An alternative method of solution to radiative transfer

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Recibido el 28 de abril de 2005; aceptado el 6 de diciembre de 2005

In this work, we propose an alternative method for solving a to radiative transfer equation in the four-stream approximation following the ideas of Jiménez-Aquino and Varela (2005). We use the Li and Ramaswamy (1995) proposal to establish the set of four coupled first-order differential equations associated with the *radiances* or radiative intensities. The method consists in transforming those four coupled differential equations into a set of four independent fourth-order differential equations associated with the quantities M^s and M^d, which represent the sum and the difference respectively of two radiative intensities. As a consequence of this fact, the solutions for the radiative intensities are then easily calculated, and no matrix method is required.

Keywords: Scattering; polarization; radiative transfer equation.

En este trabajo proponemos un método de solución alternativo a las ecuaciones de transferencia de radiación en la aproximación de cuatro flujos, siguiendo las ideas propuestas por Jiménez-Aquino and Varela (2005). Usamos la propuesta de Li and Ramaswamy (1995), para establecer un conjunto acoplado de cuatro ecuaciones diferenciales de primer orden asociados a las intensidades de radiación. El método consiste en transformar esas cuatro ecuaciones acopladas en un conjunto de cuatro ecuaciones diferenciales independientes, asociados a las cantidades M^s y M^d, las cuales representan la suma y la diferencia de dos intensidades de radiación respectivemente. Como cosecuencia de este hecho, las soluciones para las intensidades de radiación son obtenidas facilmente sin el requerimiento de algún método matricial.

Descriptores: Dispersión; polarización; ecuación de transferencia radiativa.

PACS: 42.68.Db; 42.68.Mj

1. Introduction

The problem of specifying the radiation field of an atmosphere which scatters light in accordance with well-defined physical laws originated in Lord Rayleigh's research in 1871 into the illumination and polarization of the sunlit sky. But the fundamental equations governing Rayleigh's problem had to wait seventy-five years for their formulation and solutions. However the subject was given in more tractable conditions when Arthur Schuster in 1905 studied a problem in Radiative Transfer in an attempt to explain the appearance of absorption and emission lines in stellar spectra, and Karl Schwarzschild introduced in 1906 the concept of radiative equilibrium in stellar atmospheres. Since that time the Radiative Transfer has been investigated principally by astrophysicists, though in recent years the subject has attracted the attention of physicists also, since essentially the same problem arises in the theory of the diffusion of neutrons.

With regard to the atmospheric problem, the radiative transfer also plays a very important role in the study of air pollution, earth global heating, photochemistry of tropospheric pollution, etc. For instance, in tropospheric photochemistry, one of the most important quantities related to the dissociation of certain molecules into fragments which are highly reactive, and one that contributes to the unlimited generation of ozone in the troposphere, is known as *actinic flux*, see Finlayson-Pitts *et al*., (1999). This is defined as the amount of radiation coming from all directions that strikes a given volume containing molecules and/or particles. The calculation of the actinic flux begins with the solar radiation incident at the top of the atmosphere and must include absorption and scattering of the light in the atmosphere and at the ground's surface. One way to calculate the actinic flux is through the solution to the radiative transfer equation applied to plane-parallel atmospheres, as proposed by Chandrasekhar in 1960. It is an Integro-differential equation associated with the intensity of solar radiation whose exact analytical solution has not yet been obtained. The solution has only been calculated by some numerical and analytical approximation methods. Such approximations are referred to as two-stream four-stream approaches.

Two-stream methods for radiative transfer have been widely used in radiative flux calculations, as described in several review papers such as Meador and Weaver (1980), Shettle and Weinman (1970), Zdunkowski *et al.* (1980), and King and Harshvardhan (1986), Ruíz-Suárez *et al.* (1993), etc. The popularity of two-stream approximation is due to the fact that analytical solutions for upward and downward fluxes can be derived, and numerical computations for these fluxes can be efficiently performed in a plane-parallel medium. The accuracies of the various two-stream methods have been compared by King and Harshvardhan (1986). It wasfound that the relative error in the radiative quantities can be up to 20% or higher for any of the two-stream methods, over a range of optical thicknesses and solar zenith angles. It follows that improvements to the two-stream approaches are needed if a higher accuracy in the calculations is desired. Generally, the technique for improvement is to extend the two-stream method to a four-stream or, in general, multi-stream approximation.

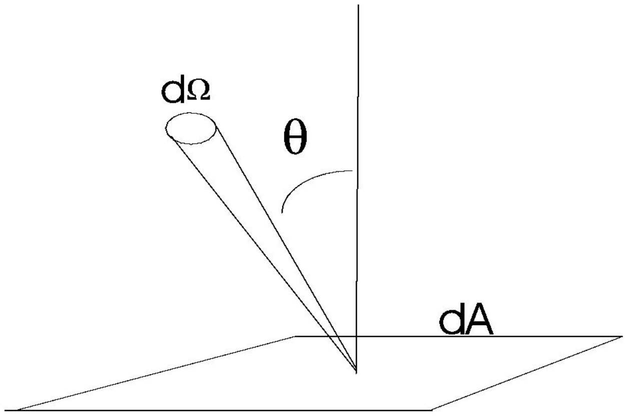

FIGURE 1. A pencil of radiation.

In 1995, an analytical method, based on the higher-order spherical harmonic expansion in both the radiative intensity and the phase function, was proposed by Li and Ramaswamy to solve the radiative transfer equation. The proposed method consists basically in reducing the radiative transfer equation to a set of coupled first-order differential equations for the radiative intensities, which, according to the truncation order in the approximations lead, to two-stream, four-stream or multi-stream approximations. In this work, we follow the Li and Ramaswamy theoretical scheme and use the four-stream approximation to establish the four coupled first-order differential equations for the radiances. Our main aim in this work is now to apply the strategy, based on the *diffusion-type equation for radiative transfer*, proposed by Jiménez-Aquino and Varela (2005) to solve those four coupled differential eaquations. The strategy consists in transforming those coupled differential equations into a set of four independent fourth-order differential equations associated with the quantities M^s and M^d, where these quantities will be defined respectively as the sum and the difference of two radiative intensities. The solutions for the radiances will be calculated through these quantities in a direct manner, without any matrix method. As will be shown, these solutions can easily be transformed into the same expressions as those calculated by Li and Ramaswamy, using some matrix methods. Finally, the conclusions are given at the end of this work.

In this work, we start with some concepts and definitions for the understanding of radiative transfer in planetary atmospheres. So, the analysis of a radiation field often requires the consideration of the amount of radiant energy dE_λ, in a time interval dt, and in a specified wavelength interval, λ to $d\lambda$, which crosses an element of area dA, and directions confined to a differential solid angle $d\Omega$, which is oriented at an angle θ to the normal of dA, as shown in Fig. 1. This energy is expressed in terms of the specific intensity (or simply intensity or *radiance*) I_λ given by

$$dE_\lambda = -I_\lambda \cos\theta \, dA \, d\Omega \, d\lambda \, dt, \tag{1}$$

where $\cos\theta \, dA$ denotes the effective area at which the energy in being intercepted. Thus the intensity is in units of energy per area per time per wavelength and per steradian. The intensity is commonly, said to be confined in a *pencil of radiation*.

The *monochromatic flux density* or *monochromatic irradiance* of radiant energy is defined by the normal component of I_λ integrated over the entire hemispheric solid angle, and may be written as

$$F_\lambda = \int_\Omega I_\lambda \cos\theta \, d\Omega \, . \tag{2}$$

In polar coordinates, we write

$$F_\lambda = \int_0^{2\pi} \int_0^{\pi/2} I_\lambda(\theta, \varphi) \cos\theta \sin\theta \, d\theta \, d\varphi \, . \tag{3}$$

Scattering is a physical process by which a particle in the path of an electromagnetic wave continuously abstracts energy from the incident wave and reradiates that energy in all directions. Therefore, the particle may be thought of as a point source of scattered energy. Scattering is often accompanied by *absorption*. Grass looks green because it scatters green light while it absorbs red and blue light. The absorbed energy is converted to some other form, and it is no longer present as red or blue light. Both scattering and absorption remove energy from a beam of light passing through the medium. The beam of light is attenuated, and this attenuation can be called *extinction*. Thus extinction is a result of scattering plus absorption. In a nonabsorbing medium, scattering is the sole process of extinction.

On the other hand, in the field of light scattering and radiative transfer, it is customary to use a term called *cross section*, analogous to the geometrical area of a particle, to denote the amount of energy removed from the original beam by the particle. When the cross section is associated with a particle dimension, its units are denoted in terms of area (cm^2). Thus the *extinction* cross section, in units of area, is the sum of the scattering and absorption cross sections, that is, $\sigma_{ext} = \sigma_{sca} + \sigma_{abs}$. However, when the cross section is in reference to unit mass, its units are given in area per mass ($cm^2 g^{-1}$). In this case, the *mass* extinction cross section is the sum of the mass absorption and mass scattering cross sections, that is, $k_{ext} = k_{sca} + k_{abs}$. Furthermore, when the extinction cross section is multiplied by the particle number density (cm^{-3}), *i.e.* $n\sigma_{ext}$, or when the mass extinction cross section is multiplied by the density ($g\ cm^{-3}$), *i.e.* ρk_{ext}, in both cases the resulting parameter has units of (cm^{-1}) and is referred to as the *extinction coefficient* and denoted by β_{ext}. The basic theories for an understanding of the scattering of particles in the atmosphere are the Rayleigh and Mie scattering theory.

In a scattering volume, which contains many particles, each particle is exposedby, and also scatters, the light that has already been scattered by other particles. For instance, a particle at some position, say A, removes the incident light just once, *i.e. single scattering*, in all directions. Meanwhile, a portion of this scattered light reaches another particle at a position, say B, where it is scattered again in all directions. This is called *secondary scattering*. Similarly, a subsequent third-order scattering involving the particle at another position, say C, takes place. Scattering more than once is called *multiple scattering*. Multiple scattering is an important process for the transfer of radiant energy in the atmosphere, especially when aerosols and clouds are involved.

2. Introduction to radiative transfer

A pencil of radiation passing through a medium will be weakened by its interaction with matter. If the intensity of radiation I_λ becomes $I_\lambda + dI_\lambda$ after passing through a thickness ds in the direction of its propagation, then

$$dI_\lambda = -\rho\, k_\lambda\, I_\lambda\, ds\,, \tag{4}$$

where ρ is the density of the material, and k_λ denotes the mass extinction cross section (in units of area per mass) for radiation of wavelength λ. The mass extinction cross section is the sum of the mass absorption and scattering cross section. Thus, the reduction in intensity is due to both absorption and scattering by the material.

On the other hand, the radiation intensity may be strengthened by emission from the material plus multiple scattering from other directions into the pencil under consideration at the same wavelength. We define the source function coefficient j_λ such that the increase in intensity due to emission and multiple scattering is given by

$$dI_\lambda = j_\lambda\, \rho\, ds, \tag{5}$$

where the source function coefficient j_λ has the same physical meaning as the mass extinction cross section. Upon combining Eqs. (4) and (5), we obtain that

$$dI_\lambda = -k_\lambda\, \rho\, I_\lambda\, ds + j_\lambda\, \rho\, ds\,. \tag{6}$$

It is convenient to define the source function $J_\lambda = j_\lambda / k_\lambda$, which in this case has units of radiant intensity. So, Eq. (6) may be rearranged to yield

$$\frac{dI_\lambda}{k_\lambda \rho\, ds} = -I_\lambda + J_\lambda\,. \tag{7}$$

This is the general radiative transfer equation without any coordinate system imposed, and it is fundamental to the discussion of any radiative transfer process.

2.1. The equation of radiative transfer for plane-parallel atmospheres

For many atmospheric radiative transfer applications, it is physically appropriate to consider that the atmosphere in localized portions is plane-parallel so that variations in the intensity and atmospheric parameters (temperature and gas profiles) are permitted only in the vertical direction (*i.e.* height and pressure). In this case, it is convenient to measure linear distances normal to the plane of stratification. If z denotes this distance, then the general equation of radiative transfer defined in Eq. (7) becomes

$$\cos\theta\, \frac{dI(z,\theta,\varphi)}{k\,\rho\, dz} = -I(z,\theta,\varphi) + J(z,\theta,\varphi)\,, \tag{8}$$

where θ denotes the inclination with respect to the upward normal, and φ the azimuthal angle in reference to the x axis. For simplicity's sake, we have omitted the subscript λ on the radiative quantities. By defining the normal optical thickness (or depth)

$$\tau = \int_z^\infty k\, \rho\, dz'\,, \tag{9}$$

measured downward from the outer boundary, we find that

$$\mu \frac{dI(\tau,\mu,\varphi)}{d\tau} = I(\tau,\mu,\varphi) - J(\tau,\mu,\varphi)\,, \tag{10}$$

where $\mu = \cos\theta$. Eq. (10) is the basic equation for the problem of multiple scattering in plane-parallel atmospheres.

2.2. Multiple scattering and absorption in planetary atmospheres

The scattering process is often coupled with absorption. To formulate the fundamental equation governing the transfer of diffuse solar radiation in plane-parallel atmospheres containing molecules and particles, the following must be considered. The term *diffuse* is associated with multiple scattering processes and is differentiated from *direct* solar radiation. The first term on the RHS of Eq. (8) describes the extinction processes and the second one the emission and multiple scattering of the diffuse radiation. For this purpose, we will consider an atmospheric layer of thickness Δz delimited by two plane-parallels and containing molecules and/or particles, as shown in Fig. 2. The differential change of diffuse intensity emerging from below the layer is due to the following processes:

(1) reduction from the extinction attenuation,

(2) increase from the single scattering of the unscattered direct solar flux from the direction $(-\mu_0, \varphi_0)$ to (μ, φ),

(3) increase from multiple scattering of the diffuse intensity from directions (μ', φ') to (μ, φ), and

(4) increase from emission within the layer in the direction (μ, φ).

Point (1) corresponds to the first term on the RHS of Eq. (8), whereas points (2)-(4) are included in the source function $J(\tau, \mu, \varphi)$.

To describe the scattering of a particle it will be necessary to introduce the *phase function*, which represents the angular distribution of the scattered radiation coming from some other direction. For instance, phase function $\wp(\mu,\varphi;-\mu_0,\varphi_0)$ describes the angular distribution of the scattered radiation to the outgoing direction (μ,φ) coming from direction $(-\mu_0,\varphi_0)$. Phase function $\wp(\mu,\varphi;\mu',\varphi')$ describes the angular distribution of the scattered radiation to the outgoing direction (μ,φ) coming from other directions (μ',φ').

On the other hand, according to the definitions given at the end of Sec. I, the quantity $n(z)\sigma\, ds$ must be understood as the number of molecules and/or particles inside the volume element of length ds and unitary cross section (1 cm^2), where $ds = dz/\cos\theta = dz/\mu$. Obviously $n(z)\sigma\, dz$ is then the number of molecules and/or particles inside the vertical column of height dz and unitary cross section. We define the extinction, scattering, and absorption coefficient as β_{ext}, β_{sca}, and β_{abs} as

$$\beta_{ext,sca,abs} = \frac{1}{\Delta z}\int\limits_{\Delta z} \sigma_{ext,sca,abs}(z) n(z)\, dz. \tag{11}$$

So, the differential change of the diffuse intensity can be written as required by Eq. (8), that is:

$$\begin{aligned}\mu\,\frac{dI(z,\theta,\varphi)}{dz} &= -n\sigma_{ext}dz\, I(z,\theta,\varphi)\\ &+n\sigma_{sca}dz\,\frac{F_{\odot}}{4\pi}\,\mathrm{e}^{-\tau\mu_0}\,\wp(\mu,\varphi;-\mu_0,\varphi_0)\\ &+\frac{n\sigma_{sca}dz}{4\pi}\int\limits_0^{2\pi}\int\limits_{-1}^{1} I(\tau,\mu',\varphi')\,\wp(\mu,\varphi;\mu',\varphi')d\varphi' d\mu'\\ &+n\sigma_{abs}dz\, B[T(z)]. \end{aligned}\tag{12}$$

Therefore, the first term on the RHS of Eq. (12) refers to the reduction of intensity because of the extinction processes; in the second term, the only expression $F_{\odot}\mathrm{e}^{-\tau\mu_0}$ is the attenuation of the direct radiation flux coming from the sun, where $F_{\odot}$ is the direct radiation flux at the top of the atmosphere, as shown in Fig. 2. This attenuation represents the Beer-Bouguer-Lambert law, which is obtained by solving Eq. (10) without the second term of the RHS. So, the second term refers to the increase in intensity because of the single scattering of the unscattered direct solar flux from direction $(-\mu_0,\varphi_0)$ to (μ,φ). The factor 4π is the normalization constant of the phase function because it must be integrated over the 4π solid angle. The third term contributes to the increase in the scattered radiation in the direction (μ,φ) coming from all other directions (μ',φ'). Finally, the last term $B[T(z)]$ is concerned with the laws of blackbody radiation, which are the basic to an understanding of the absorption and emission processes. This is the case for the transfer of thermal infrared radiation emitted for the earth and the atmosphere. However, the flux emitted for the earth and the atmosphere with an equilibrium temperature $\sim 255K$ is not sufficient for the photodissociation process of some chemical species in comparison to that emitted from the sun for $\lambda \leq 3.5$ μm. Therefore, for some solar radiative transfer problem, which is our interest in this work, we may omit the last term of Eq. (12).

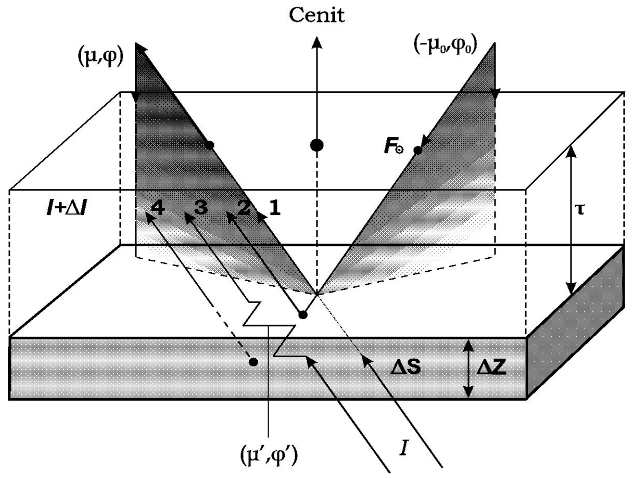

FIGURE 2. Transfer of diffuse solar intensity from below in plane-parallel layers: (1) attenuation y extinction; (2) single scattering of the unscattered solar flux; (3) multiple scattering; and (4) emission from the layer.

So, if we integrate Eq. (12) in the region Δz, and use the definition of Eq. (11), we get

$$\begin{aligned}\mu\,\frac{\Delta I(z,\theta,\varphi)}{dz} &= -\beta_{ext}\, I(z,\theta,\varphi)\\ &+\beta_{sca}\,\frac{F_{\odot}}{4\pi}\,\mathrm{e}^{-\tau\mu_0}\,\wp(\mu,\varphi;-\mu_0,\varphi_0)\\ &+\frac{\beta_{sca}}{4\pi}\int\limits_0^{2\pi}\int\limits_{-1}^{1} I(\tau,\mu',\varphi')\,\wp(\mu,\varphi;\mu',\varphi')d\varphi' d\mu'\,. \end{aligned}\tag{13}$$

If we define the *single-scattering albedo* $\tilde{\omega}$ as

$$\tilde{\omega} = \frac{\beta_{sca}}{\beta_{ext}}\,, \tag{14}$$

and the optical depth τ as

$$\tau = \int\limits_z^{\infty} \beta_{ext}\, dz'\,, \tag{15}$$

then, by taking the limit when Δz goes to zero, Eq. (13) can finally be written more concisely as

$$\mu\frac{dI(z,\theta,\varphi)}{d\tau} = I(\tau,\theta,\varphi) - J(\tau,\mu,\varphi) - J_0(\tau,\mu_0,\varphi_0), \tag{16}$$

where $J(\tau,\mu,\varphi)$ is referred to as the *internal source function* due to multiple scattering and is defined as

$$J(\tau,\mu,\varphi) = \frac{\tilde{\omega}}{4\pi}\int\limits_0^{2\pi}\int\limits_{-1}^{1} \wp(\mu,\varphi;\mu',\varphi') I(\tau,\mu',\varphi') d\varphi' d\mu', \tag{17}$$

and $J_0(\tau,\mu_0,\varphi_0)$ is referred to as the *external source function* due to single scattering of the direct radiation, and is given by

$$J_0(\tau,\mu_0,\varphi_0) = \frac{\tilde{\omega}}{4\pi}\wp(\mu,\varphi;-\mu_0,\varphi_0) F_{\odot}\,\mathrm{e}^{-u_0\tau}\,. \tag{18}$$

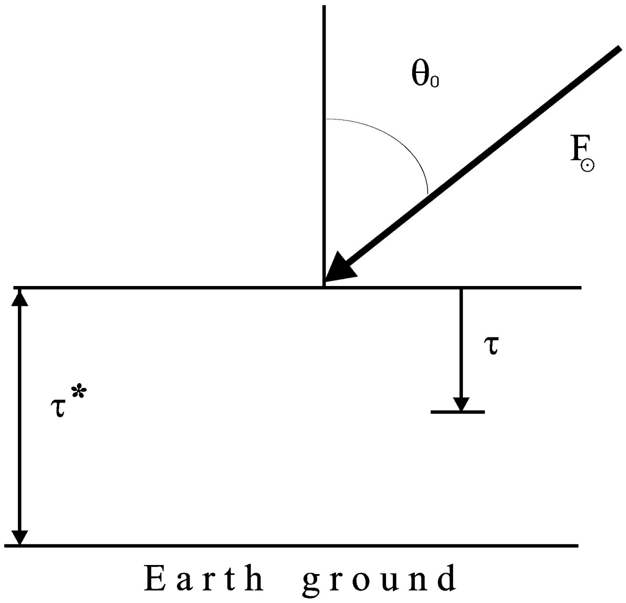

FIGURE 3. Illustration of the incident solar flux $F_\odot$ on the top of a plane-parallel atmosphere, at an angle θ_0. The parameter τ^* is the total optical dept,h and τ represents any point inside the layer.

The fundamental parameters that drive the transfer of diffuse intensity are the optical depth, the single-scattering albedo, and the phase function.

3. Li and Ramaswamy method

In the Li and Ramaswamy scheme, the phase function is expanded in terms of the spherical harmonic function in the following way:

$$\wp(\mu,\varphi;\mu',\varphi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \frac{\omega_l}{2l+1}\, Y_l^m(\mu,\varphi) Y_l^{m\,*}(\mu',\varphi')\,, \tag{19}$$

where $Y_l^m(\mu,\varphi)$ are the spherical harmonic function and $Y_l^{m\,*}(\mu',\varphi')$ its conjugate complex, such that

$$Y_l^m(\mu,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}}\, P_l^m(\mu)\, \mathrm{e}^{im\varphi}\,, \tag{20}$$

$P_l^m(\mu)$ being the associated Legendre function. The moments ω_l of the series can be calculated using the expansion of the phase function in terms of the scattering angle $\cos\Theta$ such that

$$\wp(\cos\Theta) = \sum_{l=0}^{\infty} \omega_l\, P_l(\cos\Theta)\,, \tag{21}$$

where

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\varphi-\varphi')\,, \tag{22}$$

Θ is the angle between incident and scattered radiation, and

$$P_l(\cos\Theta) = \sum_{m=-l}^{m=l} \frac{1}{2l+1}\, Y_l^m(\mu,\varphi) Y_l^{m\,*}(\mu',\varphi')\,. \tag{23}$$

The moment ω_l is determined by the orthogonality of the Legendre function

$$\omega_l = \frac{2l+1}{2}\int_{-1}^{1} \wp(\cos\Theta) P_l(\cos\Theta) d\cos\Theta\,. \tag{24}$$

It can be shown that $\omega_0 = 1$, which represents the normalization of the phase function. The quantity $\omega_1/3 = g$, is defined as the *asymmetry factor* and is given by

$$g = \frac{\omega_1}{3} = \frac{1}{2}\int_{-1}^{1} \wp(\cos\Theta) cos\Theta\; d\cos\Theta; \tag{25}$$

it is an important parameter because it characterizes the scattering pattern of a particle.

3.1. The four stream approximation

The four stream method proposed by Li and Ramaswamy starts with the following series expansion for radiative intensity:

$$I(\tau,\mu,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \sqrt{2l+1}\, I_l^m(\tau) Y_l^m(\mu,\varphi), \tag{26}$$

where $I_l^m(\tau)$ is the radiative intensity as a function of τ. Substituting Eq. (26) into Eqs. (17) and (18), and using the orthogonality property of the spherical harmonics, we obtain expressions for the source functions

$$J = \tilde\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \frac{\omega_l}{\sqrt{2l+1}}\, I_l^m(\tau) Y_l^m(\mu,\varphi)\,, \tag{27}$$

and

$$J_0 = \frac{\tilde\omega}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \frac{\omega_l}{2l+1} Y_l^m(\mu,\varphi) Y_l^{m*}(-\mu_0,\varphi_0) \times F_\odot\, \mathrm{e}^{-u_0\tau}. \tag{28}$$

Now, substituting Eqs. (26), (27) and (28) into the radiative transfer equation (17), we get the following:

$$\big[(l-m+1)(l+m+1)\big]^{1/2}\frac{dI_{l+1}^m}{d\tau} + \big[(l+m)(l-m)\big]^{1/2}\frac{dI_{l-1}^m}{d\tau} = a_l\, I_l^m - b_l^m\, \mathrm{e}^{-u_0\tau}, \tag{29}$$

where coefficients a_l and b_l^m are $a_l = [(2l+1) - \tilde\omega\omega_l]$ and $b_l^m = \tilde\omega\omega_l Y_l^{m\,*}(-\mu_0,\varphi_0)F_\odot/4\sqrt{2l+1}$. We consider a solution with a truncation of order L, which means that the spherical harmonic function $Y_l^m(\mu,\varphi)$ is restricted to order $l = 0, 1, 2, \ldots, L$. For $L = 1$, if only calculations of flux and the azimuthally averaged intensity are considered, we have (for $m = 0$)

$$\frac{dI_1^0}{d\tau} = a_0 I_0^0 - b_0^0 \, e^{-u_0\tau},$$
$$\frac{dI_0^0}{d\tau} = a_1 I_1^0 - b_1^0 \, e^{-u_0\tau}. \tag{30}$$

System (30) is the well-known two-stream approximation and is the same as that obtained in the Eddington approximation (Shettle and Weinman, 1970). The case $L = 2$ corresponds to a degenerate case and will not be considered in this work.

For $L = 3$ we obtain the four-stream closure. Again, if only the calculations of flux and the azimuthally averaged intensity are considered, then $m = 0$. In this case,

$$\frac{dI_1}{d\tau} = a_0 I_0 - b_0 \, e^{-u_0\tau}$$

$$2\frac{dI_2}{d\tau} + \frac{dI_0}{d\tau} = a_1 I_1 - b_1 \, e^{-u_0\tau}$$

$$3\frac{dI_3}{d\tau} + 2\frac{dI_1}{d\tau} = a_2 I_2 - b_2 \, e^{-u_0\tau}$$

$$3\frac{dI_2}{d\tau} = a_3 I_3 - b_3 \, e^{-u_0\tau}\,. \tag{31}$$

Since we are considering only the azimuthally independent case, and for simplicity the superscript "0" in I_i^0 and b_i^0 ($i = 0, 1, 2, 3$) are omitted. Eqs. given in (31) can be combined to yield

$$\frac{dI_0}{d\tau} = a_1 I_1 - \frac{2a_3}{3} I_3 - \big(b_1 - \frac{2b_3}{3}\big) \, e^{-u_0\tau}$$

$$\frac{dI_1}{d\tau} = a_0 I_0 - b_0 \, e^{-u_0\tau}$$

$$\frac{dI_2}{d\tau} = \frac{a_3}{3} I_3 - \frac{b_3}{3} \, e^{-u_0\tau}$$

$$\frac{dI_3}{d\tau} = -\frac{2a_0}{3} I_0 + \frac{a_2}{3} I_2 + \big(\frac{2b_0}{3} - \frac{b_2}{3}\big) \, e^{-u_0\tau}\,. \tag{32}$$

4. An alternative method of solution

To solve the set of ***Eqs. (??)***, we propose the following. Let us define the new variables

$$M_1^s = I_0 + I_2, \quad M_1^d = I_0 - I_2,$$
$$M_2^s = I_1 + I_3, \quad M_2^d = I_1 - I_3, \, , \tag{33}$$

in terms of which we will construct a set of four independent fourth-order differential equations, one for each $M_i^{s,\,d}$ with $i = 1, 2$. For this purpose, we first write the first derivative with respect to τ for each $M_i^{s,\,d}$, obtaining the following:

$$\frac{dM_1^s}{d\tau} = A_1^- M_2^s + A_1^+ M_2^d + B_1 \, e^{-u_0\tau}\,, \tag{34}$$

$$\frac{dM_1^d}{d\tau} = A_1'^- M_2^s + A_1'^+ M_2^d + B_1' \, e^{-u_0\tau}\,, \tag{35}$$

$$\frac{dM_2^s}{d\tau} = A_2^+ M_1^s + A_2^- M_1^d + B_2 \, e^{-u_0\tau}\,, \tag{36}$$

$$\frac{dM_2^d}{d\tau} = A_2'^- M_1^s + A_2'^+ M_1^d + B_2' \, e^{-u_0\tau}\,, \tag{37}$$

where we have defined the coefficients

$$A_1^\pm = \frac{a_1}{2} \pm \frac{a_3}{6}, \quad A_1'^\pm = \frac{a_1}{2} \pm \frac{a_3}{2},$$
$$A_2^\pm = \frac{a_0}{6} \pm \frac{a_2}{6}, \quad A_2'^\pm = \frac{5a_0}{6} \pm \frac{a_2}{6},$$
$$B_1 = \frac{b_3}{3} - b_1, \quad B_1' = b_3 - b_1,$$
$$B_2 = -\frac{b_0}{3} - \frac{b_2}{3}, \quad B_2' = \frac{b_2}{3} - \frac{5b_0}{3}\,. \tag{38}$$

Next, the second derivative with respect to τ of Eqs. (34)-(37) is calculated, combining these equations, so that

$$\frac{d^2M_1^s}{d\tau^2} = C_{11} M_1^s + C_{12} M_1^d + D \, e^{-u_0\tau}\,, \tag{39}$$

$$\frac{d^2M_1^d}{d\tau^2} = C_{21} M_1^s + C_{22} M_1^d + E \, e^{-u_0\tau}\,, \tag{40}$$

$$\frac{d^2M_2^s}{d\tau^2} = C_{11}' M_2^s + C_{12}' M_2^d + D' \, e^{-u_0\tau}\,, \tag{41}$$

$$\frac{d^2M_2^d}{d\tau^2} = C_{21}' M_2^s + C_{22}' M_2^d + E' \, e^{-u_0\tau}\,, \tag{42}$$

in this case, the coefficients are defined as

$$C_{11} = A_1^- A_2^+ + A_1^+ A_2'^-, \;\; C_{12} = A_1^- A_2^- + A_1^+ A_2'^+,$$
$$C_{21} = A_1'^- A_2^+ + A_1'^+ A_2'^-, \;\; C_{22} = A_1'^- A_2^- + A_1'^+ A_2'^+,$$
$$C_{11}' = A_2^+ A_1^- + A_2^- A_1'^-, \;\; C_{12}' = A_2^+ A_1^+ + A_2^- A_1'^+,$$
$$C_{21}' = A_2'^- A_1^- + A_2'^+ A_1'^-, \;\; C_{22}' = A_2'^- A_1^+ + A_2'^+ A_1'^+,$$
$$D = A_1^- B_2 + A_1^+ B_2' - u_0 B_1,$$
$$E = A_1'^- B_2 + A_1'^+ B_2' - u_0 B_1',$$
$$D' = A_2^+ B_1 + A_2^- B_1' - u_0 B_2,$$
$$E' = A_2'^- B_1 + A_2'^+ B_1' - u_0 B_2'. \tag{43}$$

The four fourth-order differential equations for each $M_i^{s,\,d}$ are easily calculated from Eqs. (39)-(42), giving us

$$\frac{d^4M_1^s}{d\tau^4} = C_{11}\frac{d^2M_1^s}{d\tau^2} + C_{12}\frac{d^2M_1^d}{d\tau^2} + u_0^2D\,\mathrm{e}^{-u_0\tau} \quad (44)$$

$$\frac{d^4M_1^d}{d\tau^4} = C_{21}\frac{d^2M_1^s}{d\tau^2} + C_{22}\frac{d^2M_1^d}{d\tau^2} + u_0^2E\,\mathrm{e}^{-u_0\tau} \quad (45)$$

$$\frac{d^4M_2^s}{d\tau^4} = C'_{11}\frac{d^2M_2^s}{d\tau^2} + C'_{12}\frac{d^2M_2^d}{d\tau^2} + u_0^2D'\,\mathrm{e}^{-u_0\tau} \quad (46)$$

$$\frac{d^4M_2^d}{d\tau^4} = C'_{21}\frac{d^2M_2^s}{d\tau^2} + C'_{22}\frac{d^2M_2^d}{d\tau^2} + u_0^2E'\,\mathrm{e}^{-u_0\tau}\,. \quad (47)$$

As we can see, these equations are not clearly independent. However, they can be transformed into four independent differential equations with the help of Eqs. (39)-(42). This is possible if the second derivative $d^2M_1^d/d\tau^2$ in (44) can be written in terms of an algebraic sum of $d^2M_1^s/d\tau^2$ and M_1^s, and also if the second derivative $d^2M_1^s/d\tau^2$ in (45) can be written in terms of an algebraic sum of $d^2M_1^d/d\tau^2$ and M_1^d. Similarly if the second derivative $d^2M_2^d/d\tau^2$ in (46) and $d^2M_2^s/d\tau^2$ in (47) satisfy the same requirements as $M_1^{s\,d}$. After some algebraic manipulations of Eqs. (44)-(47), we obtain for $M_1^{s\,d}$

$$\frac{d^4M_1^s}{d\tau^4} = \beta\frac{d^2M_1^s}{d\tau^2} + \gamma M_1^s + \delta\,\mathrm{e}^{-u_0\tau}\,, \quad (48)$$

$$\frac{d^4M_1^d}{d\tau^4} = \beta\frac{d^2M_1^d}{d\tau^2} + \gamma M_1^d + \epsilon\,\mathrm{e}^{-u_0\tau}\,, \quad (49)$$

and for $M_2^{s,\,d}$ we get

$$\frac{d^4M_2^s}{d\tau^4} = \beta'\frac{d^2M_2^s}{d\tau^2} + \gamma' M_2^s + \delta'\,\mathrm{e}^{-u_0\tau}, \quad (50)$$

$$\frac{d^4M_2^d}{d\tau^4} = \beta'\frac{d^2M_2^d}{d\tau^2} + \gamma' M_2^d + \epsilon'\,\mathrm{e}^{-u_0\tau}\,, \quad (51)$$

which are clearly four independent fourth-order differential equations, and the coefficients are defined as

$$\begin{aligned}
&\beta = C_{11} + C_{22}, \qquad \gamma = C_{12}C_{21} - C_{11}C_{22},\\
&\delta = C_{12}E - C_{22}D + u_0^2D,\\
&\epsilon = C_{21}D - C_{11}E + u_0^2E\,,
\end{aligned} \quad (52)$$

and

$$\begin{aligned}
&\beta' = C'_{11} + C'_{22}, \qquad \gamma' = C'_{12}C'_{21} - C'_{11}C'_{22},\\
&\delta' = C'_{12}E' - C'_{22}D' + u_0^2D',\\
&\epsilon' = C'_{21}D' - C'_{11}E' + u_0^2E'\,.
\end{aligned} \quad (53)$$

It can be shown that $\beta = \beta' = a_0a_1 + (4/9)a_0a_3 + (1/9)a_2a_3$ and $\gamma = \gamma' = -(1/9)a_0a_1a_2a_3$.

The solutions to Eqs. (48)-(51) are now very easy to calculate. The solutions for each $M_1^{s,\,d}$ and each $M_2^{s,\,d}$ are given by the sum of the homogeneous part plus a particular solution. In this case, they can be written more concisely as

$$\begin{bmatrix} M_1^s \\ M_1^d \end{bmatrix} = \sum_{j=1}^{4} \begin{bmatrix} F_j \\ G_j \end{bmatrix}\,\mathrm{e}^{-k_j\tau} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}\,\mathrm{e}^{-u_0\tau}\,, \quad (54)$$

and

$$\begin{bmatrix} M_2^s \\ M_2^d \end{bmatrix} = \sum_{j=1}^{4} \begin{bmatrix} F'_j \\ G'_j \end{bmatrix}\,\mathrm{e}^{-k_j\tau} + \begin{bmatrix} \xi'_1 \\ \xi'_2 \end{bmatrix}\,\mathrm{e}^{-u_0\tau}\,, \quad (55)$$

where F_j, G_j, F'_j, G'_j, ξ_1, ξ_2, ξ'_1 and ξ'_2 are constant. If we substitute the homogeneous solution for M_1^s into the homogeneous part of Eq. (48), we obtain the following:

$$\sum_{j=1}^{4}(k_j^4 - \beta k_j^2 - \gamma)F_j\,\mathrm{e}^{k_j\tau} = 0\,. \quad (56)$$

We arrive at the same expression when the homogeneous solutions for M_1^d, M_2^s and M_2^d are substituted into the homogeneous part of Eq. (49), (50) and (51) respectively, except that F_j must be replaced by G_j, F'_j and G'_j. Thus, to have a nontrivial solution for each $M_i^{s,\,d}$, we must have

$$f(k) = k^4 - \beta k^2 - \gamma = 0\,, \quad (57)$$

and therefore, the four roots for the solutions (54) and (55) will be given by

$$k_1 = [\beta + \sqrt{\beta^2 + 4\gamma}]^{1/2}/\sqrt{2},$$
$$k_2 = [\beta - \sqrt{\beta^2 + 4\gamma}]^{1/2}/\sqrt{2},$$

$k_3 = -k_1$, and $k_4 = -k_2$.

On the other hand, if the particular solutions for each $M_i^{s\,d}$ are respectively substituted into Eqs. (48), (49), (50) and (51), we obtain

$$\xi_1 = \frac{\delta}{f(u_0)},\quad \xi_2 = \frac{\epsilon}{f(u_0)},\quad \xi'_1 = \frac{\delta'}{f(u_0)},\quad \xi'_2 = \frac{\epsilon'}{f(u_0)}\,. \quad (58)$$

So, according to the results given above, the explicit solutions for $M_1^{s,\,d}$ and $M_2^{s,\,d}$ will be given by

$$M_1^s = F_1e^{-k_1\tau} + F_3e^{k_1\tau} + F_2\mathrm{e}^{-k_2\tau} + F_4e^{k_2\tau} + \xi_1\mathrm{e}^{-u_0\tau} \quad (59)$$

$$M_1^d = G_1e^{-k_1\tau} + G_3e^{k_1\tau} + G_2\mathrm{e}^{-k_2\tau} + G_4e^{k_2\tau} + \xi_2\mathrm{e}^{-u_0\tau} \quad (60)$$

$$M_2^s = F'_1\mathrm{e}^{-k_1\tau} + F'_3\mathrm{e}^{k_1\tau} + F'_2e^{-k_2\tau} + F'_4e^{k_2\tau} + \xi'_1\mathrm{e}^{-u_0\tau} \quad (61)$$

$$M_2^d = G'_1e^{-k_1\tau} + G'_3\mathrm{e}^{k_1\tau} + G'_2e^{-k_2\tau} + G'_4e^{k_2\tau} + \xi'_2\mathrm{e}^{-u_0\tau}. \quad (62)$$

According to the definitions given by Eq. (33), the solutions for the radiative intensities $I_i(\tau)$ can be written as

$$I_0(\tau) = J_1 e^{-k_1\tau} + K_1 e^{k_1\tau} + J_2 e^{-k_2\tau} + K_2 e^{k_2\tau} + \eta_0 e^{-u_0\tau} \quad (63)$$

$$I_1(\tau) = J'_1 e^{-k_1\tau} + K'_1 e^{k_1\tau} + J'_2 e^{-k_2\tau} + K'_2 e^{k_2\tau} + \eta_1 e^{-u_0\tau} \quad (64)$$

$$I_2(\tau) = R_1 e^{-k_1\tau} + L_1 e^{k_1\tau} + R_2 e^{-k_2\tau} + L_2 e^{k_2\tau} + \eta_2 e^{-u_0\tau} \quad (65)$$

$$I_3(\tau) = R'_1 e^{-k_1\tau} + L'_1 e^{k_1\tau} + R'_2 e^{-k_2\tau} + L'_2 e^{k_2\tau} + \eta_3 e^{-u_0\tau}, \quad (66)$$

where now

$$\begin{aligned}
J_1 &= (F_1 + G_1)/2, & K_1 &= (F_3 + G_3)/2,\\
J_2 &= (F_2 + G_2)/2, & K_2 &= (F_4 + G_4)/2,\\
J'_1 &= (F'_1 + G'_1)/2, & K'_1 &= (F'_3 + G'_3)/2,\\
J'_2 &= (F'_2 + G'_2)/2, & K'_2 &= (F'_4 + G'_4)/2,\\
R_1 &= (F_1 - G_1)/2, & L_1 &= (F_3 - G_3)/2,\\
R_2 &= (F_2 - G_2)/2, & L_2 &= (F_4 - G_4)/2,\\
R'_1 &= (F'_1 - G'_1)/2, & L'_1 &= (F'_3 - G'_3)/2,\\
R'_2 &= (F'_2 - G'_2)/2, & L'_2 &= (F'_4 - G'_4)/2,\\
\eta_0 &= (\xi_1 + \xi_2)/2, & \eta_1 &= (\xi'_1 + \xi'_2)/2,\\
\eta_2 &= (\xi_1 - \xi_2)/2, & \eta_3 &= (\xi'_1 - \xi'_2)/2.
\end{aligned} \quad (67)$$

To write the radiances given by Eqs. (63)-(66), in the same form as those established by Li and Ramaswamy, we must show that the coefficients of Eqs. (64)-(66) are not independent. In fact, it can be shown with the help of Eqs. (34)-(37) that each one of them is related to its corresponding coefficient given in Eq. (63). Thus, by substituting only the homogeneous solution for each $M_i^{s,d}$ into its corresponding homogeneous part in Eqs. (34)-(37), we find the following conditions for $M_1^{s,d}$:

$$\begin{aligned}
-k_1F_1 &= A_1^- F'_1 + A_1^+ G'_1, & -k_1G_1 &= A_1'^- F'_1 + A_1'^+ G'_1,\\
k_1F_3 &= A_1^- F'_3 + A_1^+ G'_3, & k_1G_3 &= A_1'^- F'_3 + A_1'^+ G'_3,\\
-k_2F_2 &= A_1^- F'_2 + A_1^+ G'_2, & -k_2G_2 &= A_1'^- F'_2 + A_1'^+ G'_2,\\
k_2F_4 &= A_1^- F'_4 + A_1^+ G'_4, & k_2G_4 &= A_1'^- F'_4 + A_1'^+ G'_4,
\end{aligned} \quad (68)$$

and for $M_2^{s,d}$, the conditions

$$\begin{aligned}
-k_1F'_1 &= A_2^+ F_1 + A_2^- G_1, & -k_1G'_1 &= A_2'^- F_1 + A_2'^+ G_1,\\
k_1F'_3 &= A_2^+ F_3 + A_2^- G_3, & k_1G'_3 &= A_2'^- F_3 + A_2'^+ G_3,\\
-k_2F'_2 &= A_2^+ F_2 + A_2^- G_2, & -k_2G'_2 &= A_2'^- F_2 + A_2'^+ G_2,\\
k_2F'_4 &= A_2^+ F_4 + A_2^- G_4, & k_2G'_4 &= A_2'^- F_4 + A_2'^+ G_4.
\end{aligned} \quad (69)$$

Combining Eqs. (68) and (69), we can show that

$$\begin{aligned}
J'_1 &= -\frac{a_0}{k_1}J_1, & K'_1 &= \frac{a_0}{k_1}K_1,\\
J'_2 &= -\frac{a_0}{k_2}J_2, & K'_2 &= \frac{a_0}{k_2}K_2,\\
R_1 &= \frac{1}{2}\left(\frac{a_0a_1}{k_1^2} - 1\right)J_1, & L_1 &= \frac{1}{2}\left(\frac{a_0a_1}{k_1^2} - 1\right)K_1,\\
R_2 &= \frac{1}{2}\left(\frac{a_0a_1}{k_2^2} - 1\right)J_2, & L_2 &= \frac{1}{2}\left(\frac{a_0a_1}{k_2^2} - 1\right)K_2,\\
R'_1 &= -\frac{3}{2a_3}\left(\frac{a_0a_1}{k_1} - 1\right)J_1, & L'_1 &= \frac{3}{2a_3}\left(\frac{a_0a_1}{k_1} - 1\right)K_1,\\
R'_2 &= -\frac{3}{2a_3}\left(\frac{a_0a_1}{k_2} - 1\right)J_2, & L'_2 &= \frac{3}{2a_3}\left(\frac{a_0a_1}{k_2} - 1\right)K_2.
\end{aligned} \quad (70)$$

Now, if we define the constants

$$\begin{aligned}
S_1 &= -a_0/k_1,\\
S_2 &= -a_0/k_2,\\
T_1 &= [(a_0a_1/k_1^2) - 1]/2,\\
T_2 &= [(a_0a_1/k_2^2) - 1]/2,\\
U_1 &= -3[(a_0a_1/k_1) - k_1]/2a_3,\\
U_2 &= -3[(a_0a_1/k_2) - k_2]/2a_3,
\end{aligned}$$

then the set of solutions for the radiative intensities (63)-(66) can be transformed as those given by Li and Ramaswamy, that is:

$$I_0(\tau) = J_1 e^{-k_1\tau} + K_1 e^{k_1\tau} + J_2 e^{-k_2\tau} + K_2 e^{k_2\tau} + \eta_0 e^{-u_0\tau} \quad (71)$$

$$I_1(\tau) = S_1[J_1 e^{-k_1\tau} - K_1 e^{k_1\tau}] + S_2[J_2 e^{-k_2\tau} - K_2 e^{k_2\tau}] + \eta_1 e^{-u_0\tau} \quad (72)$$

$$I_2(\tau) = T_1[J_1 e^{-k_1\tau} + K_1 e^{k_1\tau}] + T_2[J_2 e^{-k_2\tau} + K_2 e^{k_2\tau}] + \eta_2 e^{-u_0\tau} \quad (73)$$

$$I_3(\tau) = U_1[J_1 e^{-k_1\tau} - K_1 e^{k_1\tau}] + U_2[J_2 e^{-k_2\tau} - K_2 e^{k_2\tau}] + \eta_3 e^{-u_0\tau}. \quad (74)$$

In the Li and Ramaswamy solutions, the constants J_i, K_i, S_i, T_i, and U_i with $i = 1, 2$ are named by other letters, and the constants η_0, η_1, η_2 and η_3 are the same as those used by the authors, namely:

$$\eta_0 = \frac{1}{9f(u_0)}\left[a_1b_0 - u_0b_1(a_2a_3 - 9u_0^2) + 2u_0^2(a_3b_2 - 2a_3b_0 - 3b_3u_0)\right], \quad (75)$$

$$\eta_1 = \frac{1}{9f(u_0)}\left[(a_0 b_1 - u_0 b_0)(a_2 a_3 - 9u_0^2) - 2a_0 u_0 (a_3 b_2 - 3b_3 u_0)\right], \tag{76}$$

$$\eta_2 = \frac{1}{9f(u_0)}\left[(a_3 b_2 - 3b_3 u_0)(a_0 a_1 - u_0^2) - 2a_3 u_0 (a_0 b_1 - b_0 u_0)\right], \tag{77}$$

$$\eta_3 = \frac{1}{9f(u_0)}\left[(a_2 b_3 - 3b_2 u_0)(a_0 a_1 - u_0^2) + u_0^2 (6a_0 b_1 - 4a_0 b_3 - 6b_0 u_0)\right]. \tag{78}$$

The above solutions determine the radiative intensity (radiant energy) for a single-layer homogeneous atmosphere. The constants J_1, K_1, J_2 and K_2 can be determined using the appropriate boundary conditions. Here we will use the Marshak boundary condition (Evans, 1993). For the layer considered, at the upper boundary (optical depth $\tau = \tau_u$),

$$\int_0^{-1}\int_0^{2\pi} [I(\tau_u, \mu, \varphi) - I^-(\tau_u, \mu, \varphi)] \times Y_l^{m\,*}(\mu, \varphi)\, d\mu d\varphi = 0\,, \tag{79}$$

with $l = 1, \ldots, L; m = \pm 1, \ldots, \pm l$; and $I^-(\tau_u, \mu, \varphi)$ is the downward diffuse intensity at the upper boundary. At the lower boundary (optical depth $\tau = \tau_l$),

$$\int_0^{1}\int_0^{2\pi} [I(\tau_l, \mu, \varphi) - I^+(\tau_l, \mu, \varphi)] \times Y_l^{m\,*}(\mu, \varphi)\, d\mu d\varphi = 0\,, \tag{80}$$

where $l = 1, \ldots, L; m = \pm 1, \ldots, \pm l$; and $I^+(\tau_l, \mu, \varphi)$ is the upward diffuse intensity at the lower boundary. For a single-layer medium, at the top ($\tau = 0$), there is no downward diffuse intensity; in this case,

$$\int_0^{-1}\int_0^{2\pi} I(0, \mu, \varphi) Y_1^{0\,*}\, d\mu d\varphi \sim \frac{1}{2} I_0(0) - I_1(0) + \frac{5}{8} I_2(0) = 0\,, \tag{81}$$

and

$$\int_0^{-1}\int_0^{2\pi} I(0, \mu, \varphi) Y_3^{0\,*}\, d\mu d\varphi \sim -\frac{1}{8} I_0(0) + \frac{5}{8} I_2(0) - I_3(0) = 0\,. \tag{82}$$

At the bottom of the layer ($\tau = \tau^*$, see Fig. 3) there is no upward diffuse intensity (surface albedo is assumed to be zero); in this case,

$$\int_0^{1}\int_0^{2\pi} I(\tau^*, \mu, \varphi) Y_1^{0\,*}\, d\mu d\varphi \sim \frac{1}{2} I_0(\tau^*) + I_1(\tau^*) + \frac{5}{8} I_2(\tau^*) = 0\,, \tag{83}$$

and

$$\int_0^{1}\int_0^{2\pi} I(\tau^*, \mu, \varphi) Y_3^{0\,*}\, d\mu d\varphi \sim -\frac{1}{8} I_0(\tau^*) + \frac{5}{8} I_2(\tau^*) - I_3(\tau^*) = 0\,. \tag{84}$$

By substituting Eqs. (71)-(74) into their corresponding expressions given by (81)-(84), we establish a set of four equations with four unknowns J_1, K_1, J_2 and K_2. These quantities can be determined in a very similar way to those calculated by Jiménez-Aquino and Varela, (2002), using the same boundary conditions.

5. Concluding remarks

The alternative method of solution proposed by Jiménez-Aquino and Varela (2005) for solving the radiative transfer equation in the two-stream approximations, has been applied in solving the set of four coupled first-order differential equations (32), which arise in the Li and Ramaswamy theoretical framework. The method of solution is developed in terms of the quantities $M_i^{s,\,d}$, with $i = 1, 2$; it is clearly simple, and the radiances given by Eqs. (63)-(66) are, consequently, calculated in a direct way. The transformation of those solutions into the same expressions as those established by Li and Ramaswamy (1995) has also been very simple to calculate. The method can, of course, be easily extended to other cases.

Acknowledgments

The authors wish to thank R.M. Velasco for her comments. Financial support from Consejo Nacional de Ciencia y Tecnología (CONACYT) México is also gratefully acknowledged. Proyect 1790P-E9507.

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