Normal and anti-normal ordered expressions for annihilation and creation operators

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We obtain normal and anti-normal order expressions of the number operator to the power k by using the commutation relation between the annihilation and creation operators. We use those expressions to obtain general formulae for functions of the number operator in normal and anti-normal order.

Keywords: Normal ordering; Fock state; Stirling numbers.

Obtenemos expresiones para las formas normales y antinormales del operador de número a la k-ésima potencia, usando las relaciones de conmutación entre los operadores de creación y aniquilación. Usamos estas expresiones para obtener fórmulas de funciones del operador de número en forma normal y antinormal.

Descriptores: Ordenamiento normal; Estado de Fock; número de Stirling.

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1. Introduction

In order to solve some problems in quantum mechanics, it is needed to calculate function of the operator $\hat{n} = \hat{a}^\dagger \hat{a}$, where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators of the harmonic oscillator, respectively. For instance in ion traps [1], it is usual to associate Laguerre polynomials of order $\hat{n}$ [2].

Very recently, Fujii and Suzuki have shown ordering expressions for $\hat{n}^k$ as different types of polynomials with respect to the number operator [3]. They have shown nontrivial relations including the use of Stirling numbers of the first kind [4].

Here we, do the opposite: we obtain an expression for $\hat{n}^k$ in normal order (the antinormal order is then straightforward, as it will be given in terms of similar coefficients, [5]), *i.e.* a sum of coefficients multiplying normal ordered forms of $\hat{a}$ and $\hat{a}^\dagger$. This enables us to obtain an expression for the normal ordered form of a function of the operator $\hat{n}$, and demonstrate, as a particular example a lemma in Louissel's book for the exponential of the number operator [6].

2. Normal ordering

One may use commutation relations of the annihilation and creation operators to obtain the powers of $\hat{n}$ in normal, anti-normal, or symmetric order [6]. For instance, we can express $\hat{n}^k$ in normal order, for $k = 2$ as

$$\hat{n}^2 = [\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{1}$$

for $k = 3$ as

$$\hat{n}^3 = [\hat{a}^\dagger]^3 \hat{a}^3 + 3[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{2}$$

and for $k = 4$

$$\hat{n}^4 = [\hat{a}^\dagger]^4 \hat{a}^4 + 6[\hat{a}^\dagger]^3 \hat{a}^3 + 7[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{3}$$

where the coefficients multiplying the different powers of the normal ordered operators do not show an obvious form to be determined. In writing the above equations, we have used repeatedly the commutator $[\hat{a}, \hat{a}^\dagger] = 1$. In Table I, we generate a list of such coefficients using the Mathematica program and a chart of the Stirling numbers of the second kind. From the table, we infer that the coefficients in the above equations are precisely these numbers (see also Ref. 7), *i.e.* we obtain

$$\hat{n}^k = \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m, \tag{4}$$

with [4]

$$S_k^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \frac{m!}{j!(m-j)!} j^k. \tag{5}$$

We now write a function of $\hat{n}$ in a Taylor series as

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \hat{n}^k, \tag{6}$$

and inserting (4) in this equation, we obtain

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m. \tag{7}$$

TABLE I. A program written in Mathematica to find the coefficients for the powers of the number operator in normal order form.

```
In[1]:=
    SetAttributes[prod, {Flat, OneIdentity}]
    prod[a___, b_Plus, c___]
    prod[a___, 1, c___] :=prod[a,#,c]& /@ b
    s[n_Integer?positive] := prod @@ Flatten[Table[{Ad,A}, {n}]]
In[6]:=
    AdA[n_]:=prod @@ Join[Table[Ad,{n}], Table[A,{n}]]
    [5]
In[7]:=
    p[a___]:=x^Length[{a}]
    sx[n_]:= /. prod -> p
    sx[6]
In[10]:=
     2       4       6      8        10    12
    x  + 31 x  + 90 x  + 65 x  + 15 x   + x
In[11]:=
    c[n_]:=CoefficientList[s[n] /. x ->Sqrt[y],y]
    Table[c[n], {n,8}] //TableForm
In[12]//TableForm=
    0   1
    0   1   1
    0   1   3     1
    0   1   7     6     1
    0   1   15    25    10     1
    0   1   31    90    65     15     1
    0   1   63    301   350    140    21    1
    0   1   127   966   1701   1050   266   28   1
```

Because $S_k^{(m)} = 0$ for $m > k$, we can take the second sum in Eq. (7) to infinite and interchange the sums to have

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}. \tag{8}$$

For the same reason stated above, we may start the second sum at $k = m$,

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=m}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}. \tag{9}$$

By noting that

$$\frac{\Delta^m f(x)}{m!} = \sum_{k=m}^{\infty} \frac{f^{(k)}(x)}{k!} S_k^{(m)}, \tag{10}$$

where Δ is the difference operator, defined as [4]

$$\Delta^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} f(x+k), \tag{11}$$

we may write (9) as

$$f(\hat{n}) = \sum_{m=0}^{\infty} \frac{[\hat{a}^\dagger]^m \hat{a}^m \Delta^m}{m!} f(0) \equiv : e^{\Delta \hat{n}} : f(0) \tag{12}$$

where $:\hat{n}:$ stands for normal order.

Lemma 1 *If we choose the function $f(\hat{n}) = \exp(-\gamma\hat{n})$, we have that*

$$\Delta^m f(0) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} e^{-\gamma k}, \tag{13}$$

and then, we obtain the well-known lemma [6]

$$e^{-\gamma\hat{n}} =: e^{(e^{-\gamma}-1)\hat{n}} : . \tag{14}$$

3. Anti-normal ordering

Following the procedure introduced in the former section, we can write $\hat{n}^k$ in anti-normal order as

$$\hat{n}^k = (-1)^k \sum_{m=0}^{k} (-1)^m S_{k+1}^{(m+1)} \hat{a}^m [\hat{a}^\dagger]^m, \qquad (15)$$

and a function of the number operator as

$$f(\hat{n}) = \sum_{m=0}^{\infty} (-1)^m \hat{a}^m [\hat{a}^\dagger]^m \times \sum_{k=m}^{\infty} (-1)^k \frac{f^{(k)}(0)}{k!} S_{k+1}^{(m+1)}. \qquad (16)$$

The second sum differs from Eq. (10) in the extra $(-1)^k$, and the parameters of the Stirling numbers. We can define $u = -x$, such that $f^{(k)}(x)_{x=0} = (-1)^k f^{(k)}(u)_{u=0}$, and use the identity [4]

$$S_{k+1}^{(m+1)} = (m+1) S_k^{(m+1)} + S_k^{(m)} \qquad (17)$$

to write

$$f(\hat{n}) = \sum_{m=0}^{\infty} (-1)^m \hat{a}^m [\hat{a}^\dagger]^m \times \left((m+1) \sum_{k=m}^{\infty} \frac{f^{(k)}(u=0)}{k!} S_k^{(m+1)} + \sum_{k=m}^{\infty} \frac{f^{(k)}(u=0)}{k!} S_k^{(m)} \right), \qquad (18)$$

so we can use again Eq. (10) to finally write

$$f(\hat{n}) = (1+\Delta) \vdots e^{-\Delta \hat{n}} \vdots f(0) \qquad (19)$$

where $\vdots \hat{n} \vdots$ stands for the anti-normal order.

Lemma 2 *Let us consider again the function*

$$f(\hat{n}) = \exp(-\gamma \hat{n}).$$

This gives us that $f(x) = e^{-\gamma x}$, and $f(u) = e^{\gamma u}$. Therefore

$$\Delta^m f(u=0) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} e^{\gamma k} = (e^\gamma - 1)^m, \qquad (20)$$

such that we can obtain the exponential of the number operator in anti-normal order (lemma) as

$$e^{-\gamma \hat{n}} = e^\gamma \vdots e^{(1-e^\gamma)\hat{n}} \vdots. \qquad (21)$$

3.1. Coherent states

Let us use Eq. (21) to find averages for coherent states, $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ is the so-called displacement operator, and $|0\rangle$ is the vacuum state:

$$\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^\gamma \langle \alpha | \sum_{m=0}^{\infty} \frac{(1-e^\gamma)^m}{m!} \hat{a}^m [\hat{a}^\dagger]^m | \alpha \rangle, \qquad (22)$$

by using that

$$\langle \alpha | \hat{a}^m [\hat{a}^\dagger]^m | \alpha \rangle = \langle 0 | (\hat{a} + \alpha)^m (\hat{a}^\dagger + \alpha^*)^m | 0 \rangle = \sum_{k=0}^{m} |\alpha|^{2k} \left(\frac{m!}{(m-k)!k!} \right)^2 (m-k)! \qquad (23)$$

we may write

$$\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^\gamma \sum_{m=0}^{\infty} (1-e^\gamma)^m L_m(-|\alpha|^2), \qquad (24)$$

where $L_m(x)$ are the Laguerre polynomials of order m. We can finally write a closed expression for the sum above [8] to obtain the expected result for coherent states

$$\langle \alpha | e^{-\gamma \hat{n}} | \alpha \rangle = e^{|\alpha|^2 (e^{-\gamma} - 1)}. \qquad (25)$$

3.2. Fock states.

For Fock or number states we obtain

$$\langle n | e^{-\gamma \hat{n}} | n \rangle = e^\gamma \langle n | \sum_{m=0}^{\infty} \frac{(1-e^\gamma)^m}{m!} \hat{a}^m [\hat{a}^\dagger]^m | n \rangle = e^\gamma \sum_{m=0}^{\infty} \frac{(1-e^\gamma)^m}{m!} \frac{(m+n)!}{n!} \qquad (26)$$

rearranging the sum above with $k = n + m$ we have

$$\langle n | e^{-\gamma \hat{n}} | n \rangle = e^\gamma \sum_{k=n}^{\infty} (1-e^\gamma)^{k-n} \frac{k!}{n!(k-n)!} \qquad (27)$$

which has a closed expression, as

$$\sum_{k=n}^{\infty} x^{k-n} \frac{k!}{n!(k-n)!} = (1-x)^{-n-1}$$

see Ref. 4:

$$\langle n | e^{-\gamma \hat{n}} | n \rangle = e^{-\gamma n} \qquad (28)$$

4. Conclusions

In conclusion, we have written the normal and anti-normal order expressions of $\hat{n}^k$ by using the commutation relation between the annihilation and creation operators. The coefficients for such expressions are the Stirling numbers of the second kind [7]. We then have used the difference operator to write a function (that may be developed in Taylor series) of the number operator in normal and anti-normal order, showing consistency with the particular case of the exponential function lemma in normal order.

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