Maxwell equations in Lorentz covariant integral form

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Most textbooks of electromagnetism give comparable weights to the presentation of Maxwell equations in their integral and differential forms. The same books, when dealing with the Lorentz covariance of the Maxwell equations, limit themselves to the discussion of their differential forms, and make no reference to their integral forms. Such a gap in the didactic literature is bridged in this paper by explicitly constructing the latter via the integration of the former, for the source-dependent and source-independent cases, over a four-vector and a rank-3 tensor hypersurfaces, respectively.

Keywords: Electromagnetismo; Maxwell equations; integral and differential forms; standard and Lorentz-covariant forms.

La mayoría de los textos de electromagnetismo dan pesos comparables a la presentación de las ecuaciones de Maxwell en sus formas integrales y diferenciales. Los mismos libros, al tratar la covariancia de Lorentz de las ecuaciones de Maxwell, se limitan a la discusión de sus formas diferenciales, y no hacen referencia a sus formas integrales. Tal laguna en la literatura didáctica se elimina en este artículo, construyendo explícitamente las últimas por medio de la integración de las primeras, para los casos dependientes e independientes de las fuentes, sobre hipersuperficies cuadrivectorial y tensorial de rango 3, respectivamente.

Descriptores: Electromgnetismo; ecuaciones de Maxwell; formas integrales y diferenciales; formas estándar y covariantes de Lorentz.

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1. Introduction

Introductory [1–3], intermediate [4, 5], advanced [6–8] and graduate [9–12] books on electromagnetism usually present Maxwell equations in their integral forms first, and then make use of them and of the Gauss and Stoke theorems to obtain their differential forms. The balanced presentation of both forms at this initial stage can be contrasted with the one sided discussions of the Lorentz covariant differential forms of Maxwell equations in the same texts, in which the integral forms are practically absent. The purpose of this work is to provide a balance in the study of the Lorentz covariance of Maxwell equations in both forms, filling in an obvious gap in the books.

Section 2 contains a brief review of Maxwell equations, the connections between their integral and differential standard forms, and their Lorentz covariant differential forms, with emphasis on their physical contents and the mathematical arguments to go between their various forms. Section 3 presents the Lorentz covariant integral forms of Maxwell equations, constructed by integrating the corresponding differential forms over a four-vector and a rank-3 tensor hypersurfaces for the source-dependent and source-independent cases, respectively. Section 4 contains a discussion of the additional physical insights, including important relativistic effects, that follow from the connections between the Lorentz covariant and standard integral forms of Maxwell equations.

2. Maxwell equations in standard and Lorentz covariant differential forms

Maxwell equations are the mathematical expressions of the laws of electromagnetism. These laws are described in words first, and their successive mathematical forms are presented next.

Gauss electric law: Electric charges are sources of electric flux.

Ampére-Maxwell law: Electric currents and displacement currents are sources of magnetic circulation.

Gauss magnetic law: There are no magnetic monopoles as sources of magnetic flux.

Faraday electromagnetic induction law: The time rate of change of magnetic flux is a source of electric circulation .

The first two laws are identified as the source-dependent laws and the last two as the source-independent laws. The former are consistent with the law of conservation of electric charge, while the latter make the description of the electromangetic phenomena possible in terms of potentials, as shown later on.

Now we proceed to express the same laws in the form of Maxwell equations in the standard integral forms:

$$\oint_S \vec{E}(\vec{r},t)\cdot d\vec{a} = 4\pi Q(t) \tag{1}$$

$$\oint_C \vec{B}(\vec{r},t)\cdot d\vec{l} = \frac{4\pi}{c} I(t) + \frac{1}{c}\frac{d}{dt}\int_S \vec{E}(\vec{r},t)\cdot d\vec{a} \tag{2}$$

$$\oint_S \vec{B}(\vec{r},t)\cdot d\vec{a} = 0 \tag{3}$$

$$\oint_C \vec{E}(\vec{r},t)\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}(\vec{r},t)\cdot d\vec{a} \tag{4}$$

Here the surface integrals over closed surfaces S of Eqs. (1) and (3), of the respective electric intensity field $\vec{E}$

and magnetic induction field $\vec{B}$, correspond to the mathematical flux integrals. In Eq. (1), Q represents the electric charge within the volume limited by the closed surface of integration, while in the magnetic case, Eq. (3), the corresponding term is zero due to the non existence of magnetic monopoles. The line integrals over closed curves C of Eqs. (2) and (4), correspond to circulation integrals for the respective fields. In Eq. (2), I represents the intensity of the electric current crossing the open surface limited by the curve C, while the displacement current is associated with the time rate of change of the electric flux across the same open surface. Similarly, in Eq. (4), the right hand side dependends on the time rate of change of the magnetic flux across the open surface S limited by the curve C. Also, the positive signs on the r.h.s. of Eq. (2) are consistent with Ampere's "right hand" law determining the relative direction of the currents and the magnetic induction field, while the negative sign on the r.h.s. of Eq. (4) is required by Lenz's "left hand" law, in order to guarantee energy conservation

In order to go from the integral forms of Maxwell's equations (1)- (4) to the respective differential forms, use is made of the Gauss and Stokes theorems for any vector field $V(\vec{r},t)$:

$$\oint_S \vec{V}(\vec{r},t)\cdot d\vec{a} = \int_V \nabla\cdot\vec{V}(\vec{r},t)d\tau \tag{5}$$

$$\oint_C \vec{V}(\vec{r},t)\cdot d\vec{l} = \int_S \nabla\times\vec{V}(\vec{r},t)\cdot d\vec{a}, \tag{6}$$

also known as the flux (or divergence) and circulation (or curl) theorems.

The first one gives the flux integral as a volume integral of the divergence derivative of the field, $\nabla\cdot\vec{V}$, while the second expresses the circulation integral as the surface integral of the curl derivative of the field, $\nabla\times\vec{V}$. The point limits of Eqs. (5) and (6) lead to the geometrical interpretation of such derivatives:

$$\nabla\cdot\vec{V} = \lim_{\Delta\tau\to 0}\frac{\oint_S \vec{V}(\vec{r},t)\cdot d\vec{a}}{\Delta\tau} \tag{7}$$

$$(\nabla\times\vec{V})\cdot\hat{n} = \lim_{\Delta a\to 0}\frac{\oint_C \vec{V}(\vec{r},t)\cdot d\vec{l}}{\Delta a} \tag{8}$$

as the flux per unit volume and the circulation per unit area, respectively.

Therefore, the standard differential forms of Maxwell's equations follow immediately from Eqs. (1)- (4) and (7)- (8):

$$\nabla\cdot\vec{E}(\vec{r},t) = 4\pi\rho(\vec{r},t) \tag{9}$$

$$\nabla\times\vec{B}(\vec{r},t) = \frac{4\pi}{c}\vec{J}(\vec{r},t) + \frac{1}{c}\frac{\partial\vec{E}(\vec{r},t)}{\partial t} \tag{10}$$

$$\nabla\cdot\vec{B}(\vec{r},t) = 0 \tag{11}$$

$$\nabla\times\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial\vec{B}(\vec{r},t)}{\partial t}, \tag{12}$$

where $\rho(\vec{r},t) = \Delta Q/\Delta\tau$ is the electric charge volume density and $\vec{J}\cdot\hat{n} = \Delta I/\Delta a$ is the electric current density.

When the divergences of both sides of Eq. (10) are evaluated, the l.h.s. vanishes, and the divergence of the electric intensity field on the r.h.s. can be substituted by its value from Eq. (9) with the final result

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0, \tag{13}$$

which is the continuity equation expressing the conservation of electric charge.

The solenoidal character of the magnetic induction field as expressed by Gauss law, Eq. (11), is immediately satisfied when written as

$$\vec{B} = \nabla\times\vec{A} \tag{14}$$

in terms of the vector potential $\vec{A}$. Substitution of Eq. (14) in (12) allows us, in turn, to write

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{15}$$

in terms of the scalar potential ϕ. Equations (14) and (15) are solutions of the differential Eqs. (11) and (12), but the potentials $\vec{A}$ and ϕ are not uniquely defined. We are free to change them via the so-called gauge transformations:

$$\vec{A}\to\vec{A}+\nabla\chi \tag{16}$$

and

$$\phi\to\phi-\frac{\partial\chi}{c\partial t}, \tag{17}$$

leaving the force fields $\vec{B}$ and $\vec{E}$ with the same values, or invariant.

The differential equations satisfied by the potentials are obtained by substituting Eqs. (14) and (15) in the source-dependent Maxwell's equations (9) and (10). The resulting equations,

$$-\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = 4\pi\rho \tag{18}$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}, \tag{19}$$

are coupled equations in both potentials. They can be uncoupled by choosing the Lorentz gauge

$$\nabla\cdot\vec{A} + \frac{\partial\phi}{c\partial t} = 0, \tag{20}$$

with the result that both potentials obey the inhomogeneous wave equations

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{21}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}, \tag{22}$$

with the electric charge density ρ and the electric current density $\vec{J}$ as their respective sources. Notice that for the time independent situations, Eqs. (21) and (22 become the Poisson equations of electrostatics and magnetostatics, respectively; while the Lorentz gauge of Eq. (20) reduces to the transverse gauge.

Now we are in a position to identify the tensorial characteristics of the sources, potentials and force fields under Lorentz transformations, by writing their respective equations in obviously covariant forms. The starting point is to introduce the four-vector $x_\mu(x, y, z, ict)$ defining the space-time position of each event, where $\mu = 1, 2, 3, 4$. Then the space-time displacement between two neighboring events is given by $\Delta x_\mu(\Delta x, \Delta y, \Delta z, ic\Delta t)$. Correspondingly, the space-time rate of change involves the four-vector "directional" derivative $\partial/\partial x_\mu(\partial/\partial x, \partial/\partial y, \partial/\partial z, \partial/ic\partial t)$. Four-scalar quantities can be constructed by contracting four-vectors:

$$\sum_{\mu=1}^{4} \Delta x_\mu \Delta x_\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \tag{23}$$

is the square of space-time interval, and

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}, \tag{24}$$

is the wave operator or D'Alambertian involved in Eqs. (21) and (22). Notice the importance of the presence of the imaginary unit i in the fourth or time component in the four-vectors, which translates into the negative sign in Eqs. (23) and (24), as required by the Minkowski metric. Einstein's summation convention consists in dropping the summation sign in Eqs. (23) and (24) and simply summing the successive terms over the repeated index as $\mu = 1, 2, 3, 4$; this convention and some symbols are used in the following:

$$\Delta x_\mu \Delta x_\mu = (\Delta\sigma)^2 = -c^2 (\Delta\tau)^2, \tag{25}$$

where $\Delta\sigma$ is the norm of the space-time interval and $\Delta\tau$ is the proper time, and

$$\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu} = \Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tag{26}$$

is the D'Alambertian, the natural space-time extension of the Laplacian operator.

Einstein's special relativity principle states that all laws of physics are valid in any inertial frame of reference. Lorentz covariance implements this principle by writing the laws of physics in terms of tensor equations, thus guaranteeing that they keep the same form when changing from one frame to another. The implementation for the laws of electromagnetism is carried out next.

We start out with the conservation of charge, Eq. (13), in which the presence of the components of $\partial/\partial x_\mu$ is recognized. If the zero on the r.h.s. is to be the same in all inertial frames, the reasonable and simplest assumption is that it is a scalar, which suggests that the current and charge densities must be the components of a four-vector $J_\mu(\vec{J}, ic\rho)$. Then the four-vector form of Eq. (13) becomes

$$\frac{\partial J_\mu}{\partial x_\mu} = 0. \tag{27}$$

We continue with the Lorentz gauge condition of Eq. (20), which by the same reasoning of the previous paragraph takes the form

$$\frac{\partial A_\mu}{\partial x_\mu} = 0, \tag{28}$$

and allows the identification of the four-vector potential $A_\mu(\vec{A}, i\phi)$.

The presence of the D'Alambertian in Eqs. (21) and (22) had already been pointed out, and they become a single four-vector equation connecting the four-vector potentials and sources:

$$\Box^2 A_\mu = -\frac{4\pi}{c} J_\mu. \tag{29}$$

Equations (14) and (15) are combined in a single antisymmetric second rank tensor equation,

$$f_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \tag{30}$$

expressing the connection between the force field and the derivatives of the potentials.

Comparison of the components of Eq. (30), for μ=$1, 2, 3, 4$ and ν=$1, 2, 3, 4$, with those of Eqs. (14) and (15) leads to the identification of the electromagnetic field antisymmetric tensor

$$f_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}, \tag{31}$$

where the space-space components correspond to the magnetic field and the space-time and time-space components correspond to the electric field.

In turn, the source dependent Maxwell's equations (9) and (10) become a single four-vector equation

$$\frac{\partial f_{\mu\nu}}{\partial x_\mu} = -\frac{4\pi}{c} J_\nu \tag{32}$$

involving the four-divergence of the field tensor.

Similarly, the source-independent Maxwell's equations (11) and (12) are combined in a rank-3 tensor equation

$$\frac{\partial f_{\mu\nu}}{\partial x_\lambda} + \frac{\partial f_{\nu\lambda}}{\partial x_\mu} + \frac{\partial f_{\lambda\mu}}{\partial x_\nu} = 0 \tag{33}$$

involving the simmetrized combination of four-gradients of the antisymmetric field tensor.

The reader can verify that Eq. (32) corresponds to Eq. (9) for $\nu = 4$, and to Eq.(10) for $\nu = 1, 2, 3$; and Eq. (33) to Eq. (11) for $\lambda, \mu, \nu = P(1, 2, 3)$, to Eq. (12) for

$$\lambda, \mu, \nu = P(1, 2, 4), P(3, 1, 4)$$

and $P(2, 3, 4)$, where P symbolizes the permutations of the indices, and to the identity $0 = 0$ for all other combination of the tensor indices.

Thus the presentation of Maxwell equations in their standard integral forms, Eqs. (1)-(4), standard differential forms, Eqs. (9)-(12), and Lorentz covariant differential forms, Eqs. (32)-(33), is completed.

This section can be concluded by pointing out how we can go back and forth between the successive forms. While the steps from the standard integral forms to the standard differential forms have already been described by using the bridges of Eqs. (5)-(8), the return trip from Eqs. (9)-(12) to Eqs. (1) and (4) uses the same bridges after integrating the divergence Gauss laws over a finite volume, and the curl Ampere-Maxwell and Faraday's laws over an open surface. The steps back and forth between the sets of Eqs. (9)-(12) and (32)-(33) have already been described in the previous paragraph.

3. Maxwell equations in Lorentz covariant integral form

The absence of the Lorentz invariant integral form of Maxwell equations in the textbooks is intentionally mirrored in the previous section. Here we simply continue with the obvious task of constructing such integral forms by integrating the differential forms of Eqs. (32) and (33) over the appropriate domains. In the process the circle is completed by showing the connections between the Lorentz covariant and standard integral forms.

The key elements for the construction are the tensorial natures of the equations to be integrated and of the elements of integration, which must be properly matched in order to guarantee the return to Eqs. (1) and (4). It was already recognized that Eq. (32) is a four-vector equation and Eq. (33) is a rank-3 tensor equation. Each of them includes one standard divergence and one standard curl equation, which upon integration over a three-dimensional volume and a two-dimensional surface, respectively, become the standard flux and circulation integral forms of the corresponding Maxwell equations. The elements of integration of the Lorentz covariant differential Eqs. (32) and (33) must include the volume and surface elements in the respective standard integral forms.

On the other hand, the choices of elements of integration in the four-dimensional space time include a four-vector line element dx_μ, a rank-2 tensor surface element $dx_\mu dx_\nu$, a rank-3 tensor hypersurface element $dx_\lambda dx_\mu dx_\nu$, a four vector hypersurface element $d^3x_\mu(dydzcdt,\ dxdzcdt, dxdycdt,\ idxdydz)$, and a scalar hypervolume element $d^4x = dx_1dx_2dx_3dx_4$. The need for integration over both volume and surface elements in the standard forms automatically excludes the line and surface elements in the covariant case. From the remaining elements, the three-dimensional hypersurfaces satisfy the conditions of including the 3-D standard volume and the 2-D standard surface elements involved in the standard Maxwell equations (1)-(4). Thus, they are the natural candidates to be chosen as the domains of integration of Eqs. (32) and (33).

The integration of Eq. (32) over the four-vector hypersurface leads to the rank-2 tensor equation

$$\int \frac{\partial f_{\mu\nu}}{\partial x_\mu} d^3x_\lambda = -\frac{4\pi}{c} \int J_\nu d^3x_\lambda. \tag{34}$$

Let us analyze some of its components in order to identify those that are connected with Eqs. (1)-(2). We start with the time-time component $\nu = 4, \lambda = 4$,

$$\int \left(\frac{\partial f_{14}}{\partial x_1} + \frac{\partial f_{24}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_3} + \frac{\partial f_{44}}{\partial x_4} \right) d^3x_4 = -\frac{4\pi}{c} \int J_4 d^3x_4. \tag{35}$$

When the space-time components of the field-tensor, Eq. (28), and the explicit forms of the integration element and the value $J_4 = ic\rho$ are used, the result is

$$\int\int\int \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dxdydz = \frac{4\pi}{c} \int \rho dxdydz, \tag{36}$$

which is identified as the electric Gauss law of Eq. (1).

Next, we write one of the diagonal space-space components of Eq. (31), $\nu = \lambda = 1$, with the result

$$-\int dt \int\int (\nabla \times \vec{B})_x dydz + \int dt \int\int \frac{\partial E_x}{c\partial t} dydz = -\frac{4\pi}{c} \int dt \int\int J_x dydz, \tag{37}$$

and similarly for $\nu = \lambda = 2$ and $\nu = \lambda = 3$. When the three diagonal space-space component equations are added, the integrands of the time integration are connected by the Ampére-Maxwell law, Eq. (2).

The conclusion is that Eq. (34) corresponds to the Lorentz covariant integral form of the source-dependent Maxwell equations, with the identifications of its time-time component as the Gauss law, Eq. (1), and of the trace of its space part as the Ampére-Maxwell law, Eq. (2), before the common time integration in Eq. (37).

Similarly, the integration of Eq. (33) over the rank-3 tensor hypersurface allows us to write the rank-6 tensor equation

$$\int\int\int \left(\frac{\partial f_{\mu\nu}}{\partial x_\lambda} + \frac{\partial f_{\nu\lambda}}{\partial x_\mu} + \frac{\partial f_{\lambda\mu}}{\partial x_\nu} \right) dx_\alpha dx_\beta dx_\gamma = 0. \tag{38}$$

The only combinations of the tensor indices in the integrand have already been identified in the paragraph after Eq. (33);

and the tensor indices of the integration element must match them $\alpha, \beta, \gamma = P(\lambda, \mu, \nu)$ in order to connect with the standard form of the source-independent Maxwell equations (3) and (4). In fact, the space-space-space component becomes

$$\int\int\int \left(\frac{\partial f_{23}}{\partial x_1} + \frac{\partial f_{31}}{\partial x_2} + \frac{\partial f_{12}}{\partial x_3}\right) dx_1 dx_2 dx_3 = 0, \quad (39)$$

and, in terms of the magnetic field components,

$$\int\int\int \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx dy dz = 0, \quad (40)$$

which is recognized as the magnetic Gauss law of Eq. (3).

In the same way, the space-space-time components, $P(i, j, 4)P(i, j, 4)$ of Eq. (38) with $i \neq j$, lead to the form

$$\int\int\int \left(\frac{\partial f_{j4}}{\partial x_i} + \frac{\partial f_{4i}}{\partial x_j} + \frac{\partial f_{ij}}{\partial x_4}\right) dx_i dx_j dx_4 = 0. \quad (41)$$

In turn, by substituting the electric and magnetic field component, and identifying the integration element of area $da_k = dx_i dx_j$ for $(i, j, k) = P(1, 2, 3)$, as well as $dx_4 = icdt$, Eqs. (41) become

$$\int dt \int\int \left[\frac{\partial E_j}{\partial x_i} - \frac{\partial E_i}{\partial x_j} + \frac{\partial B_k}{c\partial t}\right] dx_i dx_j$$

$$\equiv \int dt \int_s \left[(\nabla \times \vec{E}) + \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right]_k da_k = 0 \quad (42)$$

The integrands of the time integral in this equation, summed over the index $k = 1, 2, 3$ lead to Faraday's law of Eq, (4).

Therefore, the Lorentz covariant integral form of the source-independent Maxwell equations is given by Eq. (38), in which the space-space-space components correspond to the magnetic Gauss law, and the space-space-time components in the time integrands of Eq. (42) add up to Faraday's law, Eq. (4).

4. Discussion

The absence of the Maxwell equations in Lorentz covariant integral form in textbooks of electromagnetism has been pointed out in Sec. 1, recognizing the need to fill in such a gap. Section 2 presented the laws of electromagnetism in the forms of Maxwell equations in their standard integral [Eqs. (1)-(4)] and differential [Eqs. (9)-(12)] versions, and also their Lorentz covariant Eqs. (32) and (33), versions, mirroring the presentation in the text books. In Sec. 3, the Lorentz covariant integral forms of Maxwell equations (34) and (38) were constructed by integrating Eqs. (32) and (33) over the four-vector and rank-3 tensor hypersurface elements, respectively, completing and balancing the study of the four different forms.

The remaining discussion complements the connections between Eqs. (34)-(38) and Eqs. (1)-(4), explaining some of the mathematics and physics behind them. We call the reader's attention to the facts that the steps from Eq. (34) to Eq. (1) and from Eq. (38) to Eq. (3), related to the electric and magnetic Gauss laws, are direct, while the steps from Eq. (34) to Eq. (2) and from Eq. (38) to Eq. (4), related to the circulation laws of Ampére-Maxwell and Faraday, involve the summations of the integrands in the respective time integrations.

We start with the source-dependent Maxwell equation (34), involving rank-2 tensors. Its diagonal time-time [Eq. (35)] and space-space [Eq. (37)] components lead to Eq. (1) and (2) by integrating over the time and space components of the four-vector hypersurface elements $d^3x_\mu(d\vec{a}cdt, idV)$, respectively, as detailed in Sec. 3. The 3D volume integration in Eq.(36) gives directly the connection with Eq. (1). In contrast, Eq. (37) involves both the time integration and one of the components of the 2D surface integration, thus requiring the comparison of integrands and the summation over components in order to connect with Eq. (2).

On the other hand, since Eq. (34) is an equality between rank-2 tensors, the traces of such tensors are scalars and equal to each other. The traces are obtained by contracting the indices $\nu{=}\lambda$ in Eq. (34), which is equivalent to summing Eqs. (37) with $\nu{=}1, 2, 3$ and (36) with $\nu{=}4$. Notice that, in fact, Eq. (1) involves the electric flux and the electric charge that are recognized to be Lorentz scalars. In contrast, the magnetic circulation and its source currents must be integrated over time, as shown by Eq. (37), written as

$$\int dt \oint_C \vec{B} \cdot d\vec{l} = \int dt \left[\frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a} + \frac{1}{c}\frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}\right], \quad (43)$$

in order to show the associated Lorentz scalars, and the connection with Eq. (2). Also, in other words, the time integrated magnetomotive force around curve C is the linear combination of the electric charge and the electric flux crossing surface S.

In the case of the source-independent Maxwell equations (38), the integrand is a symmetric-antisymmetric rank-3 tensor and the integration element is a symmetric rank-3 tensor. Let us recall that the differential form, Eq. (33) , corresponds to Bianchi's identity involving the antisymmetric field tensor, Eq. (31), and the symmetric combination of its gradients. The selection of matching tensor indices in Eq. (38), to be of the space-space-space and space-space-time for both sets λ, μ, ν and α, β, γ involves again $dV = dxdydz$ and $dx_i dx_j cdt$, Eqs. (39) and (41), respectively. Correspondingly, Eq. (39) connects immediately with (3), but (42) requires the comparison of time integrands and the summation in order to connect with Eq. (4).

The contraction of indices $\lambda{=}\alpha, \mu{=}\beta, \nu{=}\gamma$ in Eq. (38) also leads to our equality of Lorentz scalars, constructed as the sum of Eqs. (40) and (42) for $k = 1, 2, 3$. In fact,the magnetic flux of Eq. (3) is a scalar too, and the connection between Eqs. (42) and (4) requires the time integration involving the scalars

$$\int dt \oint\limits_C \vec{E} \cdot d\vec{l} = \int dt \left[-\frac{1}{c}\frac{d}{dt} \int\limits_S \vec{B} \cdot d\vec{a} \right] \qquad (44)$$

which are identified as the time integrated electromotive force around C and the magnetic flux crossing S.

In conclusion, the scalar invariants associated with the rank-2 tensor equation (34) and the rank-6 tensor equation (38), contain the standard integral forms of the source-dependent and source-independent Maxwell equations, respectively; directly for the Gauss laws, and as integrands in the circulation laws. Upon recognition of the scalar nature of Eqs. (1) and (43), and Eqs. (3) and (44), their validity for all inertial frames is established beyond any doubt.

Obviously, Eqs. (34) and (38) have much more mathematical and physical content than Eqs. (1) - (4). This contribution covers only the connections between the respective integral forms of Maxwell equations, involving only the "diagonal" components of the tensor equations, with the distinction between the time-time and space-space contributions. The tensors in Eqs. (34) and (38) also have off-diagonal components, and their analysis may lead to future work and results.

1. D. Halliday and R. Resnick, *Physics for Students of Science and Engineering* (Wiley, New York 1960).
2. E.M. Purcell, *Electricity and Magnetism* (Berkeley Physics Course, McGraw-Hill, New York, 1965) Vol. 2.
3. P. Lorrain and D.R. Corson, *Electromagnetism Principles and Applications*, Second Edition (Freeman, New York, 1990).
4. R.P. Feynman, R. Leighton, M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading. Mass. 1963).
5. O.D. Jeffimenko, *Electricity and Magnetism: An introduction to the theory of electric and magnetic fields* (Apple-Century-Crofts, New York, 1966).
6. J. R. Reitz and F.J. Milford, *Foundations of Electromagnetic Theory*, Second Edition (Addison-Wesley, Reading, Mass. 1967).
7. D.J. Griffiths, *Introduction to Electrodynamics*, Third Edition (Prentice Hall, Upper Saddle, N.J. 1999).
8. L. Eyges, *The Classical Electromagnetic Field* (Dover, New York, 1972).
9. W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, Mass. 1964).
10. J.D. Jackson, *Classical Electrodynamics*, Second Edition (Wiley, New York 1975).
11. M. Brédov, V. Rumiántsev, and I. Toptiguin, *Electrodinámica Clásica* (Mir, Moscú 1986).
12. W. Greiner, *Electrodynamics Springer* (Heidelberg, 1996).