Mirror potentials in classical mechanics

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It is shown that for a central potential that is an injective function of the radial coordinate, a second central potential can be found that leads to trajectories in the configuration space and the momentum space coinciding, respectively, with the trajectories in the momentum space and the configuration space produced by the original potential.

Keywords: Hamiltonian mechanics; classical mechanics.

Se muestra que, para un potencial central que sea una función inyectiva de la coordenada radial, se puede hallar un segundo potencial central que lleva a trayectorias en el espacio de configuración y en el de momentos que coinciden, respectivamente, con las trayectorias en el espacio de momentos y de configuración producidas por el potencial original.

Descriptores: Mecánica hamiltoniana; mecánica clásica.

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1. Introduction

In most examples of classical mechanics, the potential energy is a function of the coordinates only; however, this a potential determines the orbit of the mechanical system in the configuration space and also the evolution of the momenta of the particles of the system. For example, the central potential $V(r) = -k/r$ (which corresponds to the so-called Kepler problem) leads to orbits in configuration space that are conics, and the trajectory in momentum space (the *hodograph*) is (part of) a circle (see, for example, Refs. 1 to 3). So, one may ask if there exists a potential that leads to orbits in the configuration space that are (part of) circles and the hodograph is a conic.

The aim of this paper is to show that, in some cases, for a given potential, one can find a second potential (which will be referred to as the *mirror potential*), depending on the coordinates only, such that the trajectories in configuration space and in momentum space produced by the mirror potential coincide with the trajectories in momentum space and configuration space, respectively, corresponding to the original potential. Our discussion will be restricted to central potentials and we shall show that the mirror potential can be constructed whenever the original potential is an injective function of the radial distance.

The existence of the mirror potential is not a trivial matter. In fact, not every system of ordinary differential equations can be expressed in the form of the Lagrange equations (see, for example, Ref. 4 and the references cited therein). As we shall show below, with the replacement of the original potential by the mirror potential, it is necessary to change the time parametrization [see Eq. (8)]. The use of the Hamiltonian formulation simplifies the derivation enormously.

2. Mirror potentials

We shall consider a particle subjected to a central potential $V(r)$; its Hamiltonian function, expressed in terms of Cartesian coordinates, can be taken as

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V\left(\sqrt{x^2 + y^2 + z^2}\right). \qquad (1)$$

(This expression for the Hamiltonian is the standard one, but there exist many other choices, see, for example, Ref. 5.)

The equations of motion are given by the Hamilton equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i},$$

and, if we interchange the coordinates and momenta in Eq. (1), by reversing the sign of the resulting expression we obtain a new Hamiltonian $\widetilde{H}$, which, by means of the Hamilton equations, will lead to the trajectories in configuration and momentum spaces defined by H, interchanged. In other words, the substitution of the Hamiltonian

$$\widetilde{H} = -\frac{1}{2m}(x^2 + y^2 + z^2) - V\left(\sqrt{p_x^2 + p_y^2 + p_z^2}\right) \qquad (2)$$

into the Hamilton equations yields the same equations of motion as H but with the coordinates and momenta interchanged.

Since we are assuming that V does not depend on time, the evolution of the state of the system in the phase space is a curve lying on a hypersurface $\widetilde{H} = -E$, where E is some real constant (the minus sign is introduced for convenience). From the condition $\widetilde{H} = -E$, making use of Eq. (2) we then

obtain,

$$p_x^2+p_y^2+p_z^2=\left[F\left(E-\frac{x^2+y^2+z^2}{2m}\right)\right]^2,$$

where F denotes the inverse function of V, whose existence requires that $V(r)$ be an injective function. The last equation can also be written as

$$\frac{1}{2m}(p_x^2+p_y^2+p_z^2)-\frac{1}{2m}\left[F\left(E-\frac{x^2+y^2+z^2}{2m}\right)\right]^2=0, \quad (3)$$

which is a relation of the form $h=\text{const.}$, with

$$h\equiv\frac{1}{2m}(p_x^2+p_y^2+p_z^2)-\frac{1}{2m}\left[F\left(E-\frac{x^2+y^2+z^2}{2m}\right)\right]^2 \quad (4)$$

and h is now a Hamiltonian function corresponding to a central potential

$$v(r)\equiv-\frac{1}{2m}\left[F\left(E-\frac{r^2}{2m}\right)\right]^2 \quad (5)$$

that depends parametrically on E.

For instance, if $V(r)=-k/r$, where k is a constant, then $F(r)=-k/r$ and, owing to Eq. (5), the corresponding mirror potential is given by

$$v(r)=-\frac{1}{2m}\left(\frac{2mk}{2mE-r^2}\right)^2. \quad (6)$$

According to the discussion above, this potential leads to orbits in configuration space that are (arcs of) circles and the orbits in momentum space are conics. In fact, if we consider the Hamiltonian with the mirror potential (6) (expressed in polar coordinates, making use of the fact that, for a central potential, the orbit lies on a plane),

$$h=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{1}{2m}\left(\frac{2mk}{2mE-r^2}\right)^2,$$

taking $h=0$ as above and using the conservation of p_θ we have

$$p_r^2+\frac{L^2}{r^2}-\left(\frac{2mk}{2mE-r^2}\right)^2=0$$

where L is a constant. Then, the chain rule gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{\mathrm{d}\theta/\mathrm{d}t}{\mathrm{d}r/\mathrm{d}t}=\frac{L}{r^2p_r}$$

and therefore

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}=\pm\frac{2mE-r^2}{r\sqrt{(2mk/L)^2r^2-(2mE-r^2)^2}}.$$

The solution of this last equation corresponds to a circle of radius $|mk/L|$ whose center is at a distance $\sqrt{(mk/L)^2+2mE}$ from the origin.

The proof that in all cases h yields the same trajectories as $\widetilde{H}$ can be given as follows. From Eqs. (2) and (4) one readily verifies that

$$h=\frac{1}{2m}\left[F\left(-\widetilde{H}-\frac{r^2}{2m}\right)\right]^2-\frac{1}{2m}\left[F\left(E-\frac{r^2}{2m}\right)\right]^2$$

so that $\widetilde{H}=-E$ is equivalent to $h=0$; hence, on the hypersurface $\widetilde{H}=-E$,

$$\mathrm{d}h=-\frac{1}{m}F\left(E-\frac{r^2}{2m}\right)F'\left(E-\frac{r^2}{2m}\right)\mathrm{d}\widetilde{H}.$$

(The terms proportional to $\mathrm{d}r$ cancel as a consequence of the condition $\widetilde{H}=-E$.) Thus, for instance,

$$\frac{\partial h}{\partial q^i}=-\frac{1}{m}FF'\frac{\partial\widetilde{H}}{\partial q^i}=\frac{1}{m}FF'\frac{\mathrm{d}p_i}{\mathrm{d}t}=-\frac{\mathrm{d}p_i}{\mathrm{d}\tau}, \quad (7)$$

with F and F' evaluated at $E-r^2/2m$, and we have defined

$$\mathrm{d}\tau\equiv-\frac{m}{FF'}\mathrm{d}t. \quad (8)$$

In a similar way, one obtains $\partial h/\partial p_i=\mathrm{d}q^i/\mathrm{d}\tau$. That is, the trajectories generated by h coincide with those generated by $\widetilde{H}$, but have a different parametrization (see also Refs. 6 and 7).

It may be remarked that the Cartesian coordinates are not essential in the construction of the mirror potential given above; in fact, in the derivation of Eqs. (7), only the central character of the potential was required.

Another simple example is given by $V(r)=(1/2)kr^2$ (corresponding to an isotropic harmonic oscillator); in this case $F(r)=(2r/k)^{1/2}$, and therefore the mirror potential is

$$v(r)=\frac{r^2}{2m^2k}-\frac{E}{mk}, \quad (9)$$

which is essentially the original potential, and this corresponds to the fact that, for an isotropic harmonic oscillator, the trajectories in configuration space and in the momentum space are both ellipses. By contrast with the potential (6), the potential (9) only contains the parameter E in an additive constant that has no effect in the equations of motion. Furthermore, in this case, $\tau=-mkt+\text{const.}$

3. Final remarks

Apart from the possibility of extending the main result of this paper to noncentral potentials, another natural question concerns finding an analog of this result in quantum mechanics.

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