On the vector solutions of Maxwell equations in spherical coordinate systems

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Recibido el 8 de noviembre de 2004; aceptado el 26 de enero de 2005

The Maxwell equations for the spherical components of the electromagnetic fields outside sources do not separate into equations for each component alone. We show, however, that general solutions can be obtained by separation of variables in the case of azimuthal symmetry. Boundary conditions are easier to apply to these solutions, and their forms highlight the similarities and differences between the electric and magnetic cases in both time-independent and time-dependent situations. Instructive examples of direct calculation of electric and magnetic fields from localized charge and current distributions are presented.

Keywords: Maxwell equations; spherical coordinates; electric and magnetic fields; boundary-value problems.

Las ecuaciones de Maxwell para las componentes esféricas de campos electromagnéticos en regiones libres de fuentes no son separables en ecuaciones para cada una de sus componentes. Se muestra, sin embargo, que soluciones generales pueden ser obtenidas por separación de variables en el caso de la simetría azimutal. Las condiciones de borde son fáciles de aplicar para estas soluciones, y sus formas destacan las similitudes y diferencias entre los casos eléctrico y magnético, tanto para las situaciones independientes del tiempo como para las de tiempo dependientes. Se presentan ejemplos instructivos de cálculos directos de campos eléctricos y magnéticos producidos por distribuciones localizadas de cargas y corrientes.

Descriptores: Ecuaciones de Maxwell; coordenadas esféricas; campos eléctrico y magnético; problemas con condiciones de borde.

PACS: 03.50.De; 41.20.Cv; 41.20.Gz

1. Introduction

The Maxwell equations for the electromagnetic field vectors, expressed in the International System of Units (SI), are [1]

$$\begin{aligned} \nabla\cdot\mathbf{D} &= \rho, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \\ \nabla\cdot\mathbf{B} &= 0, \qquad \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}, \end{aligned} \tag{1}$$

where the source terms ρ and $\mathbf{J}$ describe the densities of electric charge and current, respectively. For a linear, isotropic medium $\mathbf{D}$ and $\mathbf{H}$ are connected with the basic fields $\mathbf{E}$ and $\mathbf{B}$ by the constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu, \tag{2}$$

where ϵ and μ are the permittivity and permeability of the medium, respectively.

The boundary conditions for fields at a boundary surface between two different media are [2]

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{D}_1-\mathbf{D}_2) &= \rho_S, \qquad \mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2) = \mathbf{0}, \\ \mathbf{n}\cdot(\mathbf{B}_1-\mathbf{B}_2) &= 0, \qquad \mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2) = \mathbf{J}_S, \end{aligned} \tag{3}$$

where ρ_S and $\mathbf{J}_S$ denote the surface charge and current densities, respectively, and the normal unit vector $\mathbf{n}$ is drawn from the second into the first region. The interior and exterior fields satisfy the homogeneous vector wave equations

$$\begin{aligned} \nabla^2\mathbf{E} - \epsilon\mu\,\frac{\partial^2\mathbf{E}}{\partial t^2} &= \mathbf{0}, \\ \nabla^2\mathbf{B} - \epsilon\mu\,\frac{\partial^2\mathbf{B}}{\partial t^2} &= \mathbf{0}, \end{aligned} \tag{4}$$

which are obtained from Eqs. (1) and (2) for regions free of charge and current by combining the two curl equations and making use of the divergence equations together with the vector identity

$$\nabla^2(\ ) = \nabla(\nabla\cdot\ ) - \nabla\times(\nabla\times\ ). \tag{5}$$

Changes in the electromagnetic fields propagate with speed $v = 1/\sqrt{\epsilon\mu}$.

Without any loss of generality, we may consider only harmonic time dependence for sources and fields:

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho(\mathbf{r})e^{-i\omega t}, \quad \mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}, \\ \mathbf{E}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r})e^{-i\omega t}, \quad \mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r})e^{-i\omega t}, \end{aligned} \tag{6}$$

where the real part of each expression is implied. Equation (4) then becomes time-independent:

$$\nabla^2\mathbf{E} + k^2\,\mathbf{E} = 0, \quad \nabla^2\mathbf{B} + k^2\,\mathbf{B} = 0, \tag{7}$$

where $k^2 = \epsilon\mu\omega^2$. These are vector Helmholtz equations for transverse fields having zero divergence. Their solutions subject to arbitrary boundary conditions are considered more complicated than those of the corresponding scalar equations, since only in Cartesian coordinates the Laplacian of a vector field is the vector sum of the Laplacian of its separated components. For spherical coordinates, as for any other curvilinear coordinate system, we are faced with a highly complicated set of three simultaneous equations, each equation involving all three components of the vector field. This complication is well known and general techniques for solving these equations have been developed, based on a dyadic Green's function which transforms the boundary conditions and source densities into the vector solution [3]. We shall show, however, that in the case of spherical boundary conditions with azimuthal symmetry, the solution can be obtained more conveniently by means of separation of variables. Several applications of physical interest can then be treated in this simplified way, so avoiding the dyadic method [4].

Actually, the usual technique for solving boundary-value problems introduces the electromagnetic potentials as intermediary field quantities. These are defined by [5]

$$\mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{8}$$

with the subsidiary Lorentz condition

$$\nabla \cdot \mathbf{A} + \epsilon\mu \, \frac{\partial \phi}{\partial t} = 0. \tag{9}$$

It is then found that these potentials satisfy the inhomogeneous wave equations

$$\begin{aligned} \nabla^2\phi - \epsilon\mu \, \frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\epsilon}, \\ \nabla^2\mathbf{A} - \epsilon\mu \, \frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu\mathbf{J}, \end{aligned} \tag{10}$$

which together with the Lorentz condition form a set of equations equivalent to the Maxwell equations. The boundary conditions for the potentials may be deduced from Eq. (3).

For fields that vary with an angular frequency ω, *i.e.*

$$\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t}, \quad \mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{-i\omega t}, \tag{11}$$

we get equations that do not depend on time in regions free of charge and current:

$$\begin{aligned} \nabla^2\phi + k^2 \, \phi &= 0, \\ \nabla^2\mathbf{A} + k^2 \, \mathbf{A} &= 0, \end{aligned} \tag{12}$$

which are like those in Eq. (7) for the electric and magnetic induction fields, so that in general we also confront, for the vector potential, the mathematical complexities mentioned above for the electromagnetic fields.

The purpose of this paper is to get general solutions of the electromagnetic vector equations in spherical coordinates with azimuthal symmetry using separation of variables in spite of having equations that mix field components. Boundary conditions are easier to apply to these solutions, and their forms highlight the similarities and differences between the electric and magnetic cases in both time-independent and time-dependent situations. The approach shows that boundary-value problems can be solved for the electric and magnetic vector fields directly, and that the process involves the same kind of mathematics as the usual approach of solving for potentials. This material in this work may be used in a beginning graduate course in classical electromagnetism or mathematical methods for physicists. It is organized as follows. In Sec. 2, we describe the method for the static case showing how the mathematical complications of solving the vector field equations are easily overcome by means of separation of variables. In Sec. 3, we extend the method to discuss the case of time-varying fields. Concluding remarks are given in Sec. 4.

2. Static fields

For steady-state electric and magnetic phenomena, the fields outside sources satisfy the vector Laplace equations

$$\nabla^2\mathbf{E} = \mathbf{0}, \quad \nabla^2\mathbf{B} = \mathbf{0}, \tag{13}$$

where only transverse components with zero divergence are involved. Supposing all the charge and current are on the bounding surfaces, solutions in different regions can be connected through the boundary conditions indicated in Eq. (3). To demonstrate the features of the treatment, we first consider boundary-value problems with azimuthal symmetry in electrostatics. The solution of stationary current problems in magnetostatics is mathematically identical.

Combining the expressions for $\nabla\times(\nabla\times\mathbf{E}) = \mathbf{0}$ and $\nabla \cdot \mathbf{E} = 0$ in spherical coordinates and assuming no φ-dependence, we find using Eq. (5) that the components of the electric field E_r and E_θ satisfy the equations

$$\begin{aligned} (\nabla^2\mathbf{E})_r =& \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2E_r) \\ &+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right) = 0, \end{aligned} \tag{14}$$

$$(\nabla^2\mathbf{E})_\theta = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rE_\theta) - \frac{1}{r}\frac{\partial^2 E_r}{\partial r\partial\theta} = 0. \tag{15}$$

Equation (14) is for E_r alone, whereas Eq. (15) involves both components. There is also a separated equation for E_φ:

$$\begin{aligned} (\nabla^2\mathbf{E})_\varphi =& \frac{1}{r}\frac{\partial^2}{\partial r^2}(rE_\varphi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} \\ &\times\left(\sin\theta\frac{\partial E_\varphi}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}E_\varphi = 0. \end{aligned} \tag{16}$$

In this paper, however, we will not be concerned about those cylindrical symmetry cases where only the φ-component of the vector field is nonzero because a scalar technique of separation of variables is already known to obtain the solution [6].

Using the transverse condition

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\,\theta}\,\frac{\partial}{\partial\theta}(\sin\,\theta\; E_\theta) = 0, \tag{17}$$

where azimuthal symmetry is assumed, Eq. (14) implies

$$\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta} = 0, \tag{18}$$

which is consistent with Eq. (15). Thus, to obtain E_θ from E_r we can consider either Eq. (17) or Eq. (18). These equations correspond to choosing a gauge when this method is applied to the vector potential.

Now, in order to solve Eq. (14) for E_r, we refer to the method of separation of variables and write the product form

$$E_r(r,\theta) = \frac{u(r)}{r^2}\ P(\theta), \tag{19}$$

which leads to the following separated differential equations:

$$\frac{d^2u}{dr^2} - \frac{n(n+1)}{r^2}u = 0, \tag{20}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta}) + n(n+1)P = 0, \tag{21}$$

where $n(n+1)$ is the separation constant. The solution of Eq. (20) is

$$u(r) = a\, r^{n+1} + \frac{b}{r^n}, \tag{22}$$

where a and b are arbitrary constants. Equation (21) is the Legendre equation of order n and the only solution which is single valued, finite and continuous over the whole interval corresponds to the Legendre polynomial $P_n(\cos\,\theta)$, n being restricted to positive integer values. Thus the general solution for E_r is

$$E_r(r,\theta) = \sum_{n=0}^{\infty}\left(a_n r^{n-1} + \frac{b_n}{r^{n+2}}\right) P_n(\cos\,\theta). \tag{23}$$

The simplest way of solving Eq. (17) for E_θ is to use the series expansion

$$E_\theta(r,\theta) = \sum_{n=0}^{\infty} v_n(r)\ \frac{d}{d\theta}P_n(\cos\,\theta), \tag{24}$$

where $v_n(r)$ are functions to be determined. By replacing Eqs. (23) and (24) into Eq. (17), it is found that

$$v_n(r) = \frac{a_n}{n}\ r^{n-1} - \frac{b_n}{n+1}\ \frac{1}{r^{n+2}} \tag{25}$$

for $n \geq 1$ with $a_o = 0$; this null factor in Eq. (23) means the absence of static field terms of the $1/r$ type, which are in reality typical of radiative fields as shown below. Clearly, the solutions given in Eqs. (23), (24) and (25) satisfy Eq. (18). The coefficients a_n and b_n are to be determined from the boundary conditions. For completeness, we include here the well-behaved general solution of Eq. (16):

$$E_\varphi(r,\theta) = \sum_{n=0}^{\infty}\left(c_n r^n + \frac{d_n}{r^{n+1}}\right)\frac{d}{d\theta}P_n(\cos\,\theta). \tag{26}$$

Thus, Eqs. (23)-(26) formally give all three components of the electric field. The same type of equations applies in magnetostatics. However, the boundary conditions of Eq. (3) will make the difference, implying in particular that $b_\circ = 0$ in the series expansion of Eq. (23) in magnetostatics; this being primarily related to the absence of magnetic monopoles.

To illustrate the use of the above formulas, we consider the simple example of the electric field due to a ring of radius a with total charge Q uniformly distributed and lying in the x-y plane. It is usually solved through the scalar potential method by using the result of the potential along the z-axis obtained from Coulomb's law [7]. The surface charge density on $r = a$, localized at $\theta = \pi/2$, is written as

$$\rho_S(a,\theta) = \frac{Q}{2\pi a^2}\ \delta(\cos\,\theta), \tag{27}$$

which may be expanded using the well-known Legendre series

$$\delta(\cos\,\theta) = \sum_{n=0}^{\infty}\frac{2n+1}{2}\ P_n(0)\ P_n(\cos\,\theta), \tag{28}$$

with $P_n(0)$ given by

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \frac{(-1)^n(2n+1)!}{2^{2n}(n!)^2}. \tag{29}$$

Taking into account the cylindrical symmetry of the system, and the requirement that the series solutions in Eqs. (23)-(25) have to be finite at the origin, vanish at infinity and satisfy the boundary conditions of Eq. (3) at $r = a$ for all values of the angle θ, namely, E_θ continuous at $r = a$ and E_r discontinuous at $r = a$, it is straightforwardly found that the spherical components of the electric field are

$$E_r(r,\theta) = \frac{Q}{4\pi\epsilon_\circ r^2}\sum_{n=0}^{\infty}P_n(0)\ P_n(\cos\,\theta) \times \begin{cases} (n+1)\left(\dfrac{a}{r}\right)^n, & r > a \\ -n\left(\dfrac{r}{a}\right)^{n+1}, & r < a \end{cases} \tag{30}$$

$$E_\theta(r,\theta) = -\frac{Q}{4\pi\epsilon_\circ r^2}\sum_{n=0}^{\infty}P_n(0)\ P_n^1(\cos\,\theta) \times \begin{cases} \left(\dfrac{a}{r}\right)^n, & r > a \\ \left(\dfrac{r}{a}\right)^{n+1}, & r < a \end{cases} \tag{31}$$

and $E_\varphi = 0$, where $P_n^1(\cos\,\theta) = (d/d\theta)\,P_n(\cos\,\theta)$ is an associated Legendre function. Note in particular that the coefficient $b_\circ$ in Eq. (23) becomes $Q/4\pi\epsilon_\circ$ for $r > a$, as expected. Also, the discontinuity of the nth component of E_r in Eq. (30) at $r = a$ is connected according to Eq. (3) with the corresponding component of the surface charge density

ρ_S obtained from Eqs. (27) and (28), exhibiting the unity of the multipole expansions of fields and sources (see Ref. 8).

To clarify the application of the formulas in the case of magnetostatics and also compare with electrostatics, we consider next the magnetic analog of the above example, that is, the magnetic induction field from a circular current loop of radius a lying in the x-y plane and carrying a constant current I. The surface current density on $r = a$ can be written as

$$\mathbf{J}_S(a,\theta,\varphi) = \frac{I}{a}\,\delta(\cos\theta)\,\hat{\varphi}, \tag{32}$$

where for the delta function is now convenient to use the expansion

$$\delta(\cos\theta) = \sum_{n=0}^{\infty} \frac{2n+1}{2n(n+1)}\,P_n^1(0)\,P_n^1(\cos\theta), \tag{33}$$

which follows from the completeness relation for the spherical harmonics after multiplication by $e^{-i\varphi}$ and integration over φ. The values for $P_n^1(0)$ are

$$P_{2n}^1(0) = 0, \quad P_{2n+1}^1(0) = \frac{(-1)^{n+1}(2n+1)!}{2^{2n}(n!)^2}. \tag{34}$$

Because of the cylindrical symmetry of the system, $B_\varphi = 0$. By requiring that the field be finite at the origin, vanish at infinity and satisfy the boundary conditions of Eq. (3) at $r = a$, the series solutions in Eqs. (23)-(25) for the magnetic case lead to the following radial and angular components of the magnetic induction field:

$$B_r(r,\theta) = -\frac{\mu_\circ I a^2}{2r^3}\sum_{n=0}^{\infty} P_n^1(0)\,P_n(\cos\theta) \times \begin{cases} \left(\dfrac{a}{r}\right)^{n-1}, & r > a \\[2ex] \left(\dfrac{r}{a}\right)^{n+2}, & r < a \end{cases} \tag{35}$$

$$B_\theta(r,\theta) = \frac{\mu_\circ I a^2}{2r^3}\sum_{n=0}^{\infty} P_n^1(0)\,P_n^1(\cos\theta) \times \begin{cases} \dfrac{1}{n+1}\left(\dfrac{a}{r}\right)^{n-1}, & r > a \\[2ex] -\dfrac{1}{n}\left(\dfrac{r}{a}\right)^{n+2}, & r < a \end{cases} \tag{36}$$

Note that, as anticipated for magnetostatic problems, the coefficient $b_\circ$ in Eq. (23) is equal to zero. Also, as expected, the discontinuity of the nth component of B_θ in Eq. (36) at $r = a$ is connected according to Eq. (3) with the corresponding component of the surface current density $J_{S\varphi}$ obtained from Eqs. (32) and (33). Another characteristic difference with the electrostatic analog is that the coefficient $P_n^1(0)$ appears instead of $P_n(0)$. This can be traced to the fact that the inhomogeneous boundary condition, as given by Eq. (3), is applied to the angular component of the magnetic induction field in Eqs. (24)-(25), as opposed to the corresponding inhomogeneous boundary condition acting on the radial component of the electric field in Eq. (23). The fields in Eqs. (35)-(36) are usually obtained through the vector potential method by using the expression of the magnetic induction field along the z-axis calculated from the Biot and Savart law [6]. An alternative technique is mere integration of the vector potential [9]. Our treatment has the advantage of introducing a considerable simplification on the procedure of applying the boundary conditions on the magnetic induction field directly.

3. Time-varying fields

By using Eqs. (1), (2), and (6) it is seen that outside sources the fields are related by

$$\mathbf{E} = \frac{i\omega}{k^2}\,\nabla\times\mathbf{B}, \tag{37}$$

so that we only need to solve Eq. (7) for $\mathbf{B}$. Alternatively, we can solve for $\mathbf{E}$, and obtain $\mathbf{B}$ through the expression

$$\mathbf{B} = -\frac{i}{\omega}\,\nabla\times\mathbf{E}. \tag{38}$$

In the following, we choose to deal with the Helmholtz equation for the magnetic induction field. The reason is to exhibit similarities and differences with the static case treated in Sec. 2.

In the case of spherical boundary surfaces with azimuthal symmetry, the B_r and B_θ components of the magnetic induction satisfy the following equations:

$$(\nabla^2\mathbf{B})_r + k^2 B_r = \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 B_r) + \frac{1}{r^2\sin\theta} \times \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B_r}{\partial\theta}\right) + k^2 B_r = 0, \tag{39}$$

$$(\nabla^2\mathbf{B})_\theta + k^2 B_\theta = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rB_\theta) - \frac{1}{r}\frac{\partial^2 B_r}{\partial r\partial\theta} + k^2 B_\theta = 0. \tag{40}$$

Similarly, for the B_φ component we would have the equation

$$(\nabla^2\mathbf{B})_\varphi + k^2 B_\varphi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rB_\varphi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} \times \left(\sin\theta\frac{\partial B_\varphi}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}B_\varphi + k^2 B_\varphi = 0. \tag{41}$$

These are analogous to Eqs. (14), (15) and (16) in connection with the vector Laplace equation. In order to solve Eq. (39) we let

$$B_r(r,\theta) = \frac{j(r)}{r}\,P(\theta), \tag{42}$$

whereupon separation yields

$$\frac{d^2 j}{dr^2} + \frac{2}{r}\frac{dj}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right] j = 0, \tag{43}$$

and Eq. (21), where the constant $n(n+1)$ is the separation parameter. Equation (43) is the spherical Bessel equation of order n with variable kr. Therefore, the general solution for B_r is

$$B_r(r,\theta) = \sum_{n=0}^{\infty} \left[a_n \frac{j_n(kr)}{r} + b_n \frac{n_n(kr)}{r} \right] P_n(\cos\theta). \quad (44)$$

Depending on boundary conditions, the spherical Hankel functions $h_n^{(1,2)}$ instead of the spherical Bessel functions j_n, n_n may be used. For B_θ we again write

$$B_\theta(r,\theta) = \sum_{n=0}^{\infty} w_n(r) \frac{d}{d\theta} P_n(\cos\theta), \quad (45)$$

and use $\nabla \cdot \mathbf{B} = 0$ to obtain now

$$\begin{aligned} w_n &= \frac{a_n}{n(n+1)r} \frac{d}{dr}[r\, j_n(kr)] \\ &+ \frac{b_n}{n(n+1)r} \frac{d}{dr}[r\, n_n(kr)], \end{aligned} \quad (46)$$

for $n \geq 1$ with $a_\circ = b_\circ = 0$. The other coefficients a_n and b_n are determined so that the boundary conditions for the vector field are exactly satisfied. In the case of the B_φ component, the general solution is

$$B_\varphi(r,\theta) = \sum_{n=0}^{\infty} [c_n j_n(kr) + d_n n_n(kr)] \frac{d}{d\theta} P_n(\cos\theta). \quad (47)$$

The same type of equations applies for the electric field.

As an example, we shall consider the problem of the magnetic induction field from a current $I = I_\circ e^{-i\omega t}$ in a circular loop of radius a lying in the x-y plane. It is the time-varying version of the case solved in Sec. 2. The surface density current on $r = a$ is then

$$\mathbf{J}_S(a,\theta,\varphi,t) = \frac{I_\circ}{a} \delta(\cos\theta)\, e^{-i\omega t}\, \hat{\varphi}, \quad (48)$$

which can be expanded using Eq. (33). The complete series solution of the Helmholtz equation for the magnetic induction field, which is finite at the origin, represents outgoing waves at infinity and satisfies the boundary conditions of Eq. (3) at $r = a$, becomes

$$\begin{aligned} B_r(r,\theta,t) &= -i \frac{\mu_\circ I_\circ ka}{2r} e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) P_n^1(0) \\ &\times P_n(\cos\theta) \begin{cases} j_n(ka)\, h_n^{(1)}(kr) \\ j_n(kr)\, h_n^{(1)}(ka) \end{cases} \end{aligned} \quad (49)$$

$$B_\theta(r,\theta,t) = -i \frac{\mu_\circ I_\circ k^2 a}{2} e^{-i\omega t} \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} P_n^1(0)$$
$$\times P_n^1(\cos\theta) \begin{cases} j_n(ka) \left[h_{n-1}^{(1)}(kr) - \frac{n}{kr} h_n^{(1)}(kr) \right] \\ h_n^{(1)}(ka) \left[j_{n-1}(kr) - \frac{n}{kr} j_n(kr) \right] \end{cases} \quad (50)$$

and $B_\varphi = 0$, where the upper line holds for $r > a$ and the lower line for $r < a$. As noted above, the coefficient $a_\circ$ in Eq. (44) indeed vanishes. Also, the discontinuity of the nth component of B_θ in Eq. (50) at $r = a$ is connected, according to Eq. (3), with the nth component of the surface current density $J_{S\varphi}$ obtained from Eqs. (48) and (33). A characteristic difference between this time-varying problem and the corresponding static case is the appearance of the spherical Bessel functions, which are solutions of the radial part of the Helmholtz equation in spherical coordinates. Using their limiting values [10], it can be seen that for $k \to 0$ the static results in Eqs. (35) and (36) are obtained, as mathematically and physically expected. On the other hand, the radiative part of the external magnetic induction field, which decreases as $1/r$, is given by

$$\begin{aligned} \mathbf{B}(r,\theta,t) &= \hat{\theta} \frac{\mu_\circ I_\circ ka}{4r} e^{i(kr-\omega t)} \sum_{n=0}^{\infty} \frac{(4n+3)(2n-1)!}{2^{2n} n!(n+1)!} \\ &\times j_{2n+1}(ka)\, P_{2n+1}^1(\cos\theta). \end{aligned} \quad (51)$$

In the dipole approximation, $ka \ll 1$, this becomes the radiative magnetic induction field from an oscillating magnetic dipole of magnetic moment $\mathbf{m} = \pi a^2 I_\circ \hat{\mathbf{z}}$:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_\circ k^2}{4\pi} (\hat{\mathbf{r}} \times \mathbf{m}) \times \hat{\mathbf{r}} \frac{e^{i(kr-\omega t)}}{r}. \quad (52)$$

The magnetic induction field in Eqs. (49) and (50) can be seen to be just that which is obtained with the more arduous technique of a dyadic Green's function expanded in vector spherical harmonics and applied to the vector potential, which, by symmetry, only has the φ-component different from zero [3]. As we have shown, a direct calculation of the electromagnetic field with r- and θ-components is more simplified if separation of variables is used.

4. Conclusion

For spherical coordinate systems, the Maxwell equations outside sources lead to coupled equations involving all three components of the electromagnetic fields. In general, the statement is that one cannot separate spherical components of the Maxwell equations, and extensive techniques for solving the vector equations have been developed which introduce vector spherical harmonics and use dyadic methods. We have shown, however, that separation of variables is still possible in the case of azimuthal symmetry, and so general solutions for each component of the electromagnetic vector fields were obtained. We have illustrated the use of these formulas with direct calculations of electric and magnetic induction fields from localized charge and current distributions, without involving the electromagnetic potentials. Boundary conditions are easier to apply to these solutions, and their forms highlight the similarities and differences between the electric and magnetic cases in both time-independent and time-dependent situations. Finally, we remark that in cylindrical coordinates, the other commonly used curvilinear coordinate system, the Maxwell equations do separate into equations for each vector component alone if there is cylindrical symmetry, so that the method of separation of variables can be used directly.

Acknowledgments

This work was partially supported by the Departamento de Investigaciones Científicas y Tecnológicas, Universidad de Santiago de Chile.

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