

The relation between momentum conservation and Newton's third law revisited

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Under certain conditions usually fulfilled in classical mechanics, the principle of conservation of linear momentum and Newton's third law are equivalent. However, the demonstration of this fact is usually incomplete in textbooks. We shall show here that to demonstrate the equivalence, we require the explicit use of the principle of superposition contained in Newton's second law. On the other hand, under some additional conditions the combined laws of conservation of linear and angular momentum, are equivalent to Newton's third law with central forces. The conditions for such equivalence apply in many scenarios of classical mechanics; once again the principle of superposition contained in Newton's second law is the clue.

Keywords: Newton's laws; conservation of linear momentum; conservation of angular momentum.

Bajo ciertas condiciones que se cumplen con frecuencia en la mecánica clásica, los principios de conservación del momento lineal y la tercera ley de Newton son equivalentes. No obstante, la demostración de este hecho está usualmente incompleta en los textos sobre el tema. En este artículo se demuestra dicha equivalencia, para lo cual requerimos del uso explícito del principio de superposición contenido en la segunda ley de Newton. Por otro lado, bajo algunas condiciones adicionales las leyes combinadas de conservación del momento angular y del momento lineal, son equivalentes a la tercera ley de Newton con fuerzas centrales. Las condiciones para esta última equivalencia también son válidas en muchos escenarios de la mecánica clásica; una vez más el principio de superposición contenido en la segunda ley de Newton es la clave para la demostración.

Descriptores: Leyes de Newton; conservación del momento lineal; conservación del momento angular.

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A very important part of the foundations of classical mechanics lies on one hand in Newton's third law or, in the other hand, on the principle of conservation of linear momentum. Thus, the link between both approaches is of the greatest interest in constructing the formalism of classical Physics. Common mechanics texts [1] usually state that Newton's third law automatically leads to the principle of conservation of linear momentum. However, the converse is also true under certain conditions; the proof in reverse order is either absent or restricted to systems of two particles in most textbooks. We start from the statement of the principle of linear momentum conservation for a closed and isolated system of particles^{*i*}, with respect to a certain inertial frame

$$\mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = \text{constant.} \quad (1)$$

Deriving with respect to time gives

$$\frac{d\mathbf{P}_1}{dt} + \frac{d\mathbf{P}_2}{dt} + \dots + \frac{d\mathbf{P}_n}{dt} = 0, \quad (2)$$

but according to Newton's second law, $d\mathbf{P}_i/dt$ refers to the total force applied to the i th particle. Further, since there are no external forces, Eq. (2) becomes

$$\begin{aligned} \sum_{j \neq 1}^n \mathbf{F}_{1j} + \sum_{j \neq 2}^n \mathbf{F}_{2j} + \dots + \sum_{j \neq n}^n \mathbf{F}_{nj} &= 0 \\ \Rightarrow \sum_{i=1}^n \sum_{j \neq i}^n \mathbf{F}_{ij} &= 0, \end{aligned} \quad (3)$$

where \mathbf{F}_{ij} stands for the force on the i th particle due to the j th particle. In the case of two particles, Eq. (3) leads to

Newton's third law automatically. However, in the case of an arbitrary number of particles, Newton's third law is only a sufficient condition in this step. The proof of necessity is the one that is usually absent in textbooks.

In order to prove the necessity, we shall use the principle of superposition stated in Newton's second law. Considering a system of n particles, let us take a couple of particles k and l . They undergo the force of each other \mathbf{F}_{kl} and \mathbf{F}_{lk} respectively, plus the internal forces due to the other particles of the system. However, according to the second law, the forces \mathbf{F}_{kl} and \mathbf{F}_{lk} are not altered by the presence of the rest of the forces (*i.e.* the other forces do not interfere with them). Therefore, if we withdrew the other particles of the system leaving the particles k and l in the same position, the forces \mathbf{F}_{kl} and \mathbf{F}_{lk} would be the same as those when all particles were interacting. Now, after the withdrawal of the other particles, our system consists of two isolated particles for which the third law is evident. Therefore, $\mathbf{F}_{kl} = -\mathbf{F}_{lk}$. We proceed in the same way for all the pairs of particles and obtain that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ for all i, j in the system. Observe that the proof of necessity requires the use of the principle of superposition contained in Newton's second law. Since we have demonstrated the necessity and sufficiency, we have proved the equivalence. Notwithstanding, this equivalence is based on many implicit assumptions:

1. *Newton's second law is valid:* As is well known, in scenarios such as quantum mechanics the concept of force is not meaningful any more.

2. *The time runs in the same way for all inertial observers:* We have used this condition since in the time derivative of Eq. (2), we do not mention what inertial system we have used to measure the time. Besides, this condition is necessary to assume that the force is equal in all inertial systems.
3. *All the momentum of the system is carried by the particles:* In this approach we are ignoring the possible storage or transmission of momentum from the fields generated by the interactions (see discussion in Ref. 3).
4. *The signals transmitting the interactions travel instantaneously:* In Eq. (1), each momentum \mathbf{P}_i is supposed to be measured at the same time. If any particle of the system changes its momentum at the time t , then to preserve the law of conservation of momentum (at the time t), it is necessary for the rest of the particles to change their momenta, **at the same time**, in such a way that they cancel the change of momentum caused by the i th particle. This fact is in turn related to the condition that all the momentum be carried by the particles (mechanical momentum). In other words, the other particles must learn of the change in momentum of the i th particle **instantaneously**.

As has been emphasized in the literature, even in the case in which all these assumptions fail, the principle of conservation of momentum is still maintained while Newton's third law is not valid any more, from which follows the advantage of formulating the empirical principles of classical mechanics in terms of the concept of momentum. Even when the assumptions given above are satisfied, the formulation in terms of momentum is advantageous [2]. Nevertheless, we emphasize that under the conditions cited above, Newton's third law is equivalent to the principle of conservation of linear momentum, but the complete proof of that statement requires the principle of superposition of forces established by Newton's second law.

On the other hand, by a similar argument we can show the equivalence of combined conservation of linear and angular momentum with Newton's third law with central forces. Starting from the conservation of angular momentum for a closed isolated system with respect to an inertial frame,

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \dots + \mathbf{L}_n = \text{constant}, \quad (4)$$

and deriving this equation, we find

$$\frac{d\mathbf{L}_1}{dt} + \frac{d\mathbf{L}_2}{dt} + \dots + \frac{d\mathbf{L}_n}{dt} = 0.$$

From the definition of \mathbf{L}_i , and taking into account that the system is isolated, the derivative of the total angular momentum reads

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^n \left(\mathbf{r}_i \times \sum_{j \neq i}^n \mathbf{F}_{ij} \right) = 0.$$

Under the assumption $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ (obtained from the conservation of linear momentum), we can show the following

identity by induction:

$$\sum_{i=1}^n \left(\mathbf{r}_i \times \sum_{j \neq i}^n \mathbf{F}_{ij} \right) = \sum_{i=1}^{n-1} \sum_{j>i}^n [(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}].$$

Clearly, Newton's third law with central forces (*i.e.* $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $(\mathbf{r}_i - \mathbf{r}_j)$ parallel to \mathbf{F}_{ij}) is a sufficient condition for the conservation of angular and linear momentum (we shall refer to the third law with central forces as the *strong version of Newton's third law* henceforth). To prove necessity, we resort again to the argument of isolating one pair of particles k, l without changing their positions. Since this two-particle system is now isolated, its total angular momentum must be constant, and remembering that the forces $\mathbf{F}_{kl} = -\mathbf{F}_{lk}$ have not changed either, we have

$$\frac{d\mathbf{L}(\text{two particles})}{dt} = (\mathbf{r}_k - \mathbf{r}_l) \times \mathbf{F}_{kl} = 0.$$

Now, since both particles have different positions and we are assuming that they are interacting ($\mathbf{F}_{kl} \neq 0$)ⁱⁱ, we obtain that $(\mathbf{r}_k - \mathbf{r}_l)$ must be parallel to \mathbf{F}_{kl} . We can proceed in a similar way for all possible pairs of particles in the system. In this case, we have used the combined laws of conservation of linear and angular momentum because Newton's third law in its weak version was assumed from the beginning. Of course, the conditions for this equivalence to hold are those cited above, but with analogous assumptions for angular momentum as well.

As before, when the conditions for this equivalence fail, the laws of conservation of linear and angular momentum are still valid, while the strong version of Newton's third law no longer holds.

An important issue arises when we consider non-isolated systems, since we have assumed that the system is isolated throughout this treatment. If we add external forces, once again the principle of superposition states that the internal forces do not interfere with them, and so Newton's third law is maintained. A similar argument holds for possible external torques and Newton's third law in its strong version.

In conclusion, we have proved that under certain conditions, the principle of linear momentum conservation is equivalent to the weak version of Newton's third law. Analogously, under similar conditions, the combined laws of conservation of linear and angular momentum are equivalent to the strong version of Newton's third law. We emphasize that for both demonstrations we should invoke the principle of superposition contained in Newton's second lawⁱⁱⁱ. Finally, it is worth mentioning that the assumptions necessary to obtain these equivalences are implicit in the original formalism of classical mechanics [4]. Therefore, such equivalences deserve more attention, at least until reaching relativity, quantum mechanics or classical (quantum) field theories.

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- i.* For a closed system, we mean a system in which particles are the same at all times, *i.e.* no particle exchange occurs with the surroundings.
 - ii.* The case of $\mathbf{F}_{kl} = 0$ could be considered as a trivial realization of the strong version of Newton's third law.
 - iii.* Of course, Newton's first law is also in the background of all this treatment, by assuming the existence of inertial frames in which the other laws of Newton and conservation of the momenta are valid.
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 4. For a revision, see for example: L. Eisenbud, *Am. J. Phys.* **26** (1958) 144; G.H. Duffey *Theoretical Physics* (Houghton Mifflin Company, Boston, 1973) p.36; E.A. Desloge, *Classical Mechanics Vol. I* (John Wiley & Sons, Inc., 1982); R.B. Lindsay and H. Margenau, *Foundations of Physics* (Dover Publications, Inc., New York, 1957).