Computational study of forced oscillations in a membrane

E. Alvarado-Anell and M. Sosa*
Instituto de Física, Universidad de Guanajuato,
Apartado Postal E-143, 37000 León, Gto., México,
**e-mail: modesto@fisica.ugto.mx*

M.A. Moreles
Centro de Investigación en Matemáticas,
Apartado Postal 402, 36240 Guanajuato, Gto., México

Recibido el 12 de enero de 2005; aceptado el 24 de noviembre de 2005

The solution of the model for small forced oscillations in membranes is described. The cases of rectangular, circular and also elliptical membranes are discussed. A simple computer animation example is provided using the MAPLE software package. The evolution of the first vibrational mode for a circular membrane is presented. The results obtained are useful for the understanding of membrane oscillations in different applications. Also, the method could be used as a help in teaching.

Keywords: Membrane; forced oscillation modes; computational simulation.

La solución del modelo para oscilaciones forzadas pequeñas en una membrana es descrita. Los casos para membranas rectangulares, circulares y elípticas son discutidos. Un ejemplo simple de animación computacional es presentado usando el paquete de software MAPLE. La evolución del primer modo vibracional para una membrana circular es presentada. Los resultados obtenidos son útiles para el entendimiento de oscilaciones en membranas, en diferentes aplicaciones. Además, el método pudiera ser de ayuda en la enseñanza.

Descriptores: Membrana; modos de oscilación forzados; simulación computacional.

PACS: 01.55; 02.70

1. Introduction

The study of free oscillations in membranes is a classical problem, widely discussed in most of the mathematical physics textbooks [1] and in many papers [2,3]. However, the study of oscillation modes in a membrane under an external force is a much more complex problem, rarely treated in most textbooks. This is due, in part, to the fact this many interesting applications of that kind of problem leads to analytical solutions too difficult to be obtained and in most of the cases impossible; only recently has it been possible to take advantage of computational tools for obtaining numerical solutions.

Recently, some works have been published concerning the oscillation modes in a circular membrane under an external force. The problem of the oscillation of an elastic circular membrane under a pressure difference has been discussed both theoretically and experimentally by Oliva *et al.* [4]. Also, a computer algebra study is shown in Ref. 3 for a force applied to a circular membrane.

On the other hand, the oscillations in an elliptical membrane are described by the Mathieu functions [2,5,6].

In this paper the forced oscillation solutions for rectangular, circular and also elliptical membranes are discussed. Also, a simple computer animation using the MAPLE software package is provided for a circular membrane.

The solutions obtained can be used for describing different physical systems. In particular, we want to emphasize that the understanding of forced oscillations in an elliptical membrane could be applied to simulate the oscillations of the tympanum under external sound pressure. For example, in Refs. 7-8 are described some experiments designed to detect the vibrations of the tympanum, by measuring either the magnetic flux changes of a permanent magnet or the reflected light by a mirror attached to the tympanic membrane. In these cases, both the magnet and the mirror act as external charges on the membrane, which must be taken into account for a better understanding of the oscillation modes.

In Secs. 2 and 3, we present a discussion of the model for the small forced oscillations of a membrane and the variable separation method, respectively. In section 4 are shown the solutions of the non-homogeneous wave equations, to describe the vibrations of rectangular, circular, and also elliptical membranes. Finally, section 5 is concerned to the numerical simulations with the vibrational modes of a membrane.

2. Model for the forced oscillations of a membrane

Consider a membrane, of constant density ρ, whose thickness can be neglected with respect to its size.

Suppose that, besides the membrane occupies an open set $\Omega \subset \Re^2$with contour $\partial\Omega$ of class C^1 and it begins to vibrate under the action of a vertical charge, say $(x_1, x_2) \rightarrow p\,(x_1, x_2)$.

Now take a small region $G_0 \subset \Omega$ with contour ∂G_0 of class C^1, and let G be the corresponding portion of the membrane, that is, G is the graph of $x \rightarrow u\,(x, t)$ restricted to G_0. Also, consider a curve Γ which is the bounding of G. Thus tension **T** acts at a point $P \in \Gamma$ and it is tangent to G and normal to Γ.

Let τ be the unitary vector of **T** and **n** the normal unitary vector exterior to G in P. Also, consider **e** the unitary vector tangent to Γ in P. Then, we find that

$$\tau = \mathbf{e} \times \mathbf{n}. \tag{1}$$

In order to calculate the vertical component of **T** in $P \in \Gamma$, consider a parametrization of Γ_0 such that

$$P_0(s) \equiv (x_1(s), x_2(s)) \quad s \in I \subset R.$$

Hence, the normal unitary vector exterior to ∂G_0 is given by

$$u = \frac{\left(x_2', -x_1'\right)}{\sqrt{x_1'^2 + x_2'^2}}, \tag{2}$$

where $P_0'(s) \equiv \left(x_1'(s), x_2'(s)\right)$ is tangent to Γ_0.

Also, consider the parametrization of Γ

$$P(s) = (P_0(s), x_3(s)),$$

where $x_3(s) = u(x_1(s), x_2(s), t)$. Then, the unitary vector **e** tangent to Γ is given by

$$\mathbf{e} = \frac{\left(x_1', x_2', x_3'\right)}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}} = \frac{\left(P_0', Du \cdot P_0'\right)}{\sqrt{\left\|P_0'\right\|^2 + \left|Du \cdot P_0'\right|^2}},$$

where *Du* denotes the gradient of u, and the normal unitary vector exterior to G in points of Γ is

$$\mathbf{n} = \frac{(Du, -1)}{\sqrt{1 + \|Du\|^2}}.$$

Therefore, from Eq. (1) we get

$$\tau = \det \begin{pmatrix} i & j & k \\ x_1' & x_2' & x_3' \\ u_{x_1} & u_{x_2} & -1 \end{pmatrix} \frac{1}{J},$$

where

$$J = \sqrt{1 + \|Du\|^2} \sqrt{\left\|P_0'\right\|^2 + \left|Du \cdot P_0'\right|^2}. \tag{3}$$

Thus,

$$\begin{aligned} J\tau \cdot k &= -\left(x_2', -x_1'\right) \cdot Du \\ &= -\sqrt{x_1'^2 + x_2'^2} \cdot Du \cdot \nu. \end{aligned} \tag{4}$$

This last expression is obtained from Eq. (2), where ν is normal to ∂G_0, that is, $\nu \cdot P_0' = 0$.

On the other hand, if $\beta = \cos\theta$, where θ is the angle between the vectors *Du* and P_0', we have

$$Du \cdot P_0' = \|Du\| \left\|P_0'\right\| \beta. \tag{5}$$

Then, substituting Eq. (5) into Eqs. (3) and (4), we have

$$\tau \cdot k = -\frac{Du \cdot \nu}{J_\beta}, \quad J_\beta = \sqrt{1 + \|Du\|^2} \sqrt{1 + \beta^2 \|Du\|^2}.$$

It can be seen that

$$1 + |\beta| \|Du\|^2 \le J_\beta \le 1 + \|Du\|^2, \tag{6}$$

where the inequality on the left side of Eq. (6) is obtained from the Cauchy-Schwarz inequality

$$1 + |\beta| \|Du\|^2 = \langle (1, \|Du\|), (1, |\beta| \|Du\|) \rangle \le J_\beta.$$

Then, because $|\beta| \le 1$, the inequality on the right side of Eq. (6) is obtained. Hence, considering that the vibrations are only in the direction of **u**, normal to the rest position of the membrane, and from Eq. (6), we find that

$$\mathbf{T} \cdot \mathbf{k} \approx -T_0 Du \cdot \nu.$$

Now, let us write the instant equilibrium equation of the portion $G$of the membrane. Assume that the vertical charges in G are given by

$$-\int_{G_0} p(x,t)dx,$$

and the vertical contribution of tension **T** is

$$-\int_{\partial G_0} T_0 Du \cdot \nu d\sigma,$$

where $d\sigma$ is the measurement along the curve ∂G_0. For an acceleration u_{tt}, from Newton's law the force is

$$\int_{G_0} \rho u_{tt} dx.$$

Therefore, $\forall t \in \Re$ and $\forall G_0 \subset \Omega$, we have

$$\int_{G_0} \rho u_{tt} dx = \int_{\partial G_0} T_0 Du \cdot \nu d\sigma + \int_{G_0} p(x,t)dx.$$

Then, by using Green's theorem

$$\int_{\partial G_0} T_0 Du \cdot \nu d\sigma = \int_{G_0} T_0 div(Du) dx,$$

we have

$$\int_{G_0} \left[\rho u_{tt} - T_0 div(Du) - p\right] dx = 0,$$

and because G_0 is arbitrary, we find

$$u_{tt} - c^2 \nabla^2 u = f \qquad \text{in } \Omega \times \Re, \tag{7}$$

where

$$c^2 = \frac{T_0}{\rho} \quad \text{and} \quad f = \frac{p}{\rho}.$$

Here, the initial conditions are given by

$$u(x,0) = \varphi_0(x) \quad x \in \Omega \tag{8}$$

and

$$u_t(x,0) = \varphi_1(x) \quad x \in \Omega \tag{9}$$

and the boundary condition is

$$u(x,t) = 0 \quad x \in \partial\Omega. \tag{10}$$

So, we find that Eqs. (7) and (8) model the small forced oscillations of a membrane.

3. The variable separation method

Consider Ω a domain in the $\Re^2$ plane, and $\partial\Omega$ its contour. Let $L^2(\Omega)$ be the integrable square function space in Ω, that is

$$\varphi \in L^2(\Omega), \quad \text{if} \quad \int_\Omega |\varphi(x,y)|^2 \, dxdy \langle +\infty.$$

In $L^2(\Omega)$, the inner product of two functions φ and ψ can be defined as

$$\langle \varphi, \psi \rangle = \int_\Omega \varphi(x,y)\, \psi(x,y)\, dxdy \tag{11}$$

and the norm of a function

$$\|\varphi\| = \langle \varphi, \psi \rangle^{1/2} = \left(\int_\Omega |\varphi(x,y)|^2 \, dxdy \right)^{1/2}. \tag{12}$$

Now consider the initial and boundary value problem (IBVP)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla u, \quad (x,y,t) \in \Omega \times [0,+\infty)$$

$$u(x,y,0) = \varphi_0(x,y)$$

and

$$\frac{\partial u}{\partial t}(x,y,0) = \varphi_1(x,y), \quad (x,y) \in \Omega$$

$$u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times [0,+\infty).$$

Here $u$is a function in the variables (x,y,t) and $\nabla^2 u$denotes the Laplacian of u on the variables (x,y)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{13}$$

For simplicity we assume that $c = 1$. The operator ∇^2 has a complete set of orthogonal eigenfunctions $\Psi_j(x,y)$, $j = 1, 2, \ldots$ that correspond to the eigenvalues $\left(-\lambda_j^2\right)$ such that the $lim_{j \to +\infty} \lambda_j^2 = +\infty$. Also, orthogonality of the system means that $\langle \Psi_i, \Psi_j \rangle = 0$ for $i \neq j$. Hence, the fact that the system is a complete set means that if $\varphi \in L^2(\Omega)$, then

$$\varphi(x,y) = \sum_{j=1}^{\infty} c_j \Psi_j(x,y), \tag{14}$$

where

$$c_j = \frac{1}{\|\Psi_j\|^2} \langle \varphi, \Psi_j \rangle = \frac{1}{\|\Psi_j\|^2} \int_\Omega \varphi_0(x,y)\, \Psi_j(x,y)\, dxdy.$$

Hence, the function $v_j(t)\, \Psi_j(x,y)$ satisfies the wave equation $\partial^2 u / \partial t^2 = \nabla^2 u$ if and only if

$$\ddot{\underset{j}{v}}(t) + \lambda_j^2 v_j(t) = 0, \tag{15}$$

that is

$$v_j(t) = a_j \cos \lambda_j t + b_j \sin \lambda_j t.$$

Thus, the solution to the IBVP can be written as

$$u(x,y,t) = \sum_{j=1}^{\infty} (a_j \cos \lambda_j t + b_j \sin \lambda_j t)\, \Psi_j(x,y), \tag{16}$$

where a_j and b_j are determined by the initial conditions

$$a_j = \frac{1}{\|\Psi_j\|^2} \langle \varphi_0, \Psi_j \rangle = \frac{1}{\|\Psi_j\|^2} \int_\Omega \varphi_0(x,y)\, \Psi_j(x,y)\, dxdy$$

and

$$b_j = \frac{1}{\|\Psi_j\|^2} \langle \varphi_1, \Psi_j \rangle = \frac{1}{\lambda_j} \frac{1}{\|\Psi_j\|^2} \int_\Omega \varphi_1(x,y)\, \Psi_j(x,y)\, dxdy.$$

The method described above is known as either the *variable separation method* or the *Fourier method*.

3.1. The non-homogeneous equation

The variable separation method described above can be used to solve the non-homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + q(x,y,t), \tag{17}$$

whose solution is expressed as

$$u(x,y,t) = \sum_{j=1}^{\infty} v_j(t)\, \Psi_j(x,y).$$

The functions $v_j(t)$ are determined by the equation

$$\ddot{\underset{j}{v}}(t) + \lambda_j^2 v_j(t) = w_j(t), \tag{18}$$

where

$$w_j(t) = \frac{1}{\|\Psi_j\|^2} \int_\Omega q(x,y,t)\, \Psi_j(x,y)\, dxdy$$

and the initial conditions being

$$v_j(0) = \frac{1}{\|\Psi_j\|^2} \int_\Omega \varphi_0(x,y)\, \Psi_j(x,y)\, dxdy$$

and

$$\dot{\underset{j}{v}}(0) = \frac{1}{\lambda_j} \frac{1}{\left\|\Psi_j\right\|^2} \int\limits_{\Omega} \varphi_1(x,y)\,\Psi_j(x,y)\,dxdy.$$

The function $v_j(t)$ is obtained as follows. Consider $G_j(t)$ as the solution of the homogeneous equation

$$\ddot{\underset{j}{G}}(t) + \lambda_j^2 G_j(t) = 0, \tag{19}$$

with conditions $G_j(0) = 0$ and $\dot{\underset{j}{G}}(0) = 1$. From here it is obtained that

$$v_j(t) = v_j(0)\cos(\lambda_j t) + \frac{\dot{\underset{j}{v}}(0)}{\lambda_j}\sin(\lambda_j t) + \int\limits_0^t G_j(t-s)\,q_j(s)\,ds. \tag{20}$$

Then, Eq. (19) can be easily solved as

$$G_j(t) = \frac{1}{\lambda_j}\sin(\lambda_j t). \tag{21}$$

4. Solutions of the non-homogeneous wave equation

The method described is now employed to solve the wave equation for the case in which a term corresponding to a force applied to a small area of the membrane is introduced. Three different cases are discussed.

4.1. Rectangular membrane

First, consider the case of a rectangular membrane. The non-homogeneous wave equation in this case can be written as

$$v^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + q(x,y,t) = \frac{\partial^2 z}{\partial t^2}, \tag{22}$$

where $q(x,y,t)$ is the external force per mass unit applied on the membrane.

From the variable separation method described earlier, the solution to Eq. (22) has the form

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\,\psi_{mn}(x,y), \tag{23}$$

where, after some algebra, we can obtain

$$u_{mn}(t) = u_{mn}(0)\cos\lambda_{mn}vt + \frac{\overset{\bullet}{\underset{mn}{u}}(0)}{\lambda_{mn}}\sin\lambda_{mn}vt + \int\limits_0^t \frac{w_{mn}(s)}{\lambda_{mn}}\sin\lambda_{mn}vs\,ds, \tag{24}$$

$$u_{mn}(0) = \int\limits_{\Omega} f(x,y)\psi_{mn}(x,y)\,dxdy, \tag{25}$$

$$\overset{\bullet}{\underset{mn}{u}}(0) = \frac{1}{\lambda_{mn}}\int\limits_{\Omega} g(x,y)\,\psi_{mn}(x,y)\,dxdy, \tag{26}$$

$$w_{mn}(s) = \int\limits_{D} \frac{q(x,y,s)}{v^2}\psi_{mn}(x,y)\,dxdy \tag{27}$$

and

$$\psi_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y, \tag{28}$$

where the functions $f(x,y)$ and $g(x,y)$ correspond to the deformation and initial velocity, respectively.

On the other hand, it is not difficult to demonstrate that the oscillation modes for a rectangular membrane with sides a, b, under an external charge M, are given by Ref. 9.

$$p_{nm}^2 = P_{nm}^2\left[1 - \frac{4M}{ab}\int\limits_0^b\int\limits_0^a \frac{1}{\rho}\sin^2\frac{n\pi x}{a}\sin^2\frac{m\pi y}{b}\,dxdy\right], \tag{29}$$

where ρ is the membrane density.

4.2. Circular membrane

The second case considered is a circular membrane. The non-homogeneous wave equation has the form

$$v^2\nabla^2 z + q(r,\theta,t) = \frac{\partial^2 z}{\partial t^2}, \tag{30}$$

where $q(r,\theta,t)$ is the external force per mass unit. The wave propagation velocity is given by $v^2 = T/\rho$, where T is the tension on the membrane and ρ is its density.

A solution of the form $Z = (r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$, subject to the conditions $Z(b,\theta,t) = 0$, $Z(r,\theta,0) = f(r,\theta)$ y $Z_t(r,\theta,0) = g(r,\theta)$, is proposed.

The general solution for Eq. (30) is

$$Z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left[U_{nk}(t)\,\Phi_{nk}(r,\theta) + V_{nk}(t)\,\Psi_{nk}(r,\theta)\right], \tag{31}$$

where

$$\Phi_{nk}(r,\theta) = \sqrt{\frac{2}{\pi}}\frac{1}{bJ_{n+1}(u_{nk})}J_n\left(\frac{u_{nk}}{b}r\right)\cos(n\theta) \tag{32}$$

and

$$\Psi_{nk}(r,\theta) = \sqrt{\frac{2}{\pi}}\frac{1}{bJ_{n+1}(u_{nk})}J_n\left(\frac{u_{nk}}{b}r\right)\sin(n\theta). \tag{33}$$

On the other hand, the functions $U_{nk}(t)$ are given by

$$U_{nk}(t) = U_{nk}(0)\cos v\lambda_{nk}t + \frac{\overset{\bullet}{U}_{nk}(0)}{\lambda_{nk}}\sin v\lambda_{nk}t + \int_0^t \frac{w_{nk}(s)}{\lambda_{nk}}\sin v\lambda_{nk}s ds, \quad (34)$$

where

$$U_{nk}(0) = \int_\Omega f(r,\theta)\,\Phi_{nk}(r,\theta)\,rdrd\theta, \quad (35)$$

$$\overset{\bullet}{U}_{nk}(0) = \frac{1}{\lambda_{nk}}\int_\Omega g(r,\theta)\,\Phi_{nk}(r,\theta)\,rdrd\theta \quad (36)$$

and

$$w_{nk}(s) = \int_D q(r,\theta,s)\,\Phi_{nk}(r,\theta)\,rdrd\theta. \quad (37)$$

Similarly, the functions $V_{nk}(t)$ are given by

$$V_{nk}(t) = V_{nk}(0)\cos v\lambda_{nk}t + \frac{\overset{\bullet}{V}_{nk}(0)}{\lambda_{nk}}\sin v\lambda_{nk}t + \int_0^t \frac{w_{nk}(s)}{\lambda_{nk}}\sin v\lambda_{nk}s ds, \quad (38)$$

where

$$V_{nk}(0) = \int_\Omega f(r,\theta)\,\Psi_{nk}(r,\theta)\,rdrd\theta, \quad (39)$$

$$\overset{\bullet}{V}_{nk}(0) = \frac{1}{\lambda_{nk}}\int_\Omega g(r,\theta)\,\Psi_{nk}(r,\theta)\,rdrd\theta \quad (40)$$

and

$$w_{nk}(s) = \int_D q(r,\theta,s)\,\Psi_{nk}(r,\theta)\,rdrd\theta. \quad (41)$$

Also, the oscillation modes for a circular membrane with radius a vibrating under an external charge M, are given by

$$p_k^2 = \lambda_k^2 c^2\left[1 - \frac{M}{\rho\pi a^2}\frac{J_0^2(\lambda_k r')}{J_1^2(\lambda_k a)}\right]. \quad (42)$$

4.3. Elliptical membrane

The third case analyzed corresponds to an elliptical membrane. The Mathieu equation is used for the analysis of this problem. Different notations appear in the literature for the Mathieu equation. Here, the notation we used has the form

$$\frac{2c^2}{\rho^2(\cosh 2u - \cos 2v)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) + q(u,v,t) = \frac{\partial^2 z}{\partial t^2}. \quad (43)$$

The solution is expressed in terms of the Mathieu functions

$$z(u,v,t) = \sum \Phi(t)\,\psi_{mn}(u,v), \quad (44)$$

where

$$\psi_{m,n}(u,v) = ce_r(v,q)\,Ce_r(v,q) + se_r(u,q)\,Se_r(u,q), \quad (45)$$

$$\Phi(t) = \Phi(0)\cos\omega t + \frac{\overset{\bullet}{\Phi}(0)}{\lambda_{mn}}\sin\omega t + \int_0^t \frac{w_{mn}(s)}{\lambda_{mn}}\sin\omega s ds, \quad (46)$$

$$\Phi(0) = \int_\Omega f(u,v)\,\psi_{mn}(u,v)\,d\Omega, \quad (47)$$

$$\overset{\bullet}{\Phi}(0) = \frac{1}{\lambda_{mn}}\int_\Omega g(u,v)\,\psi_{mn}(u,v)\,d\Omega \quad (48)$$

and

$$w_{mn}(s) = \int \frac{q(u,v,s)}{c^2}\psi_{mn}(u,v)\,d\Omega. \quad (49)$$

5. Computational simulation

Just as a guide, a piece of MAPLE code for the simulation of the vibrational modes of a circular membrane under an external force, is presented next.

The case analyzed corresponds to a circular membrane with mass m, density ρ, and radius b, where an external force $w0$ acts on a circular region of radius e, with $b \gg e$.

```
> #Define a compact notation for Bessel J function and its zeroes
> alias(J=BesselJ,j=BesselJZeros):
> #Define relevant constants and the function omega
> b:=0.5: c:=sqrt(T/rho): rho:=1.2: T:=w0: m:=0.007: M:=0.02: e:=0.08: r1:=(b/2)-e: r2:=r1+2*e: g:=981: w0:=M*g: f:=r->r:
theta0:=arctan(e/(r1+e));
> omega:=n->k(n)*c;
> k:=n->j(0,n)/b;
> #Evalue constants of integration
> A:=evalf(Int(r*f(r)*J(0,j(0,1)*r/b),r=0..b));
B:=evalf(Int(r*J(0,j(0,1)*r/b),r=0..e));
> #Define the function z to plot in cylindrical coordinates
```

```
> z1:=n->((4*J(0,omega(n)*r/c))/(b 2*J(1,omega(n)*b/c) 2))*(A*cos(omega(n)*t)+(w0*B/(m*j(0,1) 2*c 2))*(cos(omega(n)*t)-1)
    *(Heaviside(r-r1)*Heaviside(r2-r)*Heaviside(theta0-theta)));
> #Plot the animation for each node of vibration (n)
> with(plots):
> animate3d([r,theta,z1(1)], r=0..b, theta=0..2*Pi,
> t=0..2*Pi/omega(1), coords=cylindrical,frames=50);
```

Figure 1 shows the evolution of the first vibrational mode for a circular membrane. It can be seen that the oscillations in the region under an external charge have much higher amplitudes than the other parts. The animation for other different modes is easily performed by substituting in the last two lines of the code the functions $z1(1)$ and *omega*(1) by $z1(n)$ and *omega*(n), respectively.

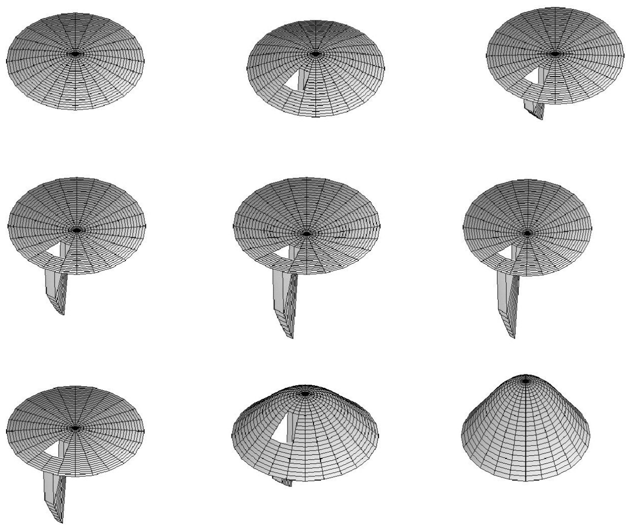

FIGURE 1. Evolution of the first vibrational mode for a circular membrane under an external force, obtained using a MAPLE code.

Finally, for the cases of both rectangular and elliptical membranes, the only changes needed in the numerical code are the corresponding coordinate systems.

6. Conclusions

The theoretical solutions of the non-homogeneous wave equations for membranes of three different shapes, that is, rectangular, circular, and elliptical, have been considered. Also, a simple numerical simulation using a MAPLE code has been provided to study the vibrational modes of a circular membrane under an applied external force. The numerical code developed can be easily adapted for studying membranes of different shapes. The results obtained could be useful for the understanding of the oscillations of membranes in different applications, especially to analyze experimental results recorded by adding a charge on the membrane. Also, the method could be used as a help in teaching.

Acknowledgments

E. Alvarado would like to thank E. Ley-Koo for many fruitful discussions. Also M. Sosa would like to thank G. López for his valuable comments. This work was supported by CONACyT, under grant No. 38749-E.

1. P.M. Morse and H. Feshbach, *Methods of theoretical physics* (McGraw-Hill, New York, 1953).
2. J. Gutiérrez-Vega, S. Chávez-Cerda, and R. Rodríguez-Dagnino, *Rev. Mex. Fis.* **45** (1999) 613.
3. R. Portugal, L. Golebiowski, and D. Frenkel, *Am. J. Phys.* **67** (1999) 534.
4. A.I. Oliva, D.C. Valdés, E. Ley-Koo, and H.G. Riveros, *Rev. Mex. Fis.* **49** (2003) 391.
5. L. Ruby, *Am. J. Phys.* **64** (1996) 39.
6. L. Chaos-Cador and E. Ley-Koo, *Rev. Mex. Fis.* **48** (2002) 67.
7. W.L.C. Rutten *et al.*, *Cryogenics* **22** (1982) 457.
8. M. Sosa, A.A.O. Carneiro, O. Baffa, and J.F. Colafemina, *Rev. Sci. Instr.* **73** (2002) 3695.
9. J.W.S. Rayleigh, *The theory of sound* (Dover Publications, New York, 1945).