An auxiliary vector space that simplifies some calculations of the Kovalevskaya top

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We discuss here Kovalevskaya's integrable case of a rigid body, with a symmetric inertia moment, with half the value for the different of inertia moment, and the position of the center of mass and fixed point both placed on a plane orthogonal to the axis of symmetry. We introduce an auxiliary vector space that is a function of two complex conjugate variables and enables ys to simplify many of the calculations necessary to separate the variables in the explicit solution of the Kovalevskaya top. This vector space plays an important role in the study of the elliptic integrals and in particular, in the use Kovalevskaya made of the theory of elliptic integrals.

Keywords: Rigid body; Kovalevskaya solution.

Se revisa el caso integrable de Kovalevskaya para el movimiento de un cuerpo rígido con una matriz de inercia simétrica, cuyo momento principal de inercia diferente tiene la mitad del valor de los otros dos momentos iguales, y cuya posici ón del centro de masa se encuentra en el plano perpendicular al eje de simetría. Se ha introducido un espacio vectorial auxiliar que es función de dos variables complejo conjugadas y permite simplificar muchos de los cálculos necesarios para separar las variables en la solución explícita del trompo de Kovalevskaya. Este espacio vectorial juega un papel importante en el estudio de las integrales elípticas y en particular, en el uso que hizo Kovalevskaya de la teoría de integrales elípticas.

Descriptores: Cuerpo rígido; solución de Kovalevskaya.

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1. Introduction

In 1888, Sophia Kovalevskaya was awarded the Bordin price by the Paris Académie des Sciences for her discovery of a new integrable case of dynamics of the rigid body in the field of constant gravity. An extract of her work [1] was published in Acta Mathematica the folowing year. In the present paper we are interested in giving a new approach to the Kovalevskaya methods, from a pedestrian point of view. Previous work on this line [2] will be improved in this publication.

One of the fundamental problems in Classical Mechanics is the movement of a rigid body. Among the few integrable cases of this rigid body motion, that due to Kovalevskaya possesses a great deal of richness and complexity. The understanding of the solutions in this case is very important in order to complete the other two better-known cases of Euler and Lagrange. We shall start with Euler's and Poisson equations for a rigid body in dimensionless form, and then find a vector basis and a constant matrix which will allow us to do the separation of the variables for the Kovalevskaya top with economy and simplicity in the algebra involved. We believe this approach to the Kovalevskaya's problem is novel and original. Lastly, the particular solution in which the Kovalevskaya constant is equal to zero will be analyzed.

Euler equations for the dynamics of a rigid body in the body frame are [3]

$$\mathbf{I}\dot{\omega} + \omega \times \mathbf{I}\omega = \mathbf{N}, \tag{1}$$

where ω is the angular velocity vector, $\mathbf{I}$ is the inertia matrix, and $\mathbf{N}$ the torque vector, and a dot on a letter denotes the time derivative.

We consider the rigid body with a fixed point. The mechanic problem consists in determining the rotation around this fixed point by means of a rotation matrixR. The velocity of this rotation matrix is related to the angular velocity in the body frame according to

$$\dot{\mathcal{R}} = \mathcal{R}\omega \times . \tag{2}$$

Assuming the force on any particle of the rigid body to be the constant gravity, one finds [2] the expression of the torque in the body frame

$$\mathbf{N} = -mgd\mathbf{a} \times \mathbf{u}, \tag{3}$$

where m is the mass of the body, g is the gravity constant acceleration, d is the distance from the fixed point to the center of mass of the body, $\mathbf{a}$ is the constant unit vector pointing from the fixed pointto the center of mass in the body frame, and $\mathbf{u}$ are the components of the time dependent unit vector in the opposite direction to the force of gravity in the body frame. This unit vector is rotated to a constant direction $\mathbf{k}$ along the vertical in the inertial frame by means of

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathcal{R}\mathbf{u}. \tag{4}$$

The time derivative of this constant is zero, and using Eq. (2) we find the Poisson equation [4]

$$\dot{\mathbf{u}} = \mathbf{u} \times \omega. \tag{5}$$

Equations (1) and (5) form a system of six equations for the six quantities ω and $\mathbf{u}$ with three constants of motion. These constants are: the unit character of vector $\mathbf{u}$:

$$1 = \mathbf{u}^{\mathrm{T}}\mathbf{u}, \tag{6}$$

the superscript T denoting the transpose vector or matrix; the conserved energy h

$$h = \frac{1}{2}\omega^{\mathrm{T}}\mathbf{I}\omega + mgd\mathbf{a}^{\mathrm{T}}\mathbf{u}; \tag{7}$$

the conserved angular momentum component along the force that is

$$\ell = u^{\mathrm{T}}\mathbf{I}\omega. \tag{8}$$

In order to solve the system of six equations some other independent constant of motion must be found. In this a case, the rotation matrix can be found using either Piña parametization [5] or any other equivalent method. Piña parametization is written in terms of the vectors k and u, which are assumed to be known, and an angle $\gamma : \mathcal{R}(\mathbf{k}, \mathbf{u}, \gamma)$. This last is determined from the expression for angular velocity in these coordinates,

$$\omega = \frac{1}{1 + \mathbf{k}^{\mathrm{T}}\mathbf{u}}\left[\dot{\mathbf{u}} \times (\mathbf{u} + \mathbf{k})\right] + \dot{\gamma}\mathbf{u}. \tag{9}$$

Note that (in a Lagrange formalism of mechanics) γ is a cyclic variable in these coordinates and the constant of motion ℓ is the conjugate momentum to γ.

In the Kovalevskaya case [1] one assumes the moment inertia of to be a symmetric one and the principal moments of inertia can be written as

$$(I_1, I_2, I_3) = D(2, 2, 1), \tag{10}$$

with two equal moments of inertia and the different moment of inertia being half the value of the other two. Kovalevskaya also assumed the unit vector a is in the plane of the two equal moments inertia, for example, we choose

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{11}$$

We write the Euler equations of motion for this Kovalevskaya case using the unit of time pD=mgd, and we find

$$2\dot{\omega}_1\omega_2\omega_3 = 0, \tag{12}$$

$$\omega_2 + \omega_1\omega_3 = u3, \tag{13}$$

and

$$\dot{\omega}_3 = -u_2. \tag{14}$$

The Poisson equations (5) are the same, with components we write as

$$\dot{u}_1 = u_2\omega_3 - u_3\omega_2, \tag{15}$$

$$\dot{u}_2 = u_3\omega_1 - u_1\omega_3, \tag{16}$$

and

$$\dot{u}_3 = u_1\omega_2 - u_2\omega_1. \tag{17}$$

The energy constant, in units of mgd, becomes

$$h = \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 + u_1. \tag{18}$$

The constant component of the angular momentum vector, written also in dimensionless quantities, is in this case

$$\ell = 2\omega_1 u_1 + 2\omega_2 u_2 + \omega_3 u_3. \tag{19}$$

To obtain the fourth constant of motion, it is convenient to use the variables $u_1 + iu_2$ and $\omega_1 + i\omega_2$ with the equations of motion

$$\frac{d}{dt}(u_1 + iu_2) = iu_3(\omega_1 + i\omega_2) - i\omega_3(u_1 + iu_2), \tag{20}$$

and

$$2\frac{d}{dt}(\omega_1 + i\omega_2) + i\omega_3(\omega_1 + i\omega_2) = iu_3. \tag{21}$$

Clearing out iu_3 between these two equations, one has

$$\frac{d}{dt}[(\omega_1 + i\omega_2)^2 - (u_1 + iu_2)] = -i\omega_3[(\omega_1 + i\omega_2)^2 - (u_1 + iu_2)]. \tag{22}$$

From this, and its complex conjugate one obtains the Kovalevskaya constant of motion

$$[(\omega_1+i\omega_2)^2-(u_1+iu_2)][(\omega_1-i\omega_2)^2-(u_1-iu_2)]=k^2. \tag{23}$$

2. The first steps in the Kovalevskaya solution

To study the motion of the Kovaleskaya top [1-7] one introduces a complex variable

$$\xi = (\omega_1 + i\omega_2)^2 - (u_1 + iu_2), \tag{24}$$

and its complex conjugate η; the Kovaleskaya constant is the product of both,

$$k^2 = \xi\eta. \tag{25}$$

The system is solved in terms of the complex variable

$$x = \omega_1 + i\omega_2, \tag{26}$$

and its complex conjugate y.

Using this notation, from the energy conservation Eq. (7) one gets ω_3^2. The ℓ constant in (8) yields $-u_3\omega_3$; and we express u_3^2 from the Eq. (6) grouped as

$$\begin{pmatrix} \omega_3^2 \\ -\omega_3 u_3 \\ u_3^2 \end{pmatrix} = \xi \begin{pmatrix} 1 \\ -y \\ y^2 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ -x \\ x^2 \end{pmatrix} + \begin{pmatrix} 2h - (x+y)^2 \\ -\ell + xy(x+y) \\ 1 - k^2 - x^2y^2 \end{pmatrix}, \tag{27}$$

that is expressed in vector form to group the Kovalevskaya equations in a form suitable for introducing an auxiliary vector space, as we shall see in the next equations.

The independent variables x and y are subjected to the equations of motion

$$2\dot{x} = -i(\omega_3 x - u_3), \quad 2\dot{y} = i(\omega_3 y - u_3). \tag{28}$$

Consequently, it is important to consider the squares and products inside parentheses on the right sides of the equations (28), namely

$$(\omega_3 x - u3)2 = \begin{pmatrix} \omega_3^2 & -\omega_3 u_3 u_3^2 \end{pmatrix} \begin{pmatrix} x^2 \\ 2^x \\ 1 \end{pmatrix}, \tag{29}$$

$$(\omega_3 x - u_3)(\omega_3 y - u_3) = \begin{pmatrix} \omega_3^2 & -\omega_3 u_3 & u_3^2 \end{pmatrix} \times \begin{pmatrix} xy \\ x+y \\ 1 \end{pmatrix}, \tag{30}$$

$$(\omega_3 y - u_3)^2 = \begin{pmatrix} \omega_3^2 & -\omega_3 u_3 & u_3^2 \end{pmatrix} \begin{pmatrix} y^2 \\ 2y \\ 1 \end{pmatrix}. \tag{31}$$

We note that vector (27) appears in transposed form in the three previous equations, contracted with three important vectors. On the right side of these equations, we recognize a basis of vectors which are linearly independent when $x \neq y$,

$$b_1 = \begin{pmatrix} x^2 \\ 2x \\ 1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} xy \\ x+y \\ 1 \end{pmatrix}, \quad b2 = \begin{pmatrix} y^2 \\ 2y \\ 1 \end{pmatrix}. \tag{32}$$

This basis is used in this and the following sections, with much economy in the algebra of the transformation of the variables of the Kovaleskaya top. This is our main contribution to this problem.

Related to this basis, we found the constant matrix

$$Q = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \tag{33}$$

which has many useful properties in this paper.

There are also the properties

$$\mathbf{b}_1^{\mathrm{T}} Q \mathbf{b}_1 = \mathbf{b}_2^{\mathrm{T}} Q \mathbf{b}_2 = \mathbf{b}_0^{\mathrm{T}} Q \mathbf{b}_2 = \mathbf{b}_0^{\mathrm{T}} Q \mathbf{b}_1 = 0. \tag{34}$$

The transpose equations also hold since Q is a symmetric matrix

$$\mathbf{b}_1^{\mathrm{T}} Q \mathbf{b}_0 = \mathbf{b}_2^{\mathrm{T}} Q \mathbf{b}_0 = 0. \tag{35}$$

Besides there are the properties

$$\mathbf{b}_0^{\mathrm{T}} Q \mathbf{b}_0 = -(x-y)^2, \tag{36}$$

and

$$\mathbf{b}_1^{\mathrm{T}} Q \mathbf{b}_2 = \mathbf{b}_2^{\mathrm{T}} Q \mathbf{b}_1 = 2(x-y)2. \tag{37}$$

In additon vectors appearing in (27) may be expressed in terms of the same vector space as

$$\frac{1}{2} Q \mathbf{b}_1 = \begin{pmatrix} 1 \\ -x \\ x^2 \end{pmatrix}, \quad \frac{1}{2} Q \mathbf{b}_2 = \begin{pmatrix} 1 \\ -y \\ y^2 \end{pmatrix}. \tag{38}$$

The algebra of the scalar products in (29–31) is now simplified to give

$$(\omega_3 x - u_3)^2 = \xi (x-y)^2 + R(x), \tag{39}$$

$$(\omega_3 y - u_3)^2 = \eta (x-y)^2 + R(y), \tag{40}$$

and

$$(\omega_3 x - u_3)(\omega_3 y - u3) = R(x, y), \tag{41}$$

where $R(x)$ is the four order polynomial

$$R(x) = -x^4 + 2hx^2 - 2\ell x + 1 - k^2, \tag{42}$$

and

$$R(x,y) = -x^2y^2 + 2hxy - \ell(x+y) + 1 - k^2. \tag{43}$$

These functions will also be expressed in terms of the vectors defined in our auxiliary space, after we present the theory of the elliptic integrals in the next section, where these functions are generalized.

An important relation that will be used to write the ξ and η variables in terms of x and y can be obtained by realizing that the square of (41) is equal to the product of (39) times (40); that is

$$R^2(x,y) = R(x)R(y) + (x-y)^2[\xi R(y) + \eta R(x)] + k^2 (x-y)^4, \tag{44}$$

where we have used (25).

The use of these properties allows us to write

$$\left(\frac{\omega_3 x - u_3}{\sqrt{R(x)}} \pm \frac{\omega_3 y - u_3}{\sqrt{R(y)}}\right)^2 = \frac{(x-y)^4}{R(x)R(y)} \left\{ \left[\frac{R(x,y) \pm \sqrt{R(x)}\sqrt{R(y)}}{(x-y)^2} \right]^2 - k^2 \right\}, \tag{45}$$

which results in the equations

$$\frac{\dot{x}}{\sqrt{R(x)}} - \frac{\dot{y}}{\sqrt{R(y)}} = \frac{i}{2}\frac{\omega_3 y - u_3}{\sqrt{R(y)}} + \frac{i}{2}\frac{\omega_3 x - u_3}{\sqrt{R(x)}} = \frac{i}{2}\frac{(x-y)^2}{\sqrt{R(x)R(y)}} \left\{ \left[\frac{R(x,y) + \sqrt{R(x)}\sqrt{R(y)}}{(x-y)^2} \right]^2 - k^2 \right\}, \tag{46}$$

and

$$\frac{\dot{x}}{\sqrt{R(x)}} + \frac{\dot{y}}{\sqrt{R(y)}} = \frac{i}{2}\frac{\omega_3 y - u_3}{\sqrt{R(y)}} - \frac{i}{2}\frac{\omega_3 x - u_3}{\sqrt{R(x)}} = \frac{i}{2}\frac{(x-y)^2}{\sqrt{R(x)R(y)}} \left\{ \left[\frac{R(x,y) - \sqrt{R(x)}\sqrt{R(y)}}{(x-y)^2} \right]^2 - k^2 \right\}, \tag{47}$$

To separate the variables Kovalevskaya uses the algebra of the elliptic integrals of first order, which we develop in the next section by using the auxiliary vector space introduced above.

3. The algebra of the elliptic integrals of first order

The transformation theory of elliptic integrals can be used also with many other purposes than just the separation of variables in the Kovaleskaya's problem. For this reason we present, in this section, the use we made of our auxiliary vector space to present this algebra in a more general context that is necessary. We postpone for the next section the choice of the particular parameters that adjust to the Kovalevskaya case.

In this section, we consider the first order elliptic integrals

$$\int \frac{dx}{\sqrt{R(x)}}, \tag{48}$$

where $R(x)$ is the fourth order polynomial

$$R(x) = Ax^4 + 4Bx^3 + 6Cx^2 + 4B'x + A'$$
$$= (x^2 \; 2x \; 1) \begin{pmatrix} A & B & C-2\lambda \\ B & C+\lambda & B' \\ C-2\lambda & B' & A' \end{pmatrix} \begin{pmatrix} x^2 \\ 2x \\ 1 \end{pmatrix}, \tag{49}$$

with λ an arbitrary parameter, multiplying the matrix Q of the previous section that does not contribute to the polynomial $R(x)$. It has been written in terms of the symmetric matrix M

$$M = \begin{pmatrix} A & B & C \\ B & C & B' \\ C & B' & A' \end{pmatrix}, \tag{50}$$

with entries that are related to the coefficients of the different powers of the polynomial. Notice that we are using the same notation for this polynomial and the polynomial used in the Kovaleskaya case that will be identified until the next section.

We want to place emphasis on the fact that our auxiliary vector space, including our Q matrix, play an important role in this theory of elliptic integrals.

We form the bilinear transformation of the integration variable

$$x = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (\alpha\delta - \beta\gamma = 1), \tag{51}$$

where α, β, γ, δ, are real or complex numbers. The integral (48) is transformed into another elliptic integral of first order

$$\int \frac{dz}{\widetilde{R}(z)}, \tag{52}$$

where $\widetilde{R}(z)$ is a third or fourth order polynomial obtained from

$$\begin{pmatrix} x^2 \\ 2x \\ 1 \end{pmatrix} = \frac{1}{(\gamma z+\delta)^2} \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} z^2 \\ 2z \\ 1 \end{pmatrix} \tag{53}$$

The factor $1/(\gamma z+\delta)^2$ does not appear (outside the root), since it is canceled by the same factor from the differential

$$dx = \frac{dz}{(\gamma z + \delta)^2}. \tag{54}$$

We note that the matrix Q remains invariant with respect to the transformation of the matrix in (53), whereas matrix M is transformed according to

$$\widetilde{M} = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} \\ \widetilde{B} & \widetilde{C} & \widetilde{B}' \\ \widetilde{C} & \widetilde{B}' & \widetilde{A}' \end{pmatrix} = \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta+\beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta\delta^2 & \end{pmatrix}$$
$$\times M \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}, \tag{55}$$

To give the coefficients of the transformed polynomial, we have

$$\widetilde{R}(z) = (z^2\ 2z\ 1)(\widetilde{M} - \lambda Q)\begin{pmatrix} z^2 \\ 2z \\ 1 \end{pmatrix}. \tag{56}$$

The determinant of the square matrix in (53) is equal to 1 and one has the invariance

$$|\widetilde{M} - \lambda Q| = |M - \lambda Q| = -4\lambda^3 + g_2\lambda + g_3, \tag{57}$$

with the two invariants

$$g_2 = AA' - 4BB' + 3C^2, \tag{58}$$

and

$$g_3 = ACA' + 2BCB' - A'B^2 - AB'^2 - C^3. \tag{59}$$

Matrix MQ^{-1} satisfies the characteristic equation for λ, when expression (57) is equal to zero; therefore we have the matrix equation

$$4MQ^{-1}MQ^{-1}M = g_2M + g_3Q, \tag{60}$$

where Q^{-1} is equal to

$$Q^{-1} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}. \tag{61}$$

We again use the auxiliary vector space with the basis of the three vectors b1, b0, b2 that were found in the previous section; the y is considered here a constant parameter.

We find the non trivial representation for matrix Q^{-1} in terms of this basis

$$Q^{-1} = \frac{1}{(x-y)^2}\left(\frac{1}{2}\mathbf{b}_1\mathbf{b}_2^{\mathrm{T}} + \mathbf{b}_2\mathbf{b}_1^{\mathrm{T}} + \frac{1}{2}\mathbf{b}_0\mathbf{b}_0^{\mathrm{T}}\right), \tag{62}$$

which will be used in (60).

We have compact expressions for polynomial $R(x)$ in terms of our auxiliary vector space as stated above:

$$R(x) = \mathbf{b}_1^{\mathrm{T}}M\mathbf{b}_1, \tag{63}$$

and also

$$R(y) = \mathbf{b}_2^{\mathrm{T}}M\mathbf{b}_2, \tag{64}$$

and define other functions in terms of the same linear algebra

$$N(x,y) = \mathbf{b}_1^{\mathrm{T}}M\mathbf{b}_2 = \mathbf{b}_2^{\mathrm{T}}M\mathbf{b}_1 = \mathbf{b}_0^{\mathrm{T}}M\mathbf{b}_0. \tag{65}$$

The Weierstrass theory of elliptic integrals proposes a more general transformation than the bilinear one, namely

$$s = \frac{N(x,y) \pm \sqrt{R(x)R(y)}}{2(x-y)^2}. \tag{66}$$

It follows that the standard form of the elliptic integral

$$\int \frac{dx}{\sqrt{R(x)}} = \int \frac{ds}{\sqrt{4s^3 - g_2s - g_3}}, \tag{67}$$

is independent of the value chosen for y.

The proof that (66) implies (67) is done in several stages. We define two other functions in the same auxiliary algebra of our basis

$$P_{02}(x,y) = P_{01}(y,x) = \mathbf{b}_0^{\mathrm{T}}M\mathbf{b}_2, \tag{68}$$

and

$$P_{01}(x,y) = P_{02}(y,x) = \mathbf{b}_0^{\mathrm{T}}M\mathbf{b}_1. \tag{69}$$

The Taylor series expansion of functions $N(x,y)$ and $P_{02}(x,y)$ in the neighborhood of $x=y$ are (to all orders)

$$N(x,y) = R(y) + \frac{x-y}{2}R'(y) + \frac{(x-y)^2}{12}R''(y), \tag{70}$$

and

$$P_{02}(x,y) = R(y) + \frac{x-y}{4}R'(y), \tag{71}$$

which become

$$P_{01}(x,y) = P_{02}(y,x) = R(x) - \frac{x-y}{4}R'(x). \tag{72}$$

We now take the differential of (66) to obtain

$$\begin{aligned} ds &= \frac{dx}{2(x-y)^4}\left\{(x-y)^2\left[N'(x,y) \pm \frac{R'(x)\sqrt{R(y)}}{2\sqrt{R(x)}}\right] - \left[N(x,y) \pm \sqrt{R(x)R(y)}\right]2(x-y)\right\} = \frac{dx}{\sqrt{R(x)}}\frac{1}{(x-y)^3} \\ &\times\left\{\sqrt{R(x)}\left[R'(y)\frac{x-y}{4} + \frac{x-y}{12}R''(y) - R(y) - \frac{x-y}{2}R'(x) - \frac{(x-y)^2}{12}R''(y)\right] \pm \sqrt{R(y)}\left[R'(x)\frac{x-y}{4} - R(x)\right]\right\} \\ &= \frac{dx}{\sqrt{R(x)}}\frac{1}{(x-y)^3}\left[-\sqrt{R(x)}P_{02}(x-y) \mp \sqrt{R(y)}P_{01}(x,y)\right] \end{aligned} \tag{73}$$

The demonstration of (67) comes from this equation provided one can prove that

$$4s^3 - g_2 s - g_3 = \frac{1}{(x-y)^6} \times \left[-\sqrt{R(x)}P_{02}(x,y) \pm \sqrt{R(y)}P_{01}(x,y)\right]^2, \quad (74)$$

which in its turn needs the proof of both,

$$N^3(x,y) + 3R(x)R(y)N(x,y) - 2P_{01}(x,y)R(y) - 2P_{02}(x,y)R(x) = g_2 N(x,y)(x-y)^4 + 2g_3(x-y)^6, \quad (75)$$

and

$$3N^2(x,y) + R(x)R(y) - 4P_{01}(x,y)P_{02}(x,y) = g_2(x-y)^4. \quad (76)$$

To deduce Eq. (75), we substitute (62) in (60) and we multiply on the left and right by vectors $\mathbf{b}_1^{\mathrm{T}}$ and $\mathbf{b}_2$, respectively.

Equation (76) results in a similar way if we multiply Eq. (62) by vectors $\mathbf{b}_1^{\mathrm{T}}$ and $\mathbf{b}_1$, respectively.

We notice that the algebra associated with the auxiliary vector space of our basis permeates all the equations of these transformations and provides a simple linear algebraic deduction of many apparently wild expressions.

When y is a root of the polynomial $R(x)$, $R(y) = 0$, then

$$N(x,y) = \frac{x-y}{2}R'(y) + \frac{(x-y)^2}{12}R''(y), \quad (77)$$

and transformation (66) becomes the bilinear transformation

$$s(x) = \frac{R'(y)}{4(x-y)} + \frac{R''(y)}{24}. \quad (78)$$

4. Separable coordinates

The previous algebra was used by Kovaleskaya to separate equations (46) and (47). Polynomial $R(z)$ in the previous section is particularized to Kovalevskaya's case by choosing the constants in the polynomial according to

$$A=-1, \quad B=0, \quad C=\frac{h}{3}, \quad B'=-\frac{\ell}{2}, \quad A'=1-k^2. \quad (79)$$

The invariants g_2 and g_3 become

$$g_2 = k^2 - 1 + h^2/3, \quad (80)$$

and

$$g_3 = (k^2-1)h/3 + \ell^2/4 - h^3/27. \quad (81)$$

We also have the property

$$N(x,y) = R(x,y) + \frac{h}{3}(x-y)^2. \quad (82)$$

Transformation (66) is used with both signs, allowing parameter y to be the complex conjugate of x, to define two new coordinates

$$s_1 = \frac{R(x,y) - \sqrt{R(x)R(y)}}{2(x-y)^2} + \frac{h}{6}, \quad (83)$$

$$s_2 = \frac{R(x,y) + \sqrt{R(x)R(y)}}{2(x-y)^2} + \frac{h}{6}. \quad (84)$$

Instead of relation (67) where y was considered constant, one has now have

$$\frac{\dot{s}_1}{\sqrt{4s_1^3 - g_2 s_1 - g_3}} = \frac{\dot{x}}{\sqrt{R(x)}} + \frac{\dot{y}}{\sqrt{R(y)}} \quad (85)$$

$$\frac{\dot{s}_2}{\sqrt{4s_2^3 - g_2 s_2 - g_3}} = -\frac{\dot{x}}{\sqrt{R(x)}} + \frac{\dot{y}}{\sqrt{R(y)}} \quad (86)$$

Transforming the right hand side of equations (46) and (47) for the new coordinates, they become

$$\frac{\dot{s}_1}{\sqrt{4s_1^3 - g_2 s_1 - g_3}} = i\frac{\sqrt{(s_1 - h/6)^2 - k^2}}{s_2 - s_1} \quad (87)$$

$$\frac{\dot{s}_2}{\sqrt{4s_2^3 - g_2 s_2 - g_3}} = i\frac{\sqrt{(s_2 - h/6)^2 - k^2}}{s_2 - s_1} \quad (88)$$

Using the polynomial

$$\phi(s) = (4s^3 - g_2 s - g_3)[(s-h/6)^2 - k^2], \quad (89)$$

the coordinates s_1 and s_2 can be separated in the symmetric form

$$\frac{\dot{s}_1}{\sqrt{\phi(s_1)}} - \frac{\dot{s}_2}{\sqrt{\phi(s_2)}} = 0, \quad (90)$$

$$\frac{s_1\dot{s}_1}{\sqrt{\phi(s_1)}} + \frac{s_2\dot{s}_2}{\sqrt{\phi(s_2)}} = 1. \quad (91)$$

5. The navel of the Kovaleskaya top

The minimum value of the Kovalevskaya constant is zero. For this particular case, each factor in equation (25) is zero, and we have

$$(\omega_1 + i\omega_2)^2 = (u_1 + iu_2). \quad (92)$$

Separating the real and imaginary parts, we have

$$u_1 = \omega^2 - 1 - \omega_2^2, \quad (93)$$

and

$$u_2 = 2\omega_1\omega_2. \quad (94)$$

Equations (12) and (13) are dependent in this case from Eqs. (15) and (16), and there remain only four independent equations for the quantities ω_1, ω_2, ω_3, and u_3

$$2\dot{\omega}_1 = \omega_2\omega_3, \quad (95)$$

$$2\dot{\omega}_2 = -\omega_1\omega_3 + u_3, \quad (96)$$

$$\dot{\omega}_3 = -2\omega_1\omega_2, \quad (97)$$

and

$$\dot{u}_3 = -\omega_2(\omega_1^2 + \omega_2^2). \tag{98}$$

The constants of motion acquire new expressions, the energy becomes

$$h = 2\omega_1^2 + \frac{1}{2}\omega_3^2, \tag{99}$$

which should be positive or zero. The constant angular momentum component is now given by

$$\ell = 2\omega_1(\omega_1^2 + \omega_2^2) + \omega_3 u_3. \tag{100}$$

As vector u has unit magnitude, we also have

$$1 - u_3^2 = (\omega_1^2 + \omega_2^2)^2, \tag{101}$$

where components u_1 and u_2 have been replaced everywhere to deduce these equations.

A simple case occurs when in addition, $h = 0$. In this case $\omega_1 = 0$, $\omega_3 = 0$, $\ell = 0$. The previous constant of motion gives us $u_3 = \sqrt{1 - \omega_2^4}$, which, when substituted in the equation of motion $2\dot{\omega}_2 = u_3$, gives the solution

$$t = 2\int \frac{d\omega_2}{\sqrt{1 - \omega_2^4}} \quad (h = 0), \tag{102}$$

which shows that ω_2 is a sinus lemniscate. In this case, since $\omega_{\mathrm{T}}(u + k) = 0$, we have $\dot{\gamma} = 0$ and γ is a constant angle.

For $h \neq 0$, the energy constant allows the parameterization, in terms of an angle μ, of the quantities ω_1 and ω_3, as follows

$$\omega_1 = \sqrt{\frac{h}{2}}\sin\mu, \tag{103}$$

and

$$\omega_3 = \sqrt{2h}\cos\mu; \tag{104}$$

the energy equation then becomes an identity.

We use Euler angle θ to express the third component of vector $\mathbf{u}$

$$u_3 = \cos\theta. \tag{105}$$

The equation (101) is simplified to

$$\sin\theta = \omega_1^2 + \omega_2^2, \tag{106}$$

since the Euler angle θ is defined in the interval $[0, \pi]$.

Substitution of these equations in the expression for ℓ produces the result

$$\ell = \sqrt{2h}\cos(\mu - \theta). \tag{107}$$

The angle $\mu - \theta$ is therefore a constant that we denote by β.

$$\mu = \theta + \beta, \quad \cos\beta = \ell/\sqrt{2h}. \tag{108}$$

Replacing the previous result in the equations of motion we obtain a generic solution with

$$\omega_2 = \dot{\mu} = \dot{\theta}. \tag{109}$$

Equations (103), (108) and (109) in (106) lead us to the differential expression

$$\dot{\mu}^2 + \frac{h}{2}\sin^2\mu = \sin(\mu - \beta), \tag{110}$$

that is integrated as

$$t = \int \frac{d\mu}{\sqrt{\sin(\mu - \beta) - h/2\sin^2\mu}}. \tag{111}$$

Once $\mu(t)$ is known, $\mathbf{u}$ and ω are known as functions of time and $\dot{\gamma}$ is obtained from equation (9)

$$\dot{\gamma} = \frac{\omega^{\mathrm{T}}(\mathbf{u} + \mathbf{k})}{1 + u3}. \tag{112}$$

A particular case occurs when in addition $\omega_2 = 0$. Quantities ω_1, ω_3, u_3, h, ℓ, $\dot{\gamma}$ are then constants. And the Kovaleskaya top pivots with a constant angular velocity around a constant unit vector in the plane formed by the symmetry axis and the vector a pointing from the fixed point to the center of mass.

6. Concluding Remarks

In this work, we have achieved the separation of the system of equations of the Kovalevskaya top in a clear, original, direct way that can be followed in a straightforward manner. To unify several equations, we have used standard properties of linear algebra in a non trivial use for the theory of Kovalevskaya's hyperelliptic integrals and the related elliptic functions. In addition, we obtain the explicit solution when the Kovalevskaya constant is zero that is the simplest solution because it corresponds to the minimal value of that constant.

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