Chirotope concept in various scenarios of physics

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We argue that the chirotope concept of oriented matroid theory may be found in different scenarios of physics, including classical mechanics, quantum mechanics, gauge field theory, p-branes formalism, two time physics and Matrix theory. Our observations may motivate the interest of possible applications of matroid theory in physics.

Keywords: p-branes; matroid theory.

Argumentamos que el concepto de chirotope de la teoría de matroides orientados se puede encontrar en diferentes escenarios de física, incluyendo mecánica clásica, mecánica cuántica, teorías de campos normados, el formalismo de p-branes, física de dos tiempos y la teoría de Matrices. Nuestras observaciones pueden motivar el interés de posibles aplicaciones de la teoría de matroides en física.

Descriptores: p-branes; teoría de matroides.

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1. Introduction

Since Whitney's work [1], the concept of matroid has been of much interest to a large number of mathematicians, specially those working in combinatorial. Technically, this interest is perhaps due to the fact that matroid theory [2] provides a generalization of both matrix theory and graph theory. However, at some deeply level, it seems that matroid theory may appear interesting to mathematicians, among other reasons, because its duality properties. In fact, one of the attractive features of a matroid theory is that every matroid has an associated dual matroid. This duality characteristic refers to any individual matroid, but matroid theory states stronger theorem at the level of axiom systems and their consequent theorems, namely if there is an statement in the matroid theory that has been proved true, then also its dual is true [3]. These duality propositions play a such on important role that matroid theory may even be called the duality theory.

It turns out that at present, the original formalism of matroid theory has been generalized in different fronts, including biased matroids [4], and greedoids [5]. However, it seems that one of the most natural extensions is oriented matroid theory [6]. In turn, the matroid bundle structure [7-11] emerges as a natural extension of oriented matroid theory. This final extension provides a very good example of the observation that two fundamental mathematical subjects which have been developed independently, are, sooner or later, fused in just one subject; in this case, fiber bundle theory becomes fused with matroid theory leading to matroid bundle structure.

The central idea of the present work is to call the attention of the physicists community about the possible importance that matroid theory may have in different scenarios of physics. For this purpose in Sec. 2 we develop a brief introduction of oriented matroid theory in such a way that we can prepare the mathematical tools which may facilitate its connection with different scenarios of physics. In particular, we introduce the definition of an oriented matroid in terms of chirotopes (see Ref. 6, Sec. 3.5). Roughly speaking a chirotope is a completely antisymmetric object that takes values in the set $\{-1, 0, 1\}$. It has been shown in Ref. 12 that the completely antisymmetric Levi-Civita symbol $\varepsilon^{i_1 \cdots i_d}$ provides us with a particular example of a chirotope. Motivated by this observation, and considering that physicists are more or less familiar with the symbol $\varepsilon^{i_1 \cdots i_d}$, we develop a brief introduction to oriented matroid theory by using the argument that the chirotope concept is in fact a generalization of the symbol $\varepsilon^{i_1 \cdots i_d}$. We hope that with such an introduction some physicists become interested in the subject.

It is worth mentioning that the concept of matroid has already been connected with Chern-Simons theory [13], string theory [14], p-branes and Matrix theory [12]. Moreover, a proposed new theory called gravitoid [15,16] has emerged from the connection between oriented matroid theory, gravity, and supergravity. Except for the link between matroids, p-branes and Matrix theory which are briefly reviewed here, all these applications of the matroid concept are not approached in this work. Instead, we add new connections such as the identification of chirotopes with the angular momentum in both classical and quantum mechanics scenarios. We also remark the fundamental importance that chirotope concept may have in two time physics [17], in electromagnetism, and Yang-Mills physics.

In a sense, all these connections are similar to the identification of tensors in different scenarios of physics. But , of course, although interesting these identifications still appear more important to the fact that tensor analysis was eventually used as a the mathematical basis of a fundamental theory: general relativity. In this case the guide was a new symmetry provided by the equivalence principle, namely general covariance. Therefore, the hope is that all these connections of matroids with different concepts of physics may eventually help to identify a new fundamental theory in which oriented matroid theory plays a basic role. But, for this to be possible we need a new symmetry as a guide. Our conjecture is that such a fundamental theory is M-theory and that the needed guide symmetry is duality. As it is known, M-theory [18,20] was suggested by various duality symmetries in string, and p-brane theory. One of the interesting aspects is that in oriented matroid theory, duality is also of fundamental importance as ordinary matroid theory (see Ref. [6] Sec. 3.4). In fact, there is also a theorem that establishes that every oriented matroid has associated dual oriented matroid. This is of vital importance for our conjecture because if we write an action in terms of a given oriented matroid, we automatically assure an action for the dual oriented matroid, and as a consequence, the corresponding partition function must have a manifest dual symmetry, as seems to be required by M-theory.

By taking this observation as motivation in this article, we put special emphasis in the chirotope concept identifying it in various scenarios of physics. In Sec. 2, the concept of oriented matroid is introduced via the chirotope concept. In Sec. 3, the identification of the angular momentum with the chirotope concept, in both classical and quantum mechanics, is made. In Sec. 4, the connection between chirotopes and p-branes is briefly reviewed. In Sec. 5, we also review briefly the connection between Matrix theory and matroids. In Sec. 6, we make some comments about the importance of the chirotope concept in two time physics. Finally in Sec. 7, we make some final remarks explaining a possible connection between the chirotope concept with electromagnetism and Yang-Mills.

2. Oriented matroid theory for physicists: a brief introduction

The idea of this section is to give a brief introduction to the concept of oriented matroid. But instead of following step by step the traditional mathematical method presented in most teaching books (see [6] and Refs. there in) of the subject, we shall follow a different route based essentially in tensor analysis.

Let us start introducing the completely antisymmetric symbol

$$\varepsilon^{i_1 \ldots i_d}, \tag{1}$$

which is, more or less, a familiar object in physics. (Here the indices $i_1, \ldots, i_d$ run from 1 to d). This is a rank-d tensor which values are $+1$ or -1 depending of even or odd permutations of

$$\varepsilon^{12 \ldots d}, \tag{2}$$

respectively. Moreover, $\varepsilon^{i_1 \cdots i_d}$ takes the value 0 unless $i_1 \ldots i_d$ are all different. In a more abstract and compact form, we can say that

$$\varepsilon^{i_1 \cdots i_d} \in \{-1, 0, 1\}. \tag{3}$$

An important property of $\varepsilon^{i_1 \cdots i_d}$ is that it has exactly the same number of indices as the dimension d of the space.

Another crucial property of the symbol $\varepsilon^{i_1 \cdots i_d}$ is that the product $\varepsilon^{i_1 \cdots i_d} \varepsilon^{j_1 \cdots j_d}$ can be written in terms of a product of the Kronecker deltas $\delta^{ij} = diag(1, \ldots, 1)$. Specifically, we have

$$\varepsilon^{i_1 \cdots i_d} \varepsilon^{j_1 \cdots j_d} = \delta^{i_1 \cdots i_d, j_1 \cdots j_d}, \tag{4}$$

where $\delta^{i_1 \cdots i_d, j_1 \cdots j_d}$ is the so called delta generalized symbol;

$$\delta^{i_1 \cdots i_d j_1 \cdots j_d} = \begin{cases} +1 \text{ if } i_1 \ldots i_d \text{ is an even permutation of } j_1 \ldots j_d, \\ -1 \text{ if } i_1 \ldots i_d \text{ is an odd permutation of } j_1 \ldots j_d, \\ \qquad 0 \quad \text{otherwise.} \end{cases} \tag{5}$$

An example may help to understand the $\delta^{i_1 \cdots i_d, j_1 \cdots j_d}$ symbol. Assume that d is equal 2. Then we have $\varepsilon^{i_1 i_2}$ and

$$\varepsilon^{i_1 i_2} \varepsilon^{j_1 j_2} = \delta^{i_1 i_2, j_1 j_2} = \delta^{i_1, j_1} \delta^{i_2, j_2} - \delta^{i_1 j_2} \delta^{i_2, j_1}. \tag{6}$$

From Eq. (4), it follows the antisymmetrized square bracket property

$$\varepsilon^{i_1 \cdots [i_d} \varepsilon^{j_1 \cdots j_d]} \equiv 0. \tag{7}$$

We recall that for any tensor $V^{i_1 i_2 . i_3}$, the object $V^{[i_1 i_2 . i_3]}$ is defined by

$$V^{[i_1 i_2 . i_3]} = \tfrac{1}{3!}(V^{i_1 i_2 . i_3} + V^{i_2 i_3 . i_1} + V^{i_3 i_1 . i_2}$$
$$- V^{i_2 i_1 . i_3} - V^{i_1 i_3 . i_2} - V^{i_3 i_2 . i_1}),$$

with obvious generalization to any dimension. The result (7) comes from the fact that any complete antisymmetric tensor, with more than d indices must vanish. Indeed, it can be shown that any completely antisymmetric tensor $F^{i_1 \cdots i_r}$ with $r > d$ must vanish, while for $r = d$, $F^{i_1 \cdots i_n}$ must be proportional to $\varepsilon^{i_1 \cdots i_d}$. In other words, up to a factor, the symbol $\varepsilon^{i_1 \cdots i_d}$ is the largest completely antisymmetric tensor that one can have in d dimensions.

Now, we would like to relate the symbol $\varepsilon^{i_1 \cdots i_d}$ with the chirotope concept of oriented matroid theory. For this purpose, we ask ourselves whether it is possible to have the analogue of the symbol $\varepsilon^{i_1 \cdots i_d}$ for $r < d$. There is not any problem for having completely antisymmetric tensors $F^{i_1 \cdots i_r}$ for $r < d$, why then not to consider the analogue of $\varepsilon^{i_1 \cdots i_d}$ for $r < d$? Let us denote by $\sigma^{i_1 \cdots i_r}$, with $r < d$, this assumed analogue of $\varepsilon^{i_1 \cdots i_d}$. What properties should we require for the object $\sigma^{i_1 \cdots i_r}$? According to our above discussion, one may say that $\varepsilon^{i_1 \cdots i_d}$ is determined by the properties (3) and (7). Therefore, we require exactly similar properties for $\sigma^{i_1 \cdots i_r}$, namely $\sigma^{i_1 \cdots i_r}$ is completely antisymmetric under interchange of any pair of indices and satisfies the two conditions,

$$\sigma^{i_1\ldots i_r} \in \{-1, 0, 1\} \tag{8}$$

and

$$\sigma^{i_1\ldots[i_r}\sigma^{j_1\ldots j_r]} \equiv 0. \tag{9}$$

A solution for (9) is provided by

$$\Sigma^{i_1\ldots i_r} = \varepsilon^{a_1\ldots a_r} v^{i_1}_{a_1} \ldots v^{i_r}_{a_r}, \tag{10}$$

where v^i_a is any $r \times d$ matrix over some field F. Other way to write Eq. (10) is

$$\Sigma^{i_1\ldots i_r} = \det(\mathbf{v}^{i_1} \ldots \mathbf{v}^{i_r}). \tag{11}$$

One may prove that (10) implies (9) as follows. Assuming (10), we get

$$\begin{aligned}\Sigma^{i_1\ldots[i_r}\Sigma^{j_1\ldots j_r]} &= \varepsilon^{a_1\ldots a_r}\varepsilon^{b_1\ldots b_r} v^{i_1}_{a_1}\ldots v^{[i_r}_{a_r} v^{j_1}_{b_1}\ldots v^{j_r]}_{b_r} \\ &= \varepsilon^{a_1\ldots[a_r}\varepsilon^{b_1\ldots b_r]} v^{i_1}_{a_1}\ldots v^{i_r}_{a_r} v^{j_1}_{b_1}\ldots v^{j_r}_{b_r}. \end{aligned} \tag{12}$$

But from Eq. (7), we know that

$$\varepsilon^{a_1\ldots[a_r}\varepsilon^{b_1\ldots b_r]} = 0, \tag{13}$$

and therefore, we find

$$\Sigma^{i_1\ldots[i_r}\Sigma^{j_1\ldots j_r]} = 0, \tag{14}$$

as required.

Since $\det(\mathbf{v}^{i_1} \ldots \mathbf{v}^{i_r})$ can be positive, negative, or zero we may have a tensor $\sigma^{i_1\ldots i_r}$ satisfying both Eqs. (3) and (7) by setting

$$\sigma^{i_1\ldots i_r} = sign \Sigma^{i_1\ldots i_r}. \tag{15}$$

Observe that if $r = d$, and v^i_a is the identity, then $\sigma^{i_1\ldots i_d}=\varepsilon^{i_1\ldots i_d}$. Therefore the tensor $\sigma^{i_1\ldots i_r}$ is a more general object than $\varepsilon^{i_1\ldots i_d}$.

Let us now analyze our results from other perspective. First, instead of saying that the indices $i_1 \ldots i_d$ run from 1 to d, we shall say that the indices $i_1 \ldots i_d$ take values in the set $E = \{1, \ldots, d\}$.In other words we set

$$i_1 \ldots i_d \in \{1, \ldots, d\}. \tag{16}$$

Now, suppose that to each element of E we associate a $r-$dimensional vector $\mathbf{v}$. In other word, we assume the map

$$i \to \mathbf{v}(i) \equiv \mathbf{v}^i. \tag{17}$$

We shall write the vector $\mathbf{v}^i$ as v^i_a, with $a \in \{1, \ldots, r\}$. With this notation the map (17) becomes

$$i \to v^i_a. \tag{18}$$

Let us try to understand the expression (10) in terms of a family-set. First note that because the symbol $\varepsilon^{a_1\ldots a_r}$ makes sense only in $r-$dimensions the indices $i_1 \ldots i_r$ combination in $\Sigma^{i_1\ldots i_r}$ corresponds to $r-$elements subsets of $E = \{1, \ldots, d\}$. This enables to define the family $\mathcal{B}$ of all possible $r-$elements subsets of E.

An example may help to understand our observations. Consider the object

$$\Sigma^{ij}. \tag{19}$$

We establish that

$$i, j \in E = \{1, 2, 3\}. \tag{20}$$

Assume that

$$\Sigma^{ij} = -\Sigma^{ji}, \tag{21}$$

that is Σ^{ij} is an antisymmetric second rank tensor This means that the only nonvanishing components of Σ^{ij} are Σ^{12}, Σ^{13}, and Σ^{23}. From these nonvanishing components of Σ^{ij}, we may propose the family-set

$$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}. \tag{22}$$

Further, suppose we associate to each value of i a two dimensional vector $\mathbf{v}(i)$. This means that the set E can be written as

$$E = \{\mathbf{v}(1), \mathbf{v}(2), \mathbf{v}(3)\}. \tag{23}$$

This process can be summarizing by means of the transformation

$$i \to v^i_a, \tag{24}$$

with $a \in \{1, 2\}$. We can connect v^i_a with an explicit form of Σ^{ij} if we write

$$\Sigma^{ij} = \varepsilon^{ab} v^i_a v^j_b. \tag{25}$$

The previous considerations proof the possible existence of an object such as $\sigma^{i_1\ldots i_r}$. In the process of proposing the object $\sigma^{i_1\ldots i_r}$, we have introduced the set E and the $r-$element subsets $\mathcal{B}$. It turns out that the pair $(E, \mathcal{B})$ plays an essential role in the definition of a matroid. But before we formally define a matroid, we would like to make one further observation. For this purpose we first notice that Eq. (9) implies

$$\sigma^{i_1\ldots i_r}\sigma^{j_1\ldots j_r} = \sum_{a=1}^{r} \sigma^{j_a i_2\ldots i_r}\sigma^{j_1 \ldots j_{a-1} i_1 j_{a+1}\ldots j_r}. \tag{26}$$

Therefore, if $\sigma^{i_1\ldots i_r}\sigma^{j_1\ldots j_r} \neq 0$, the expression (26) means that there exist an $a \in \{1, 2, \ldots, r\}$ such that

$$\sigma^{i_1\ldots i_r}\sigma^{j_1\ldots j_r} = \sigma^{j_a i_2\ldots i_r}\sigma^{j_1 \ldots j_{a-1} i_1 j_{a+1}\ldots j_r}. \tag{27}$$

This proves that Eq. (9) implies Eq. (27) but the converse is not true. Therefore, the expression (27) defines an object that it is more general than one determined by (9). Let us denote this more general object by $\chi^{i_1\cdots i_r}$. We are ready to formally define an oriented matroid (see Ref. 6, Sec. 3.5).

Let $r \geq 1$ be an integer, and let E be a finite set (ground set). An oriented matroid $\mathcal{M}$ of rank r is the pair (E,χ) where χ is a mapping (called chirotope) $\chi : E \to \{-1,0,1\}$ which satisfies the following three properties:

1) χ is not identically zero

2) χ is completely antisymmetric.

3) for all $i_1,\ldots,i_r,j_1,\ldots,j_r \in E$ such that

$$\chi^{i_1\cdots i_r}\chi^{j_1\cdots j_r} \neq 0. \tag{28}$$

There exists an a such that

$$\chi^{i_1\cdots i_r}\chi^{j_1\cdots j_r} = \chi^{j_a i_2\cdots i_r}\chi^{j_1\cdot\cdot j_{a-1}\cdot i_1 j_{a+1}\cdots j_r}. \tag{29}$$

Let $\mathcal{B}$ be the set of $r-$elements subsets of E such that

$$\chi^{i_1\cdots i_r} \neq 0,$$

for $i_1,\ldots,i_r \in E$. Then (29) implies that if $i_a \in B$, there exist $j_a \in B' \in \mathcal{B}$ such that $(B-i_a)\cup j_a \in \mathcal{B}$. This important property of the elements of $\mathcal{B}$ defines an ordinary matroid on E (see Ref. 2, Sec. 1.2).

Formally, a matroid M is a pair $(E,\mathcal{B})$, where E is a non-empty finite set, and $\mathcal{B}$ is a non-empty collection of subsets of E (called bases) satisfying the following properties:

$(\mathcal{B}\ i)$ no basis properly contains another basis;

$(\mathcal{B}\ ii)$ if B_1 and B_2 are bases and if b is any element of B_1, then there is an element g of B_2 with the property that $(B_1 - \{b\}) \cup \{g\}$ is also a basis.

M is called the underlying matroid of $\mathcal{M}$. According to our considerations every oriented matroid $\mathcal{M}$ has an associated underlying matroid M. However the converse is not true, that is, not every ordinary matroid M has an associated oriented matroid $\mathcal{M}$. In a sense, this can be understood observing that Eq. (29) not necessarily implies condition (9). In other words, the condition (29) is less restrictive than (9). It is said that an ordinary matroid M is orientable if there is an oriented matroid $\mathcal{M}$ with an underlying matroid M. There are many examples of non-oriented matroids, perhaps one of the most interesting is the so called Fano matroid F_7 (see Ref. 6, Sec. 6.6). This is a matroid defined on the ground set

$$E = \{1,2,3,4,5,6,7\},$$

whose bases are all those subsets of E with three elements except f_1=$\{1,2,3\}$, f_2=$\{5,1,6\}$, f_3=$\{6,4,2\}$, f_4=$\{4,3,5\}$, $f_5 = \{4,7,1\}$, $f_6 = \{6,7,3\}$, and $f_7 = \{5,7,2\}$. This matroid is realizable over a binary field and is the only minimal irregular matroid. Moreover, it has been shown in Refs. 13 to 16 that F_7 is connected with octionions and therefore with supergravity. However, it appears intriguing that in spite these interesting properties of F_7 this matroid is not orientable.

It can be shown that all bases have the same number of elements. The number of elements of a basis is called rank, and we shall denote it by r. Thus, the rank of an oriented matroid is the rank of its underlying matroid.

One of the simplest, but perhaps one of the most important ordinary matroids is the so called uniform matroid denoted as $U_{r,d}$, and defined by the pair $(E,\mathcal{B})$, where $E = \{1,\ldots,d\}$, and $\mathcal{B}$ is the collection of $r-$element subsets of E, with $r \leq d$.

With these definitions at hand we can now return to the object $\varepsilon^{i_1\cdots i_d}$ and reanalyze it in terms of the oriented matroid concept. The tensor $\varepsilon^{i_1\cdots i_d}$ has an associated set $E = \{1,2,\ldots,d\}$. It is not difficult to see that in this case $\mathcal{B}$ is given by $\{\{1,2,\ldots,d\}\}$. This means that the only basis in $\mathcal{B}$ is E itself. Further since $\varepsilon^{i_1\cdots i_d}$ satisfies the property (7) must also satisfy the condition (29), and therefore we have discovered that $\varepsilon^{i_1\cdots i_d}$ is a chirotope, with underlying matroid $U_{d,d}$. Thus, our original question whether is it possible to have the analogue of the symbol $\varepsilon^{i_1\cdots i_d}$ for $r < d$ is equivalent to ask wether there exist chirotopes for $r < d$, and oriented matroid theory give us an affirmative answer. An object $\chi^{i_1\cdots i_r}$ satisfying the definition of oriented matroid is a chirotope that, in fact, generalizes the symbol $\varepsilon^{i_1\cdots i_d}$.

A realization of $\mathcal{M}$ is a mapping $\mathbf{v} : E \to R^r$ such that

$$\chi^{i_1\cdots i_r} \to \sigma^{i_1\cdots i_r} = sign\Sigma^{i_1\cdots i_r}, \tag{30}$$

for all $i_1,\ldots,i_r \in E$. Here, $\Sigma^{i_1\cdots i_r}$ is given in Eq. (10). By convenience we shall call the symbol $\Sigma^{i_1\cdots i_r}$ *prechirotope*.

Realizability is a very important subject in oriented matroid theory and deserves to be discussed in some detail. However, in this paper we are more interested in a rough introduction to the subject, and for this reason we refer to the interested redear to Chap. 8 of Ref. 6, where a whole discussion of the subject is given. Nevertheless, we need to make some important remarks. First of all, it turns out that not all oriented matroids are realizable. In fact, it has been shown that the smallest non-ralizable uniform oriented matroids have the (r,d)-parameters $(3,9)$ and $(4,8)$. It is worth mentioning that given a uniform matroid $U_{r,d}$ the orientability is not unique. For instance, there are precisely 2628 (reorientations classes of) uniform $r = 4$ oriented matroids with $d = 8$. Further, precisely 24 of these oriented matroids are non-realizables.

A rank preserving weak map concept is another important notion in oriented matroid theory. This is a map between two oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set E and $r_1 = r_2$ with the property that every basis of $\mathcal{M}_2$ is a basis of $\mathcal{M}_1$. There is an important theorem that establishes that every oriented matroid is the weak map image of a uniform oriented matroid of the same rank.

Finally, we should mention that there is a close connection between Grassmann algebra and chirotopes. To understand this connection let us denote by $\wedge_r R^n$ the

$\binom{n}{r}$-dimensional real vector space of alternating r-forms on R^n. An element $\mathbf{\Sigma}$ in $\wedge_r R^n$ is said to be decomposable if

$$\mathbf{\Sigma} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge .\mathbf{v}_r, \tag{31}$$

for some $\mathbf{v}_1, \mathbf{v}_2, \ldots, .\mathbf{v}_r \in R^n$. It is not difficult to see that Eq. (31) can be written as

$$\mathbf{\Sigma} = \frac{1}{r!}\Sigma^{i_1 \cdots i_r} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}, \tag{32}$$

where $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$ are one form bases in R^n and $\Sigma^{i_1 \cdots i_r}$ given in Eq. (10). This shows that the prechirotope $\Sigma^{i_1 \cdots i_r}$ can be identified with alternating decomposable r-forms. It is known that the projective variety of decomposable forms is isomorphic to the Grassmann variety of r-dimensional linear subspaces in R^n. In turn, the Grassmann variety is the classifying space for vector bundle structures. Perhaps, related observations motivate MacPherson [7] to develop the combinatorial differential manifold concept which was the predecessor of the matroid bundle concept [7-11]. This is a differentiable manifold in which at each point, an oriented matroidis attached as a fiber.

It is appropriate to briefly comment about the origins of chirotope concept. It seems that the concept of chirotope appears for the first time in 1965 in a paper by Novoa [21] under the name of “n-ordered sets and order completeness”. The term chirotope was used by Dress [22] in connection with certain chirality structure in organic chemistry. Bokowski and Shemer [23] apply the chirotope concept in relation with the Steinitz problem. Finally, Las Vergnas [24] used the chirotope concept to construct an alternative definition of oriented matroid.

Now, the symbol $\varepsilon^{i_1 \cdots i_d}$ is very much used in different contexts of physics, including supergravity and p-branes. Therefore the question arises whether the chirotope symbol $\chi^{i_1 \cdots i_r}$ may have similar importance in different scenarios of physics. In the next sections we shall make the observation that the symbol $\Sigma^{i_1 \cdots i_r}$ is already used in different scenarios of physics, but apparently it has not been recognized as a chirotope.

3. Chirotopes in classical and quantum mechanics

It is well known that the angular momentum $\bar{L}$ in a 3-dimensional space is one of the most basic concepts in classical mechanics. Traditionally $\bar{L}$ is defined by

$$\bar{L} = \bar{r} \times \bar{p}. \tag{33}$$

In tensor notation, this expression can be written as

$$L^i = \varepsilon^{ijk} x_j p_k. \tag{34}$$

We observe the presence of the symbol ε^{ijk} which is a chirotope. In fact, this $\varepsilon-$symbol appears in any cross product $\bar{A} \times \bar{B}$ for any two vectors $\bar{A}$ and $\bar{B}$, in 3 dimensions. We still have a deeper connection between $\bar{L}$ and matroids. First, we observe that the formula (34) can also be written as

$$L^i = \frac{1}{2}\varepsilon^{ijk} L_{jk}, \tag{35}$$

where

$$L^{ij} = x^i p^j - x^j p^i. \tag{36}$$

Of course, L^i and L^{ij} have the same information.

Let us redefine x^i and p^j in the form

$$v_1^i \equiv x^i, \quad v_2^i \equiv p^i. \tag{37}$$

Using this notation the expression (36) becomes

$$L^{ij} = \varepsilon^{ab} v_a^i v_b^j, \tag{38}$$

where the indices a and b take values in the set $\{1, 2\}$. If we compare the expression (38) with (10), we recognize in (38) the form of a rank-2 prechirotope. This means that the angular momentum itself is a prechirotope. For a possible generalization to any dimension, the form (38) of the angular momentum appears more appropriate than the form (35). Thus, our conclusion that the angular momentum is a prechirotope applies to any dimension, not just 3-dimensions.

The classical Poisson brackets associated to L^{ij} is

$$\{L^{ij}, L^{kl}\} = \delta^{ik} L^{jl} - \delta^{il} L^{jk} + \delta^{jl}\delta L^{ik} - \delta^{jk}\delta L^{il}. \tag{39}$$

One of the traditional mechanisms for going from classical mechanics to quantum mechanics is described by the prescription

$$\{A, B\} \to \frac{1}{i}[\hat{A}, \hat{B}], \tag{40}$$

for any two canonical variables A and B. Therefore, at the quantum level the expression (39) becomes

$$[\hat{L}^{ij}, \hat{L}^{kl}] = i(\delta^{ik}\hat{L}^{jl} - \delta^{il}\hat{L}^{jk} + \delta^{jl}\hat{L}^{ik} - \delta^{jk}\hat{L}^{il}). \tag{41}$$

It is well known the importance of this expression in both the eigenvalues determination and the group analyses of a quantum system. Therefore, the prechirotope property of L^{ij} goes over at the quantum level.

4. Chirotopes and p-branes

Consider the action

$$S = \frac{1}{2}\int d^{p+1}\xi(\gamma^{-1}\gamma^{\mu_1 \ldots \mu_{p+1}}\gamma_{\mu_1 \ldots \mu_{p+1}} - \gamma T_p^2), \tag{42}$$

where

$$\gamma^{\mu_1 \cdots \mu_{p+1}} = \varepsilon^{a_1 \ldots a_{p+1}} V_{a_1}^{\mu_1}(\xi) \ldots V_{a_{p+1}}^{\mu_{p+1}}(\xi), \tag{43}$$

with

$$V_a^\mu(\xi) = \partial_a x^\mu(\xi). \tag{44}$$

Here γ is a lagrange multiplier and T_p is a constant measuring the inertial of the system. It turns out that the action (42) is equivalent to the Nambu-Goto type action for p-branes (see Ref. 12 and Refs there in). One of the important aspects of (42) is that it makes sense to set $T_p = 0$. In such case, (42) is reduced to the Schild type null p-brane action [26-27].

From the expression (43) we observe that, except for its locality, $\gamma^{\mu_1 \cdots \mu_{p+1}}$ has the same form as a prechirotope. The local property of $\gamma^{\mu_1 \cdots \mu_{p+1}}$ can be achieved by means of the matroid bundle concept. The key idea in matroid bundle is to replace tangent spaces in a differential manifold by oriented matroids. This is achieved by considering the linear map $f_\xi :\| star\Delta \| \rightarrow U \subset T_{\eta(\xi)}$ such that $f_\xi(\xi) = 0$, where $\| \Delta \|$ is the minimal simplex of $\| X \|$ containing $\xi \in X$, where X is a simplicial complex associated to a differential manifold. Then, $f_\xi \, \| (star\Delta)^0 \|$, where $(star\Delta)^0$ are the 0-simplices of $star\Delta$, is a configuration of vectors in $T_{\eta(\xi)}$ defining an oriented matroid $\mathcal{M}(\xi)$. One should expect that the function f_ξ induces a map

$$\Sigma^{\mu_1 \cdots \mu_r} \rightarrow \gamma^{\mu_1 \cdots \mu_{p+1}}(\xi), \tag{45}$$

where we consider that the rank r of $\mathcal{M}(\xi)$ is $r = p + 1$. Observe that the formula (45) means that the function f_ξ also induces the map $v_a^\mu \rightarrow V_a^\mu(\xi)$.

Our last task is to establish the expression (44). Consider the expression

$$F_{ab}^\mu = \partial_a V_b^\mu(\xi) - \partial_b V_a^\nu(\xi). \tag{46}$$

Thus, if the equation $F_{ab}^\mu = 0$ is implemented in (42) as a constraint then we get the solution $V_a^\mu(\xi) = \partial x^\mu / \partial \xi^a$, where x^μ is, in this context, a gauge function. In this case, one says that $v_a^\mu(\xi)$ is a pure gauge. Of course, F_{ab}^μ and $V_b^\mu(\xi)$ can be interpreted as field strength and abelian gauge potential, respectively.

5. Chirotopes and Matrix theory

Some years ago Yoneya [28] showed that it is possible to construct a Matrix theory of the Schild type action for strings. The key idea in the Yoneya's work is to consider the Poisson bracket structure

$$\{x^\mu, x^\nu\} = \frac{1}{\xi}\gamma^{\mu\nu}, \tag{47}$$

where ξ is an auxiliary field. This identification suggests to replace the Poisson structure by coordinate operators

$$\{x^\mu, x^\nu\} \rightarrow \frac{1}{i}[\hat{x}^\mu, \hat{x}^\nu]. \tag{48}$$

The next step is to quantize the constraint

$$-\frac{1}{\xi^2}\gamma^{\mu\nu}\gamma_{\mu\nu} = T_p^2, \tag{49}$$

which can be derived from the expression (42) by setting $p = 1$. According to the expressions (47), (48) and (49) one gets

$$([\hat{x}^\mu, \hat{x}^\nu])^2 = T_p^2 I, \tag{50}$$

where I is the identity operator. It turns out that the constraint (50) plays an essential role in Matrix theory. Extending the Yoneya's idea for strings, Oda [29] (see also Refs. 30 and 31) has shown that it is also possible to construct a Matrix model of M-theory from a Schild-type action for membranes. It is clear from our previous analysis of identifying the quantity $\gamma^{\mu\nu}$ with a prechirotope of a given chirotope $\chi^{\mu\nu}$, that these developments of Matrix theory can be linked with the oriented matroid theory.

6. Chirotopes and two time physics

Consider the first order lagrangian [17]

$$L = \frac{1}{2}\varepsilon^{ab}\dot{v}_a^\mu v_b^\nu \eta_{\mu\nu} - H(v_a^\mu), \tag{51}$$

where $\eta_{\mu\nu}$ is a flat metric whose signature will be determined below. Up to total derivative, this lagrangian is equivalent to the first order lagrangian

$$L = \dot{x}^\mu p_\mu - H(x, p), \tag{52}$$

where

$$x^\mu = v_1^\mu, \quad p^\mu = v_2^\mu. \tag{53}$$

Typically, one chooses H as $H = \lambda(p^\mu p_\mu + m^2)$. For the massless case we have

$$H = \lambda(p^\mu p_\mu). \tag{54}$$

From the point of view of the lagrangian (51) in terms of the coordinates v_a^μ, this choice is not good enough since the $SL(2, R)-$symmetry in the first term of expression (51) is lost. It turns out that the simplest possible choice for H which maintains the symmetry $SL(2, R)$ is

$$H = \frac{1}{2}\lambda^{ab} v_a^\mu v_b^\nu \eta_{\mu\nu}, \tag{55}$$

where λ^{ab} is a Lagrange multipliers. Arbitrary variations of λ^{ab} lead to the constraint $v_a^\mu v_b^\nu \eta_{\mu\nu} = 0$ which means that

$$p^\mu p_\mu = 0, \tag{56}$$

$$p^\mu x_\mu = 0 \tag{57}$$

and

$$x^\mu x_\mu = 0. \tag{58}$$

The key point in two time physics comes from the observation that if $\eta_{\mu\nu}$ corresponds to just one time, that is, if $\eta_{\mu\nu}$ has the signature $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$, then from (56)-(58)

it follows that p^μ is parallel to x^μ, and therefore the angular momentum

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{59}$$

associated with the Lorentz symmetry of (55) should vanish, which is an unlikely result. Thus, if we impose the condition $L^{\mu\nu} \neq 0$ and the constraints (56)-(58) we find that the signature of $\eta_{\mu\nu}$ should be, at least of the form $\eta_{\mu\nu} = diag(-1,-1,1,\ldots,1)$. In other words, only with two times the constraints (56)-(58) are consistent with the requirement $L^{\mu\nu} \neq 0$. In principle, we can assume that the number of times is grater than 2, but then one does not have enough constraints to eliminate all the possible ghosts.

As in Sec. 3, we can rewrite (59) in form

$$L^{\mu\nu} = \frac{1}{2}\varepsilon^{ab} v_a^\mu v_b^\nu, \tag{60}$$

which means that $L^{\mu\nu}$ is a prechirotope. Thus, one of the conditions for maintaining both the symmetry $SL(2,R)$ and the Lorentz symmetry in the lagrangian (51) is that the prechirotope $L^{\mu\nu}$ must be different from zero, in agreement with one of the conditions of the definition of oriented matroids in terms of chirotopes. Therefore, if our starting point in the formulation of lagrangian (51) is the oriented matroid theory then the two time physics arises in a natural way.

7. Final remarks

Besides the connection between matroid theory and Chern-Simons formalism , supergravity, string theory, p-branes and Matrix theory found previously, in this work we have added new links of matroids with different scenarios of physics such as classical and quantum mechanics and two time physics. All these physical scenarios are so diverse that one wonders why the matroid subject has passed unnoticed. This has been due, perhaps, to the fact that oriented matroid theory has evolved putting much emphasis in the equivalence of various possible axiomatizations. Just to mention some possible definitions of an oriented matroid besides a definition in terms of chirotopes, there are equivalent definitions in terms of circuits, vectors and covectors among others (see Ref. 6 for details). As a result, it turns out that most of the material in matroid theory is dedicated to existence theorems. Part of our effort in the present work has been to start this subject with just one definition, and instead of jumping from one definition to another we have tried to put the oriented matroid concept, and in particular the chirotope concept, in such a way that physicists can make some further computations with such concepts. In a sense, our view is that the chirotope notion may be the main tool for translating concepts from oriented matroid theory to a physical setting and vice versa.

It is interesting to mention that even electromagnetism seems to admit a chirotope construction. In fact, let us write the electromagnetic gauge potential as [32]

$$A_\mu = \varepsilon^{ab} e_a^i \partial_\mu e_{bi}. \tag{61}$$

where e_a^i are two bases vectors in a tangent space of a given manifold. It turns out that the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ becomes

$$F_{\mu\nu} = \varepsilon^{ab} \partial_\mu e_a^i \partial_\nu e_{bi}, \tag{62}$$

We recognize in (62) the typical form of a prechirotope (10). The idea can be generalized to Yang-Mills [32] and gravity using MacDowell-Mansouri formalism.

As we mentioned, an interesting aspect of the oriented matroid theory is that the concept of duality may be implemented at the quantum level. For instance, an important theorem in oriented matroid theory assures that

$$(\mathcal{M}_1 \oplus \mathcal{M}_2)^* = \mathcal{M}_1^* \oplus \mathcal{M}_2^*, \tag{63}$$

where $\mathcal{M}^*$ denotes the dual matroid and $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the direct sum of two oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. If we associate the symbolic actions S_1 and S_2 to the two the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively; then the corresponding partition functions $Z_1(\mathcal{M}_1)$ and $Z_2(\mathcal{M}_2)$ should lead to the symmetry $Z = Z^*$ of the total partition function $Z = Z_1 Z_2$.

Another interesting aspect of duality in oriented matroid theory is that it may allow an extension in of the Hodge duality. From the observation that the completely antisymmetric object $\varepsilon_{\mu_1\cdots\mu_d}$ is in fact a chirotope associated to the underlaying uniform matroid $U_{n,n}$, corresponding to the ground set $E = \{1,2,\ldots,n\}$ and bases subset $\mathcal{B} = \{\{1,2,\ldots,n\}\}$, it is natural to ask why not to use other chirotopes to extend the Hodge duality concept? In Ref. 12 it was suggested the idea of the object

$$^{\ddagger}\Sigma^{\mu_{p+2}\cdots\mu_r} = \frac{1}{d!}\chi^{\mu_{p+2}\cdots\mu_r}_{\mu_1\cdots\mu_{p+1}}\Sigma^{\mu_1\cdots\mu_{p+1}}, \tag{64}$$

where $\Sigma^{\mu_1\cdots\mu_{p+1}}$ is any completely antisymmetric tensor and

$$\chi_{\mu_1\cdots\mu_{p+1}\mu_{p+2}\cdots\mu_r} \equiv \chi(\mu_1,..,\mu_{p+1},\mu_{p+2},\ldots,\mu_r)$$

is a chirotope associated to some oriented matroid of rank $r \geq p+1$. In Ref. 12 the concept $^{\ddagger}\mathbf{\Sigma}$ was called dualoid for distinguishing it from the usual Hodge dual concept

$$^{*}\mathbf{\Sigma}^{\mu_{p+2}\cdots\mu_r} = \frac{1}{(p+1)!}\varepsilon^{\mu_{p+2}\cdots\mu_r}_{\mu_1\cdots\mu_{p+1}}\Sigma^{\mu_1\cdots\mu_{p+1}} \tag{65}$$

which is a particular case of (64), when $r = d+1$. It turns out that the dualiod may be of some interest in p-branes theory (see Ref. 12 for details).

Recently, it was proposed that every physical quantity is a polyvector (see Ref. 33 and references there in). The polyvectors are completely antisymmetric objects in a Clifford aggregate. It may be interesting for further research to investigate whether there is any connection between the polyvector concept and the chirotope concept.

Finally, as it was mentioned the Fano matroid is not orientable. But this matroid seems to be connected with octonions and therefore with $D = 11$ supergravity. Perhaps this suggests to look for a new type of orientability. Moreover, there are matroids, such as non-Pappus matroid, which are either realizable and orientable. The natural question is what kind of physical concepts are associated to these type of matroids. It is tempting to speculate that there must be physical concepts of pure combinatorial character in the sense of matroid theory. On the other hand, it has been proved that matroid bundles have well-defined Stiefel-Whitney classes [8], and other characteristic classes [11]. In turn, Stiefel-Whitney classes are closely related to spinning structures. Thus, there must be a matroid/supersymmetry connection and consequently matroid/M-theory connection.

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1. H. Whitney, *Am. J. Math.* **57** (1935) 509.
2. J.G. Oxley, *Martroid Theory*, (Oxford University Press, New York, 1992).
3. G. Nicoletti and N. White, *Theory of Matroids* (Cambridge University Press, Cambridge, 1993) Chap. 2, ed. by N. White.
4. T. Zaslavsky, *J. Comb. Theor. B* **47** (1989) 32; *J. Com Theor. B* 51 (1991) 46.
5. A. Bjorner and G. Ziegler, *Matroid Applications* (Cambridge University Press, Cambridge, 1993) Chap. 8, ed. by N. White.
6. A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler, *Oriented Martroids* (Cambridge University Press, Cambridge, 1993).
7. R.D. MacPherson, "Combinatorial differential manifolds: a symposium in honor of John Milnor's sixtieth birthday," pp. 203-221 in Topological methods on modern mathematics (Stony Brook, NY, 1991), edited by L.H. Goldberg and A. Phillips, Houston, 1993.
8. L. Anderson and J.F. Davis, "Mod 2 Cohomolgy of Combinatorial Grassmannians," math.GT/9911158.
9. L. Anderson, New Perspectives. in *Geom. Comb.* **38** (1999) 1.
10. I.M. Gelfand and R.D. MacPherson, *Bull. Amer. Math. Soc.* **26** (1992) 304.
11. D. Biss, "Some applications of oriented matroids to topology," PhD. thesis, MIT, 2002.
12. J.A. Nieto, *Adv. Theor. Math. Phys.* **8** (2004) 177; hep-th/0310071.
13. J.A. Nieto and M.C. Marín, *J. Math. Phys.* **41** (2000) 7997.
14. J.A. Nieto, *J. Math. Phys.* **45** (2004) 285; hep-th/0212100.
15. J.A. Nieto and M.C. Marín, *Int. J. Mod. Phys. A* **18** (2003) 5261; hep-th/0302193.
16. J.A. Nieto, *Rev. Mex. Fis.* **44** (1998) 358.
17. I. Bars Class. Quant. Grav. **18** (2001) 3113; hep-th/0008164.
18. P.K. Townsend, "Four lectures on M-theory," *Proceedings of the ICTP on the Summer School on High Energy Physics and Cosmology*, June 1996, hep-th/9612121.
19. M.J. Duff, *Int. J. Mod. Phys.A* **11** (1996) 5623, hep-th/9608117.
20. P. Horava and E. Witten, *Nucl. Phys. B* **460** (1996) 506.
21. L.G. Novoa, *Pac. J. Math.* **15** (1965) 1337.
22. A. Dress, *Bayreuther Math. Schr.* **21** (1986) 14.
23. J. Bokowski and I Shemer, *Israel J. Math* **58** (1987) 103.
24. M. Las Vergnas, *J. Combinatorial Theory, Ser. B* **25** (1978) 283.
25. M.J. Duff and J.X. Lu, *Class. Quant. Grav.* **9** (1992) 1.
26. A. Schild, *Phys. Rev. D* **16** (1977) 1722.
27. J. Gamboa, C. Ramírez, and M. Ruiz-Altaba, *Nucl. Phys. B* **338** (1990) 143.
28. T. Yoneya, *Prog. Theor. Phys.* **97** (1997) 949; hep-th/9703078.
29. I. Oda, "Matrix theory from Schild action"; hep-th/9801085.
30. N. Kitsunezaki and S. Uehara, *JHEP* **0110** (2001) 033; hep-th/0108181.
31. R. Kuriki, S. Ogushi, and A. Sugamoto, *Mod. Phys. Lett. A* **14** (1999) 1123; hep-th/9811029.
32. J.P. Ralston, *The Bohr Atom Glueballs*, hep-th/0301089.
33. C. Castro, *Found. of Phys.* **34** (2004) 107.