \hat{k} \cos kx \cosh\, kz\Big) \tag{43}$$

$$\vec{B}(x \leq 0, z) = \frac{2\pi}{c} K_0 \Big(-\hat{i} \sin kx \sinh\, kz + \hat{k} \cos kx \cosh\, kz\Big) \tag{44}$$

As in the electrostatic case of the previous section, the limit situations of the current distributions of Eq. (27) for $L \to \infty$, and of Eq. (37) for $k \to 0$, correspond to the familiar case of the plane with a uniformly distributed current producing a uniform magnetic induction field, given by the respective limits of Eqs. (35) - (36) and (43)-(44).

4. Electromagnetic fields from time-harmonically-varying currents distributed harmonically on an infinite plane

Maxwell's equations in their differential and boundary conditions forms, for time-harmonically-varying sources and fields, $e^{-i\omega t}$ with frequency ω

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{45}$$

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma \tag{46}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} - \frac{i\omega}{c} \vec{E}, \tag{47}$$

$$(\vec{B}_2 - \vec{B}_1) \times \hat{n} = \frac{4\pi}{c} \vec{K} \tag{48}$$

$$\nabla \times \vec{E} = \frac{i\omega}{c} \vec{B}, \tag{49}$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0 \tag{50}$$

$$\nabla \cdot \vec{B} = 0, \tag{51}$$

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \tag{52}$$

correspond to the electric Gauss's law, the Ampére-Maxwell law, Faraday's law, and the magnetic Gauss's law, respectively. Notice that the boundary condition forms coincide with their counterparts of Sects. 2 and 3. For sources harmonically distributed on an infinite plane, $\rho(\vec{r}) = 0$ and $\vec{J}(\vec{r}) = 0$ for all points outside such a plane. Now, the electric intensity and the magnetic induction fields are both solenoidal, Eqs. (45) and (51), but the curl of one is proportional to the other, Eqs. (47) and (49). The latter are coupled first order differential equations, which can be decoupled, as shown in the Appendix, becoming Helmholtz's equations. Consequently, $\vec{E}$ and $\vec{B}$ must be constructed as solenoidal fields, satisfying Helmholtz equation and the curl equations (47) and (49), as well as the boundary conditions of Eqs. (46)-(52). The components of the electric and magnetic fields are constructed as the appropriate combinations of the Helmholtz equation solutions defined by Eqs. (A.12) and (A.13).

4.1. Cosine distribution of tranverse current in the $x - y$ plane

The current is chosen to be in the $z = 0$ plane, in the direction of the y axis, and varying as a cosine in the x coordinate, so that its linear density

$$\vec{K}(x, y, z = 0, t) = \hat{j} K_0 \cos \frac{\pi x}{L} e^{-i\omega t} \tag{53}$$

has the same space distribution as in Sect. 3A, Eq. (27), but they are different in their time (in)dependence. The time harmonic variation of the source is inherited by the fields, as already incorporated in Eqs. (47) and (49), and the harmonic space variation of the source selects $k_1 = \pi/L$ in the solutions of Helmholtz equations, (A.12) and (A.13). The last equation determines the possible values of k_3:

$$k_3^2 = \frac{\omega^2}{c^2} - \frac{\pi^2}{L^2} \gtrless 0, \tag{54}$$

which determine the character of the solutions of Eq. (A.12) as traveling or evanescent waves, respectively. Here we concentrate on the study of the traveling waves. The invariance of the source under displacements in the y direction is also inherited by the electric and magnetic fields. From the experience of Sect. 3A we start by proposing the expressions of the magnetic induction field above and below the source plane:

$$\vec{B}(x,z\geq 0)=\left(\hat{i}B^a_{0x}\cos k_1x+\hat{k}B^a_{0z}\sin k_1x\right)e^{ik_3z} \quad (55)$$

$$\vec{B}(x,z\leq 0)=\left(\hat{i}B^b_{0x}\cos k_1x+\hat{k}B^b_{0z}\sin k_1x\right)e^{-ik_3z}, \quad (56)$$

as the counterparts of Eqs. (28)-(29). Notice the common x dependence and the difference in the traveling and decaying exponential functions in the z-coordinate, between the respective fields. The vanishing of the divergences of both expressions, Eq. (51), leads to the restrictions on the respective amplitudes:

$$\vec{B}^a_{0x}k_1 = B^a_{0z}ik_3 \quad (57)$$

$$\vec{B}^b_{0x}k_1 = -B^b_{0z}ik_3. \quad (58)$$

The continuity of the normal components of the magnetic field at the source plane, Eq. (52), requires that

$$B^a_{0z} = B^b_{0z}. \quad (59)$$

If follows from Eq. (57)-(59) that the amplitudes in the $\hat{i}$ direction are connected as

$$B^b_{0x} = -B^a_{0x}. \quad (60)$$

Their value is determined by the discontinuity of the tangential components of the field at the source plane, Eq. (48), as

$$B^a_{0x} = \frac{2\pi}{c}K_0 \quad (61)$$

checking along the way the anticipated selection of the value of k_1 by the harmonicity of the source distribution.

Consequently, the final forms of Eqs. (55)- (56) become

$$\vec{B}(x,z\geq 0)=\frac{2\pi}{c}k_0\left(\hat{i}\cos\frac{\pi x}{L}-\hat{k}\frac{i\pi}{k_3L}\sin\frac{\pi x}{L}\right)e^{ik_3z} \quad (62)$$

$$\vec{B}(x,z\leq 0)=\frac{2\pi}{c}k_0\left(-\hat{i}\cos\frac{\pi x}{L} -\hat{k}\frac{i\pi}{k_3L}\sin\frac{\pi x}{L}\right)e^{-ik_3z} \quad (63)$$

By taking the curl of Eqs. (61) and (62), and using Eq. (47), the other component of the electromagnetic field is evaluated as

$$\vec{E}(x,z\gtrless 0) = -\frac{2\pi}{c}K_0\hat{j}\frac{\omega}{k_3c}\cos\frac{\pi x}{L}e^{\pm ik_3z}. \quad (64)$$

Notice that this electric intensity field is divergenceless, Eq. (45), has vanishing normal components at the source plane, Eq. (46), and its tangential components at the source plane are continuous, Eq. (50). The reader can also check that its curl reproduces Eqs. (62) and (63), by using Eq. (49).

The time averaged Poynting vector associated with the electromagnetic field is directly evaluated for both regions above and below the source plane

$$\langle\vec{S}\rangle_\pm = \frac{c}{8\pi}\left(\frac{2\pi}{c}K_0\right)^2\frac{\omega}{k_3c}\left[\hat{i}\frac{i\pi}{k_3L}\cos\frac{\pi x}{L}\sin\frac{\pi x}{L} \pm\hat{k}cos^2\frac{\pi x}{L}\right] \quad (65)$$

The power radiated by each rectangle of the source, with dimensions $\Delta x = 2L$ and $\Delta y = 1$, is also evaluated:

$$p = \int_0^1\int_0^{2L}\left[\langle\vec{S}\rangle_+\cdot\hat{k}-\langle\vec{S}\rangle_-\cdot\hat{k}\right]dxdy = \frac{c}{8\pi}\left(\frac{2\pi}{c}K_0\right)^2\frac{\omega}{k_3c}2L. \quad (66)$$

It is also recognized that the imaginary terms in Eqs. (62), (63) and (65) describe induction effects.

4.2. Cosine distribution of longitudinal current in the $x-y$ plane

The current in the $z = 0$ plane is chosen to be in the x direction, and varying as a cosine in the x coordinate, so that its linear density is given by

$$\vec{K}(x,y,z=0) = \hat{i}K_0\cos\frac{\pi x}{L}. \quad (67)$$

The important difference between the transverse current of the previous subsection and the longitudinal current under consideration resides in their vanishing and non-vanishing divergences, respectively.

The latter translates into the presence of a surface charge distribution as determined from the continuity equation

$$\nabla\cdot\vec{K}+\frac{\partial\sigma}{\partial t} = 0, \quad (68)$$

which in the present situation takes the form:

$$\sigma(x,y,z=0) = \frac{i}{\omega}K_0\frac{\pi}{L}\sin\frac{\pi x}{L}e^{-i\omega t}. \quad (69)$$

From the experiences of Sect. 2, the expressions for the electric intensity field above and below the source plane are written as

$$\vec{E}(x,z\geq 0)=\left(\hat{i}E^a_{0x}\cos\frac{\pi x}{L}+\hat{k}E^a_{0z}\sin\frac{\pi x}{L}\right)e^{ik_3z} \quad (70)$$

$$\vec{E}(x,z\leq 0)=\left(\hat{i}E^b_{0x}\cos\frac{\pi x}{L}+\hat{k}E^b_{0z}\sin\frac{\pi x}{L}\right)e^{-ik_3z} \quad (71)$$

The divergenceless of these fields, Eq. (45), requires that

$$E^a_{0x}\frac{\pi}{L} = E^a_{0z} ik_3 \tag{72}$$

$$E^b_{0x}\frac{\pi}{L} = -E^b_{0z} ik_3. \tag{73}$$

The continuity of its tangential components, Eq. (50), leads to

$$E^a_{0x} = E^b_{0x} \tag{74}$$

As a consequence of Eqs, (72)- (74),

$$E^b_{0z} = -E^a_{0z} \tag{75}$$

Then the discontinuity of its normal components, Eq. (46), determines the value of the corresponding amplitude,

$$E^a_{0z} = 2\pi\, \sigma_0 = \frac{2\pi i\pi}{\omega L} K_0 \tag{76}$$

using the surface charge amplitude of Eq. (69), and corroborating the choice of $k_1 = \pi/L$.

Correspondingly, the final forms of Eqs. (70) and (71) become

$$\vec{E}(x, z \geq 0) = \frac{2\pi}{\omega} K_0 \left[- \hat{i} k_3 \cos\frac{\pi x}{L} + \hat{k}\frac{i\pi}{L}\sin\frac{\pi x}{L}\right] e^{ik_3 z} \tag{77}$$

$$\vec{E}(x, z \leq 0) = \frac{2\pi}{\omega} K_0 \left[- \hat{i} k_3 \cos\frac{\pi x}{L} - \hat{k}\frac{i\pi}{L}\sin\frac{\pi x}{L}\right] e^{-ik_3 z} \tag{78}$$

Their curls lead to the other component of the electromagnetic field, via Eq. (49), with the result

$$\vec{B}(x, z \gtrless 0) = \mp \hat{j}\frac{2\pi}{c} K_0 \cos\frac{\pi x}{L} e^{\pm ik_3 z} \tag{79}$$

It is obvious that this magnetic induction field is divergenceless, Eq. (51), its components normal to the source plane vanish, Eq. (52), and its tangential components are discontinuous, Eq. (48), and consistent with the current source of Eq. (67). Again, the reader can check that its curl reproduces Eqs. (77)- (78), via Eq. (47).

Now the time averaged Poynting vector becomes

$$\langle\vec{S}\rangle_{\pm} = \frac{c}{8\pi}\left(\frac{2\pi}{c}K_0\right)^2 \frac{c}{\omega}\left[\hat{i}\frac{i\pi}{L}\sin\frac{\pi x}{L}\cos\frac{\pi x}{L} \pm \hat{k} k_3 \cos^2\frac{\pi x}{L}\right] \tag{80}$$

Correspondingly, the power radiated by each periodic rectangle of the source plane is given by

$$p = \frac{c}{8\pi}\left(\frac{2\pi}{c}K_0\right)^2 \frac{k_3 c}{\omega} 2L. \tag{81}$$

The limit situations when $L \to \infty$ for the sources in Eq. (53) and (66) correspond to the familiar uniform current distributions on the plane. The corresponding plane wave electromagnetic fields are also obtained as the limits of Eqs. (62)- (64) and (77)- (79), respectively.

Although, the solutions studied in this section correspond to traveling waves, they can be easily converted into evanescent wave solutions through analytic continuation. This is accomplished by the substitution of k_3 by ik_3, in Eqs. (55)- (56) and (70)- (71), and subsequent ones, going from positive to negative values of k_3^2 in Eq. Eqs. (54).

5. Discussion

The cosine distributions of the sources of the electrostatic, magnetostatic and electromagnetic fields constructed in Sects. 2-4, respectively, were chosen to ensure that the corresponding two-dimensional systems reduce to the familiar one-dimensional systems with uniform source distributions on the plane, as $L \to \infty$ or $k \to 0$. The examples are meant to be illustrative, but the methods are general and can be applied equally to sine distributions.

The common features shared by the electrostatic, magnetostatic and electromagnetic systems studied in Sects. 2-4 are successively: harmonic sources on a plane; two dimensional, divergenceless, curlless, and therefore harmonic electrostatic and magnetostatic fields; two dimensional, divergenceless, traveling wave electromagnetic fields; the harmonicity of the field is determined by the harmonicity of the source.

The static limit of the systems of Sect. 4 is obtained as $\omega \to 0$. In such a limit, Helmholtz Eq. Eqs. (A.11) reduces to Laplace Eq. (A.3), and $k_3^2 = -k_1^2$ in Eq. (A.13), so that their respective solutions Eq. (A.12) $\to$ (A.4). Traveling wave solutions are excluded and only evanescent wave solutions can be constructed. Specifically, for $k_3 \to (i\pi/L)$, Eqs. (62)-(63) reduce to Eqs. (35)-(36), while the electric intensity field in Eq. (64) vanishes, reproducing the magnetostatic situation of Sec. 3. Similarly, the reader can obtain the electrostatic limit of Eqs. (77)-(78), the electric intensity field produced by the sine distributed charge of Eq. (69), and the vanishing magnetic induction field, Eq. (79).

It can also be pointed out that the electrostatic, magnetostatic and electromagnetic fields of Sects. 2-4 were constructed directly as solutions of the respective Maxwell equations. The usual route of using the electrostatic, magnetic vector, and scalar and vector potentials, respectively, has been bypassed. Of course, such a route could also be followed. From the route already covered in this work, it is also possible to identify the respective potentials:

$$\phi(x, z \gtrless 0) = 2\pi\sigma_0 \frac{L}{\pi}\cos\frac{\pi x}{L} e^{\pm\frac{\pi z}{L}} \tag{82}$$

$$\phi(x \gtrless 0, z) = \frac{2\pi\sigma_0}{k}\sin kx \cosh kz \tag{83}$$

leading to the electrostatic fields of Sec. 2,

$$\vec{A}(x, z \gtrless 0) = \hat{j}\frac{2\pi}{c}K_0\frac{L}{\pi}\cos\frac{\pi x}{L}e^{\mp\frac{\pi z}{L}} \quad (84)$$

$$\vec{A}(x \gtrless 0, z) = \mp\hat{j}\frac{2\pi}{c}K_0\frac{1}{k}\sin kx\cosh kz \quad (85)$$

leading to the magnetostatic fields of Sec. 3, and

$$\vec{A}(x, z \gtrless 0, t) = \hat{j}\frac{2\pi}{c}K_0\frac{i}{k_3}\cos\frac{\pi x}{L}e^{\pm ik_3 z - i\omega t} \quad (86)$$

$$\vec{A}(x, z \gtrless 0, t) = -i\frac{2\pi c}{\omega^2}k_0\left[-\hat{i}k_3\cos\frac{\pi x}{L} \pm \hat{k}\frac{i\pi}{L}\sin\frac{\pi x}{L}\right]e^{\pm ik_3 z - i\omega t} \quad (87)$$

leading to the electromagnetic fields of Sec. 4. Notice the continuity of all the potentials and one of their derivatives, and the discontinuities of their other derivative, at the source plane. The latter are in correspondence with the normal components of the electric intensity field, Eq. (46), and the tangential components of the magnetic induction field, Eq. (48).

A Appendix

The first order differential equations of electrostatics, Eqs. (1) and (3), lead to Poisson's equation by taking the curl of Eq. (3) and using Eq. (1):

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = 0 \quad (A.1)$$

$$\therefore \quad \nabla^2\vec{E} = 4\pi\nabla\rho. \quad (A.2)$$

For $\rho = 0$, Eq. (A.2) becomes the Laplace equation and $\vec{E}$ must be a harmonic function.

The solutions of Laplace's equation in two dimensions, needed in Sec. 2 and 3,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) f(x, z) = 0 \quad (A.3)$$

have the general separable form:

$$f(x, z) = (A\cos kx + B\sin kx)(ce^{-kz} + De^{kz}) \quad (A.4)$$

These forms determine the choices of harmonic distributions of the sources on the plane, and also the corresponding electrostatic and magnetostatic fields. Also notice that for a given solution $f(x, z)$, its partial derivatives with respect to x and to z are also harmonic functions.

The first order differential equations of magnetostatics, Eqs. (23) and (25), also lead to Poisson's equation for the magnetic induction field, by taking the curl of Eq. (23) and using Eq. (25):

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2\vec{B} = \frac{4\pi}{c}\nabla \times \vec{J} \quad (A.5)$$

$$\therefore \quad \nabla^2\vec{B} = -\frac{4\pi}{c}\nabla \times \vec{J} \quad (A.6)$$

For $\vec{J} = 0$, Eq. (A.11) becomes the Laplace equation and $\vec{B}$ must be a harmonic function.

The Maxwell curl equations (47) and (49) are coupled first order differential equations, which can be decoupled by taking the curl of each one of them, and using them once more as well as Eqs. (45) and (51), with the end results that both $\vec{E}$ and $\vec{B}$ satisfy Helmholtz's equations:

$$\begin{aligned}\nabla \times (\nabla \times \vec{B}) &= \nabla(\nabla \cdot \vec{B}) - \nabla^2\vec{B} \\ &= \frac{4\pi}{c}\nabla \times \vec{J} - \frac{i\omega}{c}\nabla \times \vec{E} \\ &= \frac{4\pi}{c}\nabla \times \vec{J} + \frac{\omega^2}{c^2}\vec{B} \end{aligned} \quad (A.7)$$

$$\therefore \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{B} = -\frac{4\pi}{c}\nabla \times \vec{J} \quad (A.8)$$

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} \\ &= \frac{i\omega}{c}\nabla \times \vec{B} \\ &= \frac{i\omega}{c}\left(\frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{E}\right) \end{aligned} \quad (A.9)$$

$$\therefore \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 4\pi\nabla\rho - \frac{i\omega 4\pi}{c^2}\vec{J} \quad (A.10)$$

For $\rho = 0$ and $\vec{J} = 0$ Eqs. (A.8) and (A.10) become the homogeneous Helmholtz equation.

The solutions of Helmoltz equation in two dimensions, needed in Sect. 5,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right)g(x, z) = 0 \quad (A.11)$$

have the general separable form

$$g(x, z) = (A\cos k_1 x + B\sin k_1 x) \times (Ce^{ik_3 z} + De^{-ik_3 z}) \quad (A.12)$$

where

$$k_1^2 + k_3^2 = \frac{\omega^2}{c^2} \quad (A.13)$$

For a given solution $g(x, z)$, its partial derivatives with respect to x and z are also solutions of Helmholtz's equation.

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