Electromagnetic fields from high frequency currents harmonically distributed on infinitely long circular cylinders

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High frequency currents harmonically distributed around the surface of infinitely long circular cylinders, in the directions of the generatrices and of the circles, are identified, and their associated electromagnetic fields are constructed. The latter are obtained as exact traveling wave solutions of Maxwell equations in differential and boundary condition forms. The study of these systems is appropriate for Electromagnetic Theory courses.

Keywords: Maxwell's equations; exact solutions; radiation; cylindrical antennas.

Se identifican corrientes de alta frecuencia distribuidas alrededor de la superficie de cilindros circulares infinitamente largos, en las direcciones de las generatrices y de los círculos, y se construyen sus campos electromagnéticos asociados. Los últimos se obtienen como soluciones exactas de ondas viajantes de las ecuaciones de Maxwell en sus formas diferenciales y de condiciones a la frontera. El estudio de estos sistemas es apropiado para cursos de teoría electromagnética.

Descriptores: Ecuaciones de Maxwell; soluciones exactas; radiación; antenas cilíndricas.

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1. Introduction

The electrostatic field of a straight line or a circular cylinder with uniform distributions of charge, and the magnetostatic fields of circular cylinders with uniform distributions of electric current in the directions of the generatrices or of the circles, are studied in the introductory course of Electricity and Magnetism [1,2]. These fields can be directly evaluated via Gauss's law and Ampère's law, respectively. Some natural extensions of these familiar results have been studied as two-dimensional harmonic expansions of electrostatic and magnetostatic fields for conical cylindrical geometries [3], appropriate for the level of Electromagnetic Theory courses. Another work for this level has recently covered the study of electrostatic, magnetostatic and electromagnetic fields for harmonically distributed sources on infinite planes [4].

On the other hand, the number of examples of radiating systems studied in the Electromagnetic Theory courses is quite limited, including the electric and magnetic oscillating dipoles and the oscillating current uniformly distributed on an infinite plane [5,6]. Reference [4] and the present article are written to provide additional examples of types of radiating systems with exact solutions and accessible to the intermediate and advanced undergraduate level students. The rest of this work is organized in the following way: Sec. 2 contains the formulation of the problem of constructing the electromagnetic fields produced by the high frequency currents distributed on the surface of infinitely long circular cylinders; Subsec. 2A covers the cases of currents in the direction of the generatrices with the harmonic distributions around the cylinder; Subsec. 2B corresponds to the cases of currents in the direction of the circles with harmonic distributions around the cylinder. In Sec. 3, the Poynting vector, angular distribution and power associated with the electromagnetic radiation emitted by the respective source cylinders are evaluated. Section 4 consists of a didactic discussion of the specific results of Secs. 2 and 3 and some of their extensions and limit situations, which may be helpful to teachers and students interested in radiating systems. The Appendix contains mathematical results connecting Maxwell equations and the Helmholtz equation, and their solutions in circular cylindrical coordinates, which are used in Sec. 2.

2. Radiation fields from high frequency currents distributed on infinitely long circular cylinders

Maxwell equations are the mathematical expression of the laws of Electromagnetism, which connect the space and time variations of the electric intensity and magnetic induction fields with the electric charges and currents as their sources. Here, the equations are written in their differential and boundary condition form, for sources and fields with a time harmonic variation $e^{-i\omega t}$of frequency ω:

$$\nabla \cdot \vec{E} = 4\pi\rho \qquad (\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma \qquad \text{(1a,b)}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{E} \qquad (\vec{B}_2 - \vec{B}_1)\times\hat{n} = \frac{4\pi}{c}\vec{K} \qquad \text{(2a,b)}$$

$$\nabla \times \vec{E} = \frac{i\omega}{c}\vec{B} \qquad (\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0 \qquad \text{(3a,b)}$$

$$\nabla \cdot \vec{B} = 0 \qquad (\hat{B}_2 - \hat{B}_1)\cdot\hat{n} = 0 \qquad \text{(4a,b)}$$

They correspond successively to: 1) Gauss's law, stating that electrical charges are sources of electric flux; and that the normal components of the electric intensity field at a boundary surface have a discontinuity proportional to the surface charge density. 2) Ampère – Maxwell law identifying two sources of magnetic circulation, electric currents and time rate of change of electric flux (Maxwell displacement current); and that the tangential component of the magnetic induction field at a boundary surface have a discontinuity proportional to the perpendicular surface linear current density. 3) Faraday's law, which recognizes that the time rate of change of the magnetic flux is a source of electric circulation; and that the tangential components of the electric field at a boundary surface are continuous. 4) Gauss's law stating the nonexistence of magnetic monopoles; and the continuity of the normal components of the magnetic induction field at a boundary surface.

For the problems studied in this work, the currents are distributed on the surface of cylinders, and consequently the volume charge density ρ and the current density $\vec{J}$ vanish at all points inside and outside the cylinders. Therefore, both the electric intensity and the magnetic induction fields must be divergenceless, Eqs. (1a) and (4a), $\vec{E}$ is proportional to the curl of $\vec{B}$, Eq. (2a), and $\vec{B}$ is proportional to the curl of $\vec{E}$, Eq. (3a). These four equations are shown in the Appendix to be equivalent to the Helmholtz equation, Eqs. (A3) and (A6). In conclusion, both fields are constructed as solutions of the Helmholtz equation with vanishing divergences, and one being the curl of the other and vice versa; and, additionally, they must satisfy the boundary conditions of Eqs. (1b) – (4b).

2.1. Currents along generatrices and harmonically distributed around cylinders

The geometry of the surfaces where the currents are distributed suggests the use of circular cylindrical coordinates to describe the sources and fields. In terms of such coordinates $(\rho = \sqrt{x^2+y^2}, \varphi = \tan^{-1} y/x, z)$and its associated unit vectors

$$\hat{\rho} = \hat{i}\cos\varphi + \hat{j}\sin\varphi, \quad \hat{\varphi} = -\hat{i}\sin\varphi + \hat{j}\cos\varphi, \quad \hat{k} \tag{5}$$

the linear current density is restricted to the surface of a cylinder of radius $\rho = a$, in the $\hat{k}$ direction, and has a cosine or sine distribution

$$\vec{K}_c(\rho = a, \varphi, z) = \hat{k} K_0 \cos m\varphi \tag{6}$$

$$\vec{K}_s(\rho = a, \varphi, z) = \hat{k} K_0 \sin m\varphi. \tag{7}$$

Since the sources are invariant under translations in the z direction the electric and magnetic fields depend only on the circular coordinates ρ and φ. It is necessary to distinguish between the solutions of Maxwell Eqs. (1a) – (4a) and the Helmholtz Eq. (A8) inside and outside the source cylinder. In particular, we start by proposing

$$\vec{B}^c(\rho \le a, \varphi) = \hat{\rho} B^{c<}_{0\rho}(\rho)\sin m\varphi + \hat{\varphi} B^{c<}_{0\varphi}(\rho)\cos m\varphi \tag{8}$$

$$\vec{B}^c(\rho \ge a, \varphi) = \hat{\rho} B^{c>}_{0\rho}(\rho)\sin m\varphi + \hat{\varphi} B^{c>}_{0\varphi}(\rho)\cos m\varphi \tag{9}$$

for the case of the source of Eq. (6). The radial functions $B^{c<}(\rho)$ must be connected with the regular Bessel functions in Eq. (A9), and $B^{c>}(\rho)$ must be connected with outgoing waves of Eq. (A11) with $n = 1$. The connections can be identified through the condition that the fields of Eqs. (8) and (9) must be divergenceless, Eq. (4a):

Since

$$\begin{aligned}\nabla \cdot \vec{B} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B^c_{0\rho}(\rho)\sin m\varphi\right) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(B^c_{0\varphi}(\rho)\cos m\varphi\right) \\ &= \frac{1}{\rho}\sin m\varphi\left[\frac{\partial}{\partial\rho}\left(\rho B^c_{0\rho}(\rho)\right) - m B^c_{0\varphi}(\rho)\right]\end{aligned} \tag{10}$$

must vanish, the radial functions inside and outside of the source cylinder are chosen, respectively, as

$$B^{c<}_{0\rho}(\rho) = B^{c<}_0 m\frac{J_m(k\rho)}{\rho} \quad B^{c<}_{0\varphi}(\rho) = B^{c<}_0 \frac{d}{d\rho}J_m(k\rho) \tag{11}$$

and

$$B^{c>}_{0\rho}(\rho) = B^{c>}_0 m\frac{H^{(1)}_m(k\rho)}{\rho} \quad B^{c>}_{0\varphi}(\rho) = B^{c>}_0 \frac{d}{d\rho}H^{(1)}_m(k\rho). \tag{12}$$

The coefficients $B^{c<}_0$ and $B^{c>}_0$ will be determined by the boundary conditions of Eqs. (4b) and (2b), that the normal component at the boundary must be continuous,

$$B^{c<}_{0m} m\frac{J_m(ka)}{a} = B^{c>}_{0m} m\frac{H^{(1)}_m(ka)}{a} \tag{13}$$

and the tangential components have a discontinuity proportional to the linear current density

$$\left[B^{c>}_0 \frac{dH^{(1)}_m(k\rho)}{d\rho} - B^{c<}_0\frac{dJ_m(k\rho)}{d\rho}\right]_{\rho=a} = \frac{4\pi}{c}K_0. \tag{14}$$

The coefficients in Eq. (13) can be written as

$$B^{c<}_0 = B^c_0 H^{(1)}_m(ka), \quad B^{c>}_0 = B^c_0 J_m(ka)$$

and Eq. (14) determines the value of the remaining unknown coefficient,

$$kB^c_0\left[J_m(ka)\frac{dH^{(1)}_m(ka)}{d(ka)} - H^{(1)}_m(ka)\frac{dJ_m(ka)}{d(ka)}\right] = \frac{4\pi}{c}K_0. \tag{15}$$

The bracket in this equation is identified as the Wronskian of Eq. (A23), so that

$$B^c_0 = \frac{4\pi}{ck}K_0\frac{\pi ka}{2i} = -\frac{2\pi}{c}K_0 i\pi a \tag{16}$$

Thus, the magnetic induction fields of Eqs. (8) and (9) take their final forms:

$$\vec{B}^c(\rho \leq a, \varphi) = -\frac{2\pi}{c} K_0 i\pi a H_m^{(1)}(ka) \times \left[\hat{\rho} m \frac{J_m(k\rho)}{\rho} \sin m\varphi + \hat{\varphi} \frac{dJ_m(k\rho)}{d\rho} \cos m\varphi\right] \quad (17)$$

$$\vec{B}^c(\rho \geq a, \varphi) = -\frac{2\pi}{c} K_0 i\pi a J_m(ka) \times \left[\hat{\rho} m \frac{H_m^{(1)}(k\rho)}{\rho} \sin m\varphi + \hat{\varphi} \frac{dH_m^{(1)}(k\rho)}{d\rho} \cos m\varphi\right] \quad (18)$$

The curls of these magnetic fields lead to the companion electric fields inside and outside the source cylinder, via Eq. (2a),

$$\vec{E}^c(\rho \leq a, \varphi) = -\frac{2\pi}{c} K_0 \pi ka H_m^{(1)}(ka) \hat{k} J_m(k\rho) \cos m\varphi \quad (19)$$

$$\vec{E}^c(\rho \geq a, \varphi) = -\frac{2\pi}{c} K_0 \pi ka J_m(ka) \hat{k} H_m^{(1)}(k\rho) \cos m\varphi. \quad (20)$$

Notice the continuity of the tangential components at the source cylinder. The reader can also check that the curls of Eqs. (19) and (20) lead back to Eqs. (17) and (18), via Faraday's law, Eq. (3a).

For the source of Eq. (7), the same procedure can be followed replacing the sine distribution for the cosine distribution in Eq. (6), and performing the exchange of the trigonometric functions in Eqs. (8) and (9). For the sake of brevity, here we give the resulting electromagnetic fields, counterparts of Eqs. (17) – (20):

$$\vec{B}^s(\rho \leq a, \varphi) = -\frac{2\pi}{c} K_0 i\pi a H_m^{(1)}(ka) \left[-\hat{\rho} m \frac{J_m(k\rho)}{\rho} \cos m\varphi + \hat{\varphi} \frac{dJ_m(k\rho)}{d\rho} \sin m\varphi\right] \quad (21)$$

$$\vec{B}^s(\rho \geq a, \varphi) = -\frac{2\pi}{c} K_0 i\pi a J_m(ka) \left[-\hat{\rho} m \frac{H_m^{(1)}(k\rho)}{\rho} \cos m\varphi + \hat{\varphi} \frac{dH_m^{(1)}(k\rho)}{d\rho} \sin m\varphi\right] \quad (22)$$

$$\vec{E}^s(\rho \leq a, \varphi) = -\frac{2\pi}{c} K_0 \pi ka H_m^{(1)}(ka) \hat{k} J_m(k\rho) \sin m\varphi \quad (23)$$

$$\vec{E}^s(\rho \geq a, \varphi) = -\frac{2\pi}{c} K_0 \pi ka J_m(ka) \hat{k} H_m^{(1)}(k\rho) \sin m\varphi. \quad (24)$$

The continuity of the normal components of $\vec{B}$, the discontinuity of its tangential components at the source cylinder in Eqs. (21) and (22), and the corresponding continuity of components of $\vec{E}$ in Eqs. (23) and (24) can be directly ascertained.

2.2. Currents along circles and harmonically distributed around cylinders

The currents in this subsection share the same location and the same harmonic distributions as those in the previous section, but differ in their direction of motion which is now circular:

$$\vec{K}_c(\rho = a, \varphi, z) = \hat{\varphi} K_0 \cos m\varphi \quad (25)$$

$$\vec{K}_s(\rho = a, \varphi, z) = \hat{\varphi} K_0 \sin m\varphi. \quad (26)$$

The difference leads to recognize the presence of surface charge densities on the cylinders, as determined by the continuity equation,

$$\nabla \cdot \vec{K} + \frac{\partial \sigma}{\partial t} = 0.$$

In fact, the respective surface charge densities are immediately evaluated to be

$$\sigma_c(\rho = a, \varphi, z, t) = \frac{i}{\omega a} K_0 m \sin m\varphi e^{-i\omega t} \quad (27)$$

$$\sigma_s(\rho = a, \varphi, z, t) = -\frac{i}{\omega a} K_0 m \cos m\varphi e^{-i\omega t} \quad (28)$$

where the complementary harmonicities of charges and currents should be noticed.

In the case of the sources of Eqs. (25) and (27) we construct first the electric intensity fields inside and outside of the source cylinder:

$$\vec{E}^c(\rho \leq a, \varphi) = \hat{\rho} E_{0\rho}^{c<}(\rho) \sin m\varphi + \hat{\varphi} E_{0\varphi}^{c<}(\rho) \cos m\varphi \quad (29)$$

$$\vec{E}^c(\rho \geq a, \varphi) = \hat{\rho} E_{0\rho}^{c>}(\rho) \sin m\varphi + \hat{\varphi} E_{0\varphi}^{c>}(\rho) \cos m\varphi. \quad (30)$$

The condition that their divergences must vanish

$$\nabla \cdot \vec{E} = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho E_{0\rho}^c(\rho)\right) \sin m\varphi + E_{0\varphi}^c(\rho) \frac{1}{\rho} \frac{d}{d\varphi} (\cos m\varphi) = 0. \quad (31)$$

can be assured by the choices

$$E^{c<}_{0\rho}(\rho) = E^{c<}_0 m \frac{J_m(k\rho)}{\rho}$$
$$E^{c<}_{0\varphi}(\rho) = E^{c<}_0 \frac{d}{d\rho} J_m(k\rho) \tag{32}$$

and

$$E^{c>}_{0\rho}(\rho) = E^{c>}_0 m \frac{H^{(1)}_m(k\rho)}{\rho}$$
$$E^{c>}_{0\varphi}(\rho) = E^{c>}_0 \frac{d}{d\rho} H^{(1)}_m(k\rho). \tag{33}$$

Then, the coefficients $E^{c<}_0$ and $E^{c>}_0$ are connected and determined through the successive use of Faraday's law, Eq. (3b):

$$E^{c<}_0 \frac{dJ_m(ka)}{da} = E^{c>}_0 \frac{dH^{(1)}_m(ka)}{da} \tag{34}$$

implying

$$E^{c<}_0 = E^c_0 \frac{dH^{(1)}_m(ka)}{da}, \quad E^{c>}_0 = E^c_0 \frac{dJ_m(ka)}{da}; \tag{35}$$

and Gauss's law, Eq. (1b):

$$E^c_0 \frac{m}{a}\left[\frac{dJ_m(ka)}{da} H^{(1)}_m(ka) - \frac{dH^{(1)}_m(ka)}{da} J_m(ka)\right] = 4\pi\sigma_c = \frac{4\pi i K_0 m}{\omega a} \tag{36}$$

where, in the last step, the surface charge amplitude of Eq. (27) is used. The quantity inside the brackets is identified as the Wronskian of Eq. (A23) leading to the value of the coefficient

$$E^c_0 = -\frac{4\pi i K_0}{\omega}\frac{\pi a}{2i}. \tag{37}$$

Thus, Eqs. (29) and (30) finally become

$$\vec{E}^c(\rho \le a, \varphi) = -\frac{2\pi}{\omega} K_0 \pi a \frac{dH^{(1)}_m(ka)}{da} \times \left[\hat{\rho} m \frac{J_m(k\rho)}{\rho}\sin m\varphi + \hat{\varphi}\frac{dJ_m(k\rho)}{d\rho}\cos m\varphi\right] \tag{38}$$

$$\vec{E}^c(\rho \ge a, \varphi) = -\frac{2\pi}{\omega} K_0 \pi a \frac{dJ_m(ka)}{da} \times \left[\hat{\rho} m \frac{H^{(1)}_m(k\rho)}{\rho}\sin m\varphi + \hat{\varphi}\frac{dH^{(1)}_m(k\rho)}{d\rho}\cos m\varphi\right] \tag{39}$$

The curls of Eqs. (38) and (39) lead to the companion magnetic induction fields, via Eq. (3a):

$$\vec{B}^c(\rho \le a, \varphi) = -\frac{i2\pi}{c} K_0 \pi a \frac{dH^{(1)}_m(ka)}{da} \hat{k} J_m(k\rho)\cos m\varphi \tag{40}$$

$$\vec{B}^c(\rho \ge a, \varphi) = -\frac{i2\pi}{c} K_0 \pi a \frac{dJ_m(ka)}{da} \hat{k} H^{(1)}_m(k\rho)\cos m\varphi. \tag{41}$$

Notice that their normal components vanish, thus satisfying Eq. (4b), and their tangential components at the source cylinder are discontinuous reproducing the linear current density of Eq. (25), via Eq. (2b). They are also divergenceless, Eq. (4a), and their curls reproduce Eqs. (38) and (39), via (2a).

For the sources of Eqs. (26) and (28), the corresponding electromagnetic fields are simply given in their final forms:

$$\vec{B}^s(\rho \le a, \varphi) = \frac{i2\pi}{c} K_0 \pi a \frac{dH^{(1)}_m(ka)}{da} \hat{k} J_m(k\rho)\sin m\varphi \tag{42}$$

$$\vec{B}^s(\rho \ge a, \varphi) = \frac{i2\pi}{c} K_0 \pi a \frac{dJ_m(ka)}{da} \hat{k} H^{(1)}_m(k\rho)\sin m\varphi \tag{43}$$

$$\vec{E}^s(\rho \le a, \varphi) = \frac{2\pi}{\omega} K_0 \pi a \frac{dH^{(1)}_m(ka)}{da}\left[\hat{\rho} m \frac{J_m(k\rho)}{\rho}\cos m\varphi - \hat{\varphi}\frac{dJ_m(k\rho)}{d\rho}\sin m\varphi\right] \tag{44}$$

$$\vec{E}^s(\rho \ge a, \varphi) = \frac{2\pi}{\omega} K_0 \pi a \frac{dJ_m(ka)}{da}\left[\hat{\rho} m \frac{H^{(1)}_m(k\rho)}{\rho}\cos m\varphi - \hat{\varphi}\frac{dH^{(1)}_m(k\rho)}{d\rho}\sin m\varphi\right]. \tag{45}$$

The reader can check that these fields satisfy Eqs. (1)–(4) in their differential and boundary condition forms.

3. Poynting vector, angular distribution and power of the electromagnetic radiation

In this section, the successive evaluations of the energy density flux, angular distribution, and the total power radiated per unit length by the antennas are presented in detail for the electromagnetic fields of Eqs. (18)–(20). The corresponding results for the electromagnetic fields of Eqs. (22)–(24), (39)–(41) and (43)–(45) are also described, compared and commented upon.

The Poynting vector,

$$\vec{S} = \frac{c\vec{E} \times \vec{B}}{8\pi}, \tag{46}$$

represents the energy per unit area per unit time associated with the electric and magnetic fields in each point in space and instant of time [5]. For the electromagnetic fields of our interest the time average of such an energy density flux is evaluated as

$$\left\langle \vec{S} \right\rangle = \frac{c\vec{E}^* \times \vec{B}}{8\pi}. \tag{47}$$

In the specific case of the electromagnetic fields described by Eqs. (18) and (20), the result for any position (ρ, φ) is

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi}\left[\frac{2\pi}{c}K_0\pi ka J_m(ka)\right]^2 H_m^{(1)*}(k\rho)\cos m\varphi \times \left[\hat{\varphi}\frac{im}{k\rho}H_m^{(1)}(k\rho)\sin m\varphi - \hat{\rho}i\frac{dH_m^{(1)}(k\rho)}{d(k\rho)}\cos m\varphi\right] \tag{48}$$

For the far zone, where $k\rho \to \infty$, the asymptotic form of the Hankel functions is given by Eq. (A17), and the time-averaged Poynting vector takes the form:

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi}\left[\frac{2\pi}{c}K_0\pi ka J_m(ka)\right]^2 \frac{2}{\pi k\rho} \left[\hat{\varphi}\frac{im}{k\rho}\sin m\varphi\cos m\varphi + \hat{\rho}\cos^2 m\varphi\right], \tag{49}$$

Where only the dominant terms are kept, including the derivative of the exponential factor in Eq. (A17). It is clear that the azimuthal component will become negligible compared with the radial component, so that the radiation propagates radially away from the antenna. The azimuthal component is associated with the electromagnetic induction of the fields.

The angular distribution of the radiated field at a large distance from the antenna is obtained by evaluating the energy crossing the area element $\Delta\vec{a} = \hat{\rho}\rho d\varphi\Delta z$ per unit time:

$$dP = \left\langle \vec{S} \right\rangle \cdot d\vec{a} = \frac{c}{8\pi}\left[\frac{2\pi}{c}K_0\pi ka J_m(ka)\right]^2 \frac{2}{\pi k}\cos^2 m\varphi d\varphi\Delta z. \tag{50}$$

Correspondingly, the total radiated average power is obtained by integration over all the azimuthal directions with the final result

$$P = \frac{c}{8\pi}\left[\frac{2\pi}{c}K_0\pi ka J_m(ka)\right]^2 \frac{\Delta z}{\pi k} = \frac{1}{2}\frac{k}{c}\left[J_m(ka)\right]^2\left[K_0\pi ka\right]^2\Delta z = \frac{1}{2}R_{rad}\Delta z I_0^2, \tag{51}$$

where $K_0\pi a$ is identified as the amplitude current I_0 (in stat Amperes) around the circular cylinder, and

$$R_{rad} = \frac{k}{c}\left[J_m(ka)\right]^2 \tag{52}$$

is the radiative resistance per unit length of the antenna in stat ohms/cm [5].

The longitudinal currents of Eqs. (6) and (7) produce the longitudinal electric fields of Eqs. (19)–(20) and (23)–(24), respectively. The respective radiation fields described by Eqs. (20) and (24) are correspondingly linearly polarized in the longitudinal direction. The sources and electric fields under discussion differ in their respective $\cos m\varphi$ and $\sin m\varphi$ dependences, leading to the corresponding differences in the associated magnetic induction fields contained in Eqs. (17)–(18) and (21)–(22), and in the counterparts of Eqs. (48)–(50). The latter exhibit the difference between the $\cos^2 m\varphi$ and $\sin^2 m\varphi$ angular distributions, which have the same shape with different orientations. On the other hand, Eqs. (51) and (52) are valid for both situations.

When we go from the longitudinal currents, Eqs. (6) and (7), to the solenoidal currents, Eqs. (25) and (26), we notice the same space dependence of the electric fields, Eqs. (19)–(20) and Eqs. (23)–(24), and of the magnetic fields, Eqs. (40)–(41) and Eqs. (42)–(43); and also of the magnetic fields, Eqs. (17)–(18) and (21)–(22), and of the electric fields, Eqs. (38)–(39) and (44)–(45). The differences reside in the normalization factors, with the substitution of $kZ_m(ka)$ by $idZ_m(ka)/da$. Consequently, the radiation fields due to the solenoidal currents are linearly polarized in the $\hat{\varphi}$ direction and show $\cos^2 m\varphi$ and $\sin^2 m\varphi$ angular distributions. Their common radiation resistance per unit length of the antenna is

$$R_{rad} = \frac{k}{c}\left|\frac{dH_m^{(1)}(ka)}{d(ka)}\right|^2$$

4. Discussion

The exact solutions of Maxwell's equations (1)–(4), for the axial currents of Eqs. (6)–(7) and the solenoidal currents of Eqs. (25)–(26), have been explicitly constructed in Sec. 2. The respective electromagnetic fields are given by Eqs. (17)–(20), (21)–(24), (38)–(41) and (40)–(45), respectively. They are solenoidal fields, satisfying the Helmholtz equation, as well as Maxwell's and Faraday's laws, inside and outside the source cylinders. Notice that the inside solutions depend on the ordinary Bessel functions, while the outside solutions correspond to Hankel functions of the first order. The latter describe asymptotically the outgoing circular cylindrical waves of the electromagnetic radiation emitted by the respective antenna, which additionally are linearly polarized in the direction of the respective source current. The angular distributions, powers and radiation resistances are determined by the harmonicity of the source current, as evaluated and discussed in Sec. 3. The inside solutions and the quasistatic solutions near the source cylinders are obtained by taking the long wavelength limit, $k\rho \to 0$, of the exact solutions via Eqs. (A12)–(A14).

The static limit corresponds to $\omega = 0$, $k = 0$, with the surviving radial dependences of Eqs. (A12)–(A14). For the stationary currents along the generatrices, Eqs. (6)–(7), only the magnetostatic fields persist, and the electric fields vanish. On the other hand, the circular currents, with $m \neq 0$ of Eqs. (25)–(26) cannot be stationary, but the static charge densities of Eqs. (27) and (28) are well defined in the limit $K_0 = 0$, such that $iK_0 m/\omega a = \sigma_0$, in which cases the electrostatic fields persist, and the magnetic fields vanish. For the uniformly distributed current of Eq. (25) with $m = 0$, the uniform magnetic induction field in the axial direction inside the cylinder is recovered from Eq. (40). Likewise, for the uniformly distributed charge of Eq. (28) with $m = 0$, the radial electric intensity field, inversely proportional to the radial distance, outside the cylinder follows from Eq. (45).

A APPENDIX

Maxwell Eqs. (1a)–(4a) are coupled first order partial differential equations. They can be decoupled by taking the curls of Eqs. (2a) and (2b), respectively, and using the equations themselves as needed:

$$\nabla \times (\nabla \times \vec{B}) = \frac{4\pi}{c}\nabla \times \vec{J} - \frac{i\omega}{c}\nabla \times \vec{E} \tag{A.1}$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \frac{4\pi}{c}\nabla \times \vec{J} + \frac{\omega^2}{c^2}\vec{B} \tag{A.2}$$

$$\therefore \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{B} = \frac{4\pi}{c}\nabla \times \vec{J} \tag{A.3}$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \frac{i\omega}{c}\nabla \times \vec{B} \tag{A.4}$$

$$4\pi\nabla\rho - \nabla^2 \vec{E} = \frac{\omega^2}{c^2}\vec{E} + \frac{4\pi i\omega}{c^2}\vec{J} \tag{A.5}$$

$$\therefore \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 4\pi\nabla\rho - \frac{4\pi i\omega}{c^2}\vec{J}. \tag{A.6}$$

From Eq. (A1) to (A2), Eq. (3a) has been used, and in the next step, Eq. (4a) is also used. Correspondingly, in the next set of equations use is made of Eqs. (1a) and (2a). Equations (A3) and (A6) are the inhomogeneous Helmholtz equations for the respective fields.

For the sources distributed on the surface of the cylinders, $\rho = 0$ and $J = 0$ for points inside and outside the cylinders, and each of the components of the fields in Eqs. (A3) and (A6) must satisfy the homogeneous Helmholtz equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) f(\vec{r}) = 0. \tag{A.7}$$

For a given solution of this equation in Cartesian coordinates $f(x, y, z)$ it is recognized that its partial derivatives with respect to x, y and z are also solutions of the same equation.

For the sources of Sec. 2, which are independent of the z coordinate, the electromagnetic fields inherit the same independence, and only depend on the ρ and φ coordinates. Correspondingly, we concentrate on the solutions of the Helmholtz equation (A7) in circular coordinates:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + k^2\right) f(\rho, \varphi) = 0 \tag{A.8}$$

where $k = \omega/c$ is the wave number. The solutions are separable as products of radial Bessel functions and sine or cosine functions of integer multiples of the angular coordinate [7,8]:

$$f(\rho, \varphi) = [A_m J_m(k\rho) + B_m Y_m(k\rho)] \times (C_m \cos m\varphi + D_m \sin m\varphi), \quad m = 0, 1, 2, \ldots \tag{A.9}$$

Here, $J_m(k\rho)$ is the ordinary regular Bessel function, $Y_m(k\rho)$ is the Bessel function of the second kind, or Neumann function, singular at $\rho = 0$. Instead of the trigonometric functions in the angular coordinate, their complex combinations

$$\cos m\varphi \pm i \sin m\varphi = e^{\pm im\varphi} \tag{A.10}$$

representing rotating waves can be used. Similarly, the complex combinations of the stationary Bessel functions,

$$J_m(k\rho) \pm iY_m(k\rho) = H_m^{(n)}(k\rho), \; n = 1, 2 \tag{A.11}$$

which are Bessel functions of the third and fourth kind, or Hankel functions of the first and second kind, represent circular cylindrical outgoing and incoming waves, respectively.

It is important to be familiar with the behavior of the Bessel functions close to the origin, $z \to 0$,

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \; \nu \neq -1, -1, -3, \ldots \tag{A.12}$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{z}{2}\right)^{-\nu}, \quad Re\nu > 0 \tag{A.13}$$

$$Y_0(z) \sim iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim \frac{2}{\pi}\ln z, \tag{A.14}$$

and asymptotically, $z \to \infty$,

$$J_\nu(z) \to \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{A.15}$$

$$Y_\nu(z) \to \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{A.16}$$

$$H_\nu^{(n)}(z) \to \sqrt{\frac{2}{\pi z}}e^{\pm i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}, \; n = 1, 2. \tag{A.17}$$

They also satisfy recurrence relations

$$Z_{\nu-1}(z) + Z_{\nu+1}(z) = \frac{2\nu}{z}Z_\nu(z) \tag{A.18}$$

$$Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2Z'_\nu(z) \tag{A.19}$$

$$Z'_\nu(z) = Z_{\nu-1}(z) - \frac{\nu}{z}Z_\nu(z) \tag{A.20}$$

$$Z'_\nu(z) = -Z_{\nu+1}(z) + \frac{\nu}{z}Z_\nu(z) \tag{A.21}$$

where $Z_\nu(z)$ is any of the four kinds of Bessel functions. Some of their Wronskians are:

$$W\left\{J_\nu(z), Y_\nu(z)\right\} = J_\nu(z)Y'_\nu(z) - J'_\nu(z)Y_\nu(z) = \frac{2}{\pi z} \tag{22}$$

$$W\left\{J_\nu(z), H^{(1)}_\nu(z)\right\} = J_\nu(z)H^{(1)\prime}_\nu(z) - J'_\nu(z)H^{(1)}_\nu(z) = \frac{2i}{\pi z} \tag{23}$$

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