An analysis on the inversion of polynomials

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In this work the application and the intervals of validity of an inverse polynomial, according to the method proposed by Arfken [1] for the inversion^i of series, is analyzed. It is shown that, for the inverse polynomial there exists a restricted domain whose longitude depends on the magnitude of the acceptable error when the inverse polynomial is used to approximate the inverse function of the original polynomial. A method for calculating the error of the approximation and its use in determining the restricted domain is described and is fully developed up to the third order. In addition, five examples are presented where the inversion of a polynomial is applied in solving different problems encountered in basic courses on physics and mathematics. Furthermore, expressions for the eighth and ninth coefficients of a ninth-degree inverse polynomial, which are not encountered explicitly in other known references, are deduced.

Keywords: Invertion of polynomial; equation solving; intervals of validity.

En este trabajo se analiza la aplicación y los intervalos de validez de un polinomio inverso, según el método propuesto por Arfken [1] para la inversión de series. Se muestra que existe un dominio restringido cuya longitud depende de la magnitud del error aceptable; esto se ejemplifica por simplicidad con un polinomio de tercer grado, aunque el procedimiento es aplicable a polinomios de cualquier grado. Se deduce una expresión para determinar el error del polinomio inverso; así mismo, se presentan cinco ejemplos, con diferentes grados de dificultad, donde se aplica la inversión polinomial para resolver diversos problemas que pueden presentarse en física y matemáticas. Se deducen las expresiones para los coeficientes octavo y noveno de un polinomio inverso de grado nueve, las cuales no se encuentran explícitamente en otras referencias conocidas.

Descriptores: Inversión polinomial; solución de ecuaciones; intervalo de validez.

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1. Introduction

Some physical and mathematical problems can be solved easily if the functions involved are approximated, to some level of accuracy, by simpler and continuous functions, such as algebraic functions, better known as polynomials. Probably the best known approximating polynomials are those due to Taylor, which are widely used to approximate any differentiable function and are commonly represented as:

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

where

$$a_n = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=0},$$

when $f(x)$ is expanded around $x = 0$; more general expressions can be found in most of the calculus texts [2].

The main advantage to this approach lies in the fact that when adding, subtracting, multiplying, deriving and integrating polynomials, other simple and also continuous polynomial functions are obtained. Some authors even deduce a polynomial that is the multiplicative inverse [3] of another polynomial; this inverse polynomial transforms a quotient of polynomials into a product of them, giving another polynomial as a result.

The most interesting fact, however, is that another inverse polynomial [1], but now in the sense of a composition of functions, can be defined; undoubtedly this new polynomial has the same characteristics as any inverse function [2] (at least within the order of the approximation given by the polynomial). This is important, because inverse functions are useful for solving many mathematical and physical problems; the simplest process for solving an equation is an algebraic solution; this process is one of the simpler applications of the inverse function.

For example, first degree equations have a simple method of solution by solving the independent variable to obtain the inverse function. If we define

$$y = f(x) = mx + b,$$

then we have:

$$x = \frac{y - b}{m} = f^{-1}(y),$$

a solution that corresponds to the inverse function of $f(x)$.

For second degree equations

$$Ax^2 + Bx + C = 0,$$

a classic solution, given by the well-known formula of Ferrari [4]:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

is obtained by completing the square of the second degree equation and extracting its square root. The last operation is the inverse function of raising to the second power.

There exists a procedure for solving third degree equations of a kind given by the following expression:

$$x^3 + Px^2 + Qx + R = 0.$$

Although this procedure is not general, the roots of some of these equations can be found. The method due to Tartaglia [4,5] consists in making a change of variable (*x = u + Q/3*) to transform the complete third degree equation into a reduced third degree equation:

$$u^3 + qu + r = 0.$$

The roots of the reduced equation are given by:

$$\begin{aligned} u_1 &= \sqrt[3]{\frac{r}{2} + \sqrt{D}} + \sqrt[3]{\frac{r}{2} - \sqrt{D}}, \\ u_2 &= -\frac{1}{2}u_1 + \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{r}{2} + \sqrt{D}} - \sqrt[3]{\frac{r}{2} - \sqrt{D}}\right) i, \\ u_3 &= -\frac{1}{2}u_1 - \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{r}{2} + \sqrt{D}} - \sqrt[3]{\frac{r}{2} - \sqrt{D}}\right) i, \end{aligned}$$

where i is the imaginary unit, and the quantity D (called the *Discriminant*), is given by:

$$D = \left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2.$$

This solution is valid only if $D \geq 0$. Finally, the roots of the complete equation are obtained as

$$x_j = u_j + {}^{Q}\!/_{3}, \qquad \text{for } j = 1, 2, 3.$$

For fourth degree equations, Ferrari [4,5] developed a procedure similar to the above. The procedure, however, is too involved to be described here in detail.

For fifth degree equations and higher, there is no specific procedure for solving them; furthermore, Evariste Galois [6] showed almost two hundred years ago that these equations are not solvable by radicals. Thus, numerical methods are a useful option.

This introduction shows that the approximate solution of even third degree polynomials and higher is well worth while. In this paper, we propose an approximate solution by finding the inverse polynomial through the procedure described by Arfken [1]; this is included in Sec. 2. It is shown that an important advantage of this method over the numerical methods is that the inversion procedure gives analytical solutions, instead of only numerical data. In addition, in Sec. 3 we show that the inverse polynomial is valid only over a restricted domain defined by the accepted error. Finally, in Sec. 4, additional expressions for obtaining the inverse polynomial up to the ninth degree are given. We also include five examples to show how the method is used in practice. All of them are conducted only to the third degree, to avoid lengthy deductions and expressions. The last two examples show how to solve simple undergraduate problems.

2. Iverting a polynomial

To obtain the inverse of a polynomial:

$$P(x) = \sum_{i=1}^{N} a_i x^i = y \tag{1}$$

according to one of the procedures mentioned by Arfken [1], a polynomial solution of the same degree is proposed as:

$$Q(y) = \sum_{j=1}^{N} b_j y^j = x \tag{2}$$

Substituting (1) in (2), the composition $Q(P(x))$ is carried out:

$$\begin{aligned} Q(y) &= \sum_{i=1}^{N} b_i \left[P(x)\right]^i \\ &= \sum_{j=1}^{N} b_j \left[\sum_{i=1}^{N} a_i x^i\right]^j \\ &= \sum_{k=1}^{N^2} c_k x^k = x, \end{aligned} \tag{3}$$

where

$$c_k = c_k(a_i,\, b_j).$$

Equating the coefficients for each power on both sides of the equation, we have

$$c_1 = 1,\ c_k = 0,\ \text{for } k = 2,\ 3, ... \tag{4}$$

a system of N^2 equations for N unknown coefficients b_j; thus, the system is over determined. In order to find a consistent solution, we need only consider the first Nequations and neglect the remaining ones. Solving the resulting system, we obtain the coefficients of the inverse polynomial.

As a particular case, let us consider a third degree polynomial, such as

$$P(x) = a_1 x + a_2 x^2 + a_3 x^3 = y. \tag{5}$$

Then the proposed inverse polynomial denoted by $Q(y)$ will have the form

$$Q(y) = b_1 y + b_2 y^2 + b_3 y^3 = x. \tag{6}$$

The composition $Q(P(x))$ is carried out by substituting (5) in (6), so we get

$$\begin{aligned} Q(P(x)) &= b_1(a_1 x + a_2 x^2 + a_3 x^3) \\ &\quad + b_2(a_1 x + a_2 x^2 + a_3 x^3)^2 \\ &\quad + b_3(a_1 x + a_2 x^2 + a_3 x^3)^3 = x. \end{aligned} \tag{7}$$

After expanding and grouping, we obtain

$$\begin{aligned}Q(P(x)) &= a_1b_1x + (a_2b_1 + a_1^2b_2)x^2 \\ &+ (a_3b_1 + 2a_1a_2b_2 + a_1^3b_3)x^3 \\ &+ (a_2^2b_2 + 2a_1a_3b_2 + 3a_1^2a_2b_3)x^4 \\ &+ (2a_2a_3b_2 + 3a_1a_2^2b_3 \\ &+ 3a_1^2a_3b_3)x^5 + (a_3^2b_2 + 6a_1a_2a_3b_3 + a_2^3b_3)x^6 \\ &+ (3a_2^2a_3b_3 + 3a_1a_3^2b_3)x^7 \\ &+ 3a_2a_3^2b_3x^8 + a_3^3b_3x^9 \\ &= \sum_{i=1}^{N^2} c_ix^i = x \end{aligned} \tag{8}$$

Equating the coefficients of the same power in x from both sides of Eq. (8), it is required that all coefficients be identical to zero, except the first. That is,

$$\begin{aligned} c_1 &= a_1b_1 = 1 \\ c_2 &= a_1b_2 + a_2b_1^2 = 0 \\ c_3 &= a_3b_1^3 + 2a_2b_1b_2 + a_1b_3 = 0 \\ c_4 &= a_2^2b_2 + 2a_1a_3b_2 + 3a_1^2a_2b_3 = 0 \\ c_5 &= 2a_2a_3b_2 + 3a_1a_2^2b_3 + 3a_1^2a_3b_3 = 0 \\ c_6 &= a_3^2b_2 + 6a_1a_2a_3b_3 + a_2^3b_3 = 0 \\ c_7 &= 3a_2^2a_3b_3 + 3a_1a_3^2b_3 = 0 \\ c_8 &= 3a_2a_3^2b_3 = 0 \\ c_9 &= a_3^3b_3 = 0 \end{aligned} \tag{9}$$

With the first three relationships, we form a system of three equations and three unknowns. The rest of the equations are ignored, otherwise we would have a system with more equations than unknowns whose solution is overdetermined. The system can be solved to obtain the coefficients b_i as functions of the coefficients a_i of the original polynomial; the solution to the system of three equations is

$$\begin{aligned} b_1 &= \frac{1}{a_1} \\ b_2 &= -\frac{a_2}{a_1^3} \\ b_3 &= \frac{2a_2^2 - a_3a_1}{a_1^5}. \end{aligned} \tag{10}$$

It is well worth noting that, according to this solution, if the polynomial does not have a linear term, the inverse polynomial would not exist; this conclusion is valid for any degree polynomial.

Now if we make the composition of Q and P, with the b_i coefficients given by Eqs. (10) and neglecting the terms of any higher degree than the third, we obtain the identity function; *i.e.* $Q(P(x)) = x$, showing consistency in the solution, within the limits of the proposed approximation.

In order to get better insight into the consequences of this result, let us analyze numerically the next example in particular.

Example 1. According to Eqs. (6) and (10), the inverse polynomial of

$$P(x) = x + x^2 + x^3 \tag{11}$$

is

$$Q(x) = x - x^2 + x^3. \tag{12}$$

Evaluating $P(x)$ in x = 0 [see Eq. (11)], we have P(0) = 0, and when evaluating $Q(P(x))$ = 0 = x, then $P(x)$ and $Q(x)$ are inverse polynomials in x = 0. If now we evaluate P(1) = 3, substituting in Eq. (12) we get $Q(3)$ = 21, so $Q(y)$ is not the inverse of $P(x)$, or at least not in x = 1. Thus the solution is not a general one; there exists a region where the solution can be considered acceptable.

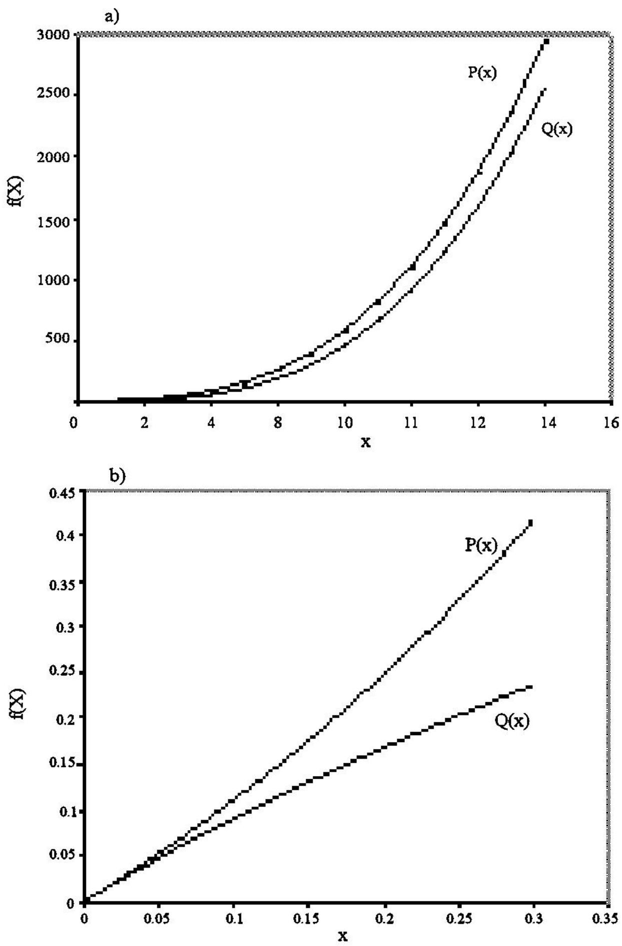

FIGURE 1. The two polynomials P(x) and Q(x) are shown. a) in the interval of x ∈ [0,16] symmetry near zero is not appreciated. b) x ∈ [0,0.3] symmetry is observed.

TABLE I. Numerical data for P(x) and Q(x) from Example 1 and the error of the composition P(Q(x)) for $-1 \leq x \leq 1$.

x	$P(x)$	$Q(P(x))$	*Error (%)*
-1	-1	-3	200
-0.9	-0.819	-2.03911426	126.568251
-0.8	-0.672	-1.42704845	78.381056
-0.7	-0.553	-1.02792138	46.845911
-0.6	-0.456	-0.75875482	26.459136
-0.5	-0.375	-0.56835938	13.671875
-0.4	-0.304	-0.42451046	6.127616
-0.3	-0.237	-0.30648105	2.160351
-0.2	-0.168	-0.20096563	0.482816
-0.1	-0.091	-0.10003457	0.034571
0	0	0	0
0.1	0.111	0.10004663	0.046631
0.2	0.248	0.20174899	0.874496
0.3	0.417	0.31562271	5.207571
0.4	0.624	0.47759462	19.398656
0.5	0.875	0.77929687	55.859375
0.6	1.176	1.41940378	136.567296
0.7	1.533	2.78559744	297.942491
0.8	1.952	5.57940941	597.426176
0.9	2.439	10.9992095	1122.13439
1	3	21	2000

In Fig. 1, the polynomials $P(x)$ and $Q(x)$ are shown graphically. In Fig. 1a the curves do not correspond to two functions with one the inverse of the other, because if they did so, the graphs should be symmetrical with respect to the identity function. Certainly they are not. In Fig. 1b certain symmetry is appreciated, at least in a small interval.

3. Error Analysis

Considering the previous example, it is convenient to ask: how big are the neglected terms? When is the inverse polynomial acceptable, and what is the error? To answer the first of these questions, we evaluate the terms neglected in the inversion of a third degree polynomial. By considering all the terms in equation (8), and by substituting the expressions for b_1, b_2 and b_3, obtained in Eqs. (10), we get

$$(a_2^2b_2 + 2a_1a_3b_2 + 3a_1^2a_2b_3)x^4 = \frac{5a_2^3 - 5a_1a_2a_3}{a_1^3}x^4 \equiv T_1x^4$$

$$2a_2a_3b_2 + 3a_1a_2^2b_3 + 3a_1^2a_3b_3)x^5 = \frac{6a_2^4 + a_1a_2^2a_3 - 3a_1^2a_3^2}{a_1^4}x^5 \equiv T_2x^5$$

$$(a_2^3b_3 + a_3^2b_2 + 6a_1a_2a_3b_3)x^6 = \frac{2a_2^5 + 11a_1a_2^3a_3 - 7a_1^2a_2a_3^2}{a_1^5}x^6 \equiv T_3x^6$$

$$(3a_2^2a_3b_3 + 3a_1a_3^2b_3)x^7 = \frac{6a_2^4a_3 + 3a_1a_2^2a_3^2 - 3a_1^2a_3^3}{a_1^5}x^7 \equiv T_4x^7$$

$$\left(3a_2a_3^2b_3\right)x^8 = \frac{6a_2^3a_3^2 - 3a_1a_2a_3^3}{a_1^5}x^8 \equiv T_5x^8$$

$$a_3^3b_3x^9 = \frac{2a_2^2a_3^3 - a_1a_3^4}{a_1^5}x^9 \equiv T_6x^9 \quad (13)$$

These are, in an approximate way, the six neglected terms that can be calculated with a knowledge of the particular values of the coefficients of the polynomial to invert; clearly they are dependent on the particular point x of the polynomial domain.

These terms are negligible provided that a_1 is large; or that the domain is restricted, for instance to $|x| < 1$, because x^n, for n = 4, 5, 6, 7, 8, 9, is small.

To answer the second question, we build Table 1, where the errors are presented for different values of x.

We can use the data in Table I to see that, for $-0.2 \leq x \leq 0.2$, the error in the approximate inverse polynomial is less than 1%; for $-0.3 \leq x \leq 0.3$, the error is not too much higher than 5%; but, for the interval $-0.5 \leq x \leq 0.5$, the error is greater than 10%. For an error smaller than 1%, by interpolation we find that x must be within the interval (-0.294, 0.241). In Fig. 2, the plots of both polynomials are shown in this interval; it is easy to observe the symmetry regarding the identity function (central line), thus, it can be assumed that $P(x)$ and $Q(x)$ are inverse polynomials in this interval.

If we evaluate the neglected terms, we find that their sum is exactly the error of $P(Q(x))$. As can be appreciated from Eqs. (13), the error diminishes, increasing the a_1 value, but it increases when a_2 and a_3 are increased.

Another way to state the problem is as follows: if one wants a error smaller than or similar to e% of x, then

$$\left|\frac{T_1x^4+T_2x^5+T_3x^6+T_4x^7+T_5x^8+T_6x^9}{x}\right| \leq \frac{e}{100} \quad (14)$$

or

$$\left|T_1x^3+T_2x^4+T_3x^5+T_4x^6+T_5x^7+T_6x^8\right| \leq \frac{e}{100} \quad (15)$$

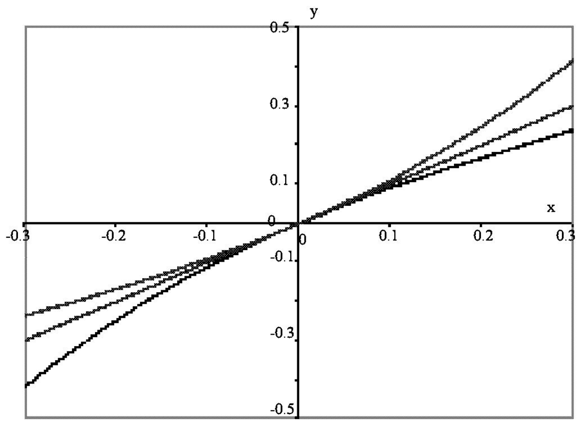

FIGURE 2. The polynomials P(x) and Q(x) are shown in an interval where certain symmetry is appreciated.

The maximum value for the error should be given in x_m, where x_m is the greatest value of x in the restricted domain. For example if x_m = 0.1, then the error is

$$\big|10^{-3}T_1 + 10^{-4}T_2 + 10^{-5}T_3 + 10^{-6}T_4 + 10^{-7}T_5 + 10^{-8}T_6\big| \leq 10^{-2}e. \quad (16)$$

Knowing the coefficients T_k, with k = 1,..., 6, we can readily estimate the error of an inverse third degree polynomial.

Example 2. Let us consider again the polynomial (11) in example 1, we have that $a_1 = a_2 = a_3$ = 1; then, using Eqs. (13) we find that the coefficients T are:

T_1= 0, T_2 = 4, T_3 = 6, T_4 = 6, T_5 = 3 and T_6 = 1.

So, if we allow a maximum percent error equal to 2%, x must satisfy the following relationship:

$$-0.02 \leq 4x^4 + 6x^5 + 6x^6 + 3x^7 + x^8 \leq 0.02 \quad (17)$$

A plot of the inequality is shown in Fig. 3; it is not difficult to see that the left inequality is always satisfied, and the right side inequality can be expressed as:

$$f(x) \equiv 4x^4 + 6x^5 + 6x^6 + 3x^7 + x^8 - 0.02 \leq 0. \quad (18)$$

In the same figure, it can be seen that the solution interval is the region where the plot of $f(x)$ is below the x axis. That interval is determined by the roots of

$$4x^4 + 6x^5 + 6x^6 + 3x^7 + x^8 - 0.02 = 0. \quad (19)$$

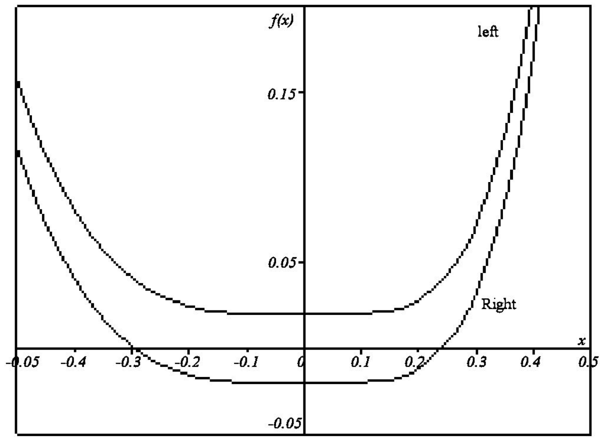

FIGURE 3. Plot of the inequality (17), where the interval solution is observed

Solving this polynomial by the method of the Regula Falsi [7] with an error smaller then 0.01%, the roots obtained are: -0.2937164 and 0.2418198, therefore, inequality (18) is satisfied when

$$-0.2937164 \leq x \leq 0.2418198,$$

which coincides with the interval found from Table I.

Example 3. An application of inverting a polynomial is to find roots of a polynomial. Let us suppose that we want to find a root of the polynomial

$$y = P(x) = x^3 + x^2 + x - \frac{1}{4} = 0, \quad (20)$$

which includes a zero order term. In general, (20) has the form

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = y \quad (21)$$

In order to find the solution, it is convenient to rewrite Eq. (21) as

$$z = (y - a_0) = a_3x^3 + a_2x^2 + a_1x. \quad (22)$$

As we already know, the inverse polynomial is of the form

$$x = b_1z + b_2z^2 + b_3z^3. \quad (23)$$

Substituting the value of z, we have:

$$x = (y - a_0) - (y - a_0)^2 + (y - a_0)^3 \quad (24)$$

Developing the binomials we get

$$x = b_3y^3 + (b_2 - 3b_3a_0)y^2 + (b_1 - 2b_2a_0 + 3b_3a_0^2)y + (-b_1a_0 + b_2a_0^2 - b_3a_0^3) \quad (25)$$

The third degree polynomial in Eq. (20) has coefficients a_0= -1/4, a_1= 1, a_2= 1 and a_3= 1, according to the relationships (10), the coefficents of the inverse polynomial are b_1= 1, b_2= -1 and b_3= 1.

Substituting these values in (25), we have

$$x = y^3 - \frac{1}{4}y^2 + \frac{11}{16}y + \frac{13}{64} \quad (26)$$

One root of the polynomial is obtained by making y = 0 in the inverted polynomial Eq. (26). Doing this, we get

x = 13/64 ≈ 0.203125. On the other hand, numerically solving this polynomial again with the method of the *Regula Falsi* [7], with a smaller error than 0.01%, the root is x = 0.201392 ± 0.01%. The value obtained from the inverse polynomial differs from that found numerically by less than 1%, a value that corresponds to the evaluation of the error through the expressions in (14), and that may be acceptable depending on the specific problem.

Example 4. It is important to recognize that this method can be applied to solve some common physical problems. For example, the Fraunhofer intensity pattern [8] of a single slit is given by

$$I = I_m \left(\frac{\sin(x)}{x}\right)^2. \tag{27}$$

Usually the width of the pattern is evaluated through the points where

$$\frac{\sin(x)}{x} = 0$$

(see Fig 4); this is given by $x = \pm n\pi$ with n an integer. The width of the pattern is found by making $n = \pm 1$, and by subtraction, $\Delta x = 2\pi$. In many cases, however, the problem is not so easy, and one of the important parameters is the Half Width of the Full Maximum, commonly abbreviated as HWFM. This is obtained by taking

$$I(x^{\pm}) = \frac{1}{2} I_m. \tag{28}$$

Substituting (28) in (27) one obtains

$$\left(\frac{\sin(x^{\pm})}{x^{\pm}}\right)^2 = \frac{1}{2}. \tag{29}$$

In this equation, $x^{\pm}$ cannot be directly solved analytically; we can, however, use polynomial inversion approach, for example, starting from the McLaurin the polynomial expansion and then inverting this polynomial. Then it is solved following the procedure of the previous examples.

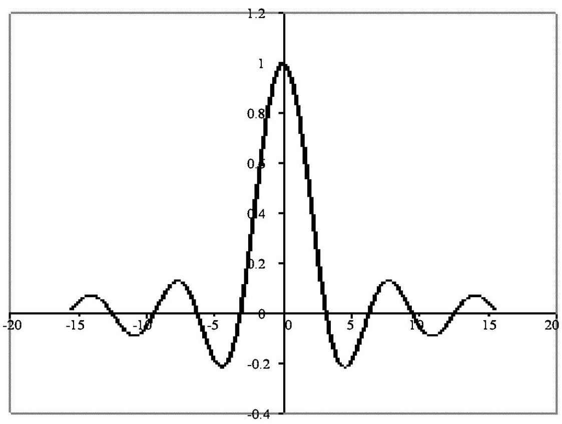

FIGURE 4. Plot of the function $f(x) = \frac{\sin(x)}{x}$, it is not difficult to see that f(x) = 0, in x =± nπ, with n integer.

Approximating the sine function with the first three terms of its McLaurin series,

$$\left(\frac{\sin(x)}{x}\right) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}. \tag{30}$$

Then, substituting Eq. (30) in to Eq. (29)

$$\frac{\sin(x)}{x} - \frac{1}{\sqrt{2}} = \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} = 0, \tag{31}$$

and defining the following polynomial:

$$P(x) = \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^6}{5040} - 0.2929, \tag{32}$$

and making the following changes of variable: $s = x^2$ and $w = P(x)$+ 0.2929, we get:

$$w = (P(x) + 0.2929) = \frac{s}{6} - \frac{s^2}{120} + \frac{s^3}{5040} = 0 \tag{33}$$

The coefficients of the inverse polynomial, according to the relationships in (10) are:

$$b_1 = 6, \quad b_2 = 1.8 \quad \text{and} \quad b_3 = 0.8232;$$

therefore, the inverse polynomial is

$$s = 6w + 1.8w^2 + 0.8232w^3 \tag{34}$$

or, using (33),

$$s = 6\,(P(x) + 0.2929) \\ + 1.8\,(P(x){+}0.2929)^2 + 0.8232\,(P(x){+}0.2929)^3 \tag{35}$$

Developing,

$$s = 1.9369 + 7.2662P(x) \\ + 2.5233P(x)^2 + 0.8232P(x)^3, \tag{36}$$

and evaluating in $P(x)$ = 0, we have s=1.9369 or $x^{\pm} = x = \pm 1.3917$.

Evaluating the error through the expressions in (14) applied to the polynomial (33) we obtain an error smaller than 0.3%.

Solving Eq. (29) for the method of the *Regula Falsi*, we find x = 1.3915, a value that differs from the one obtained for the inverse polynomial by less than 0.02%.

Example 5. In the study [10] of the radiation emitted by a black body, it is often necessary to find the wavelength λ, for which the density of energy radiated by a black body has a maximum at a given temperature. As is well known, the density of energy (see Fig. 5) is given by:

$$E(\lambda) = \frac{8\pi k^5 T^5}{c^4 h^4} \frac{x^5}{e^x - 1},$$

where: $x = hc/\lambda kT$.

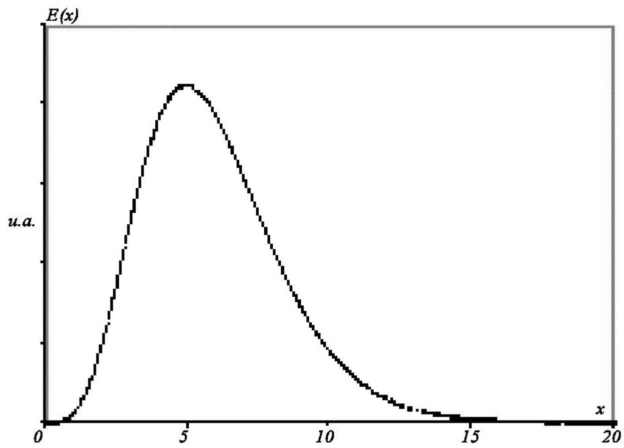

FIGURE 5. Plot of density of energy emitted by a black body.

To find the maximum, it is necessary to find the roots of:

$$\frac{dE}{dx} = 0.$$

Expanding the last expression, the problem is reduced to solving the following equation:

$$e^{-x} + \frac{x}{5} - 1 = 0 \tag{37}$$

This is a transcendental equation, not easily solved by elementary algebra. The solution can be approximated, however, by expanding in Taylor's series, around $x = a$.We obtain to the third degree

$$\begin{aligned} y(x) &= e^{-x} + \frac{x}{5} - 1 \\ &\approx -\frac{e^{-a}}{6}x^3 + \left\{ \left(\frac{1}{2} + \frac{a}{2} \right) e^{-a} \right\} x^2 \\ &- \left\{ \left(1 + a + \frac{a^2}{2} \right) e^{-a} - \frac{1}{5} \right\} x \\ &+ \left\{ \left(1 + a + \frac{a^2}{2} + \frac{a^3}{6} \right) e^{-a} - 1 \right\}, \end{aligned}$$

defining:

$$\begin{aligned} a_3 &= -\frac{e^{-a}}{6}; \\ a_2 &= \left\{ \left(\frac{1}{2} + \frac{a}{2} \right) e^{-a} \right\}; \\ a_1 &= -\left\{ \left(1 + a + \frac{a^2}{2} \right) e^{-a} - \frac{1}{5} \right\}; \\ a_0 &= \left\{ \left(1 + a + \frac{a^2}{2} + \frac{a^3}{6} \right) e^{-a} - 1 \right\} \end{aligned}$$

$y(x)$ can be expressed approximately as:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{38}$$

At this point, the value of a is arbitrary. If we try to invert the polynomial (38) for a particular choice of a, it is very highly probable that the resulting solution will have a large error. So, in order to find what is the best a value in such a way that the solution for x has a low error, we first make some estimations of error for several values of a. After several trials, we found that a = 8 is a good choice.

By substituting this value in Eq. (37), we have

$$\begin{aligned} y = &-5.591 \times 10^{-5} x^3 + 0.00150958 x^2 \\ &+ 0.18624603 x - 0.95761989 \end{aligned}$$

and:

$$\begin{aligned} z &= y + 0.95761989 \\ &= -5.591 \times 10^{-5} x^3 + 0.00150958 x^2 + 0.18624603 x. \end{aligned}$$

The coefficients of the inverse polynomial, according to the relationships (10), are:

$$b_1 = -1.25, \quad b_2 = 0.9765625 \quad \text{and} \quad b_3 = -1.11897786$$

Therefore:

$$x = 0.06680501 z^3 - 0.23366602 z^2 + 5.3692419 z$$

or

$$\begin{aligned} x = &\ 0.06680501(y + 0.95761989)^3 \\ &- 0.23366602(y + 0.95761989)^2 \\ &+ 5.3692419(y + 0.95761989) \end{aligned}$$

Expanding and evaluating in y = 0, we get the root of (38) as

$$x = 4.9860789.$$

This is a very good approximation to the root of the function (37), because if we solve numerically through the method of the *Regula Falsi* [7], with a smaller error than 0.01%, the root is: 4.965332 $\pm$ 0.01%. Thus, the value obtained by inverting the polynomial differs from the numerical result in less than 0.5%.

Significantly, we found that, for a values between 6 and 10, the result is very similar to the above.

These examples have shown how to estimate the validity of the solutions of the method of polynomial inversion, for an approximating polynomial of third degree with a error smaller than 2%, but these conditions can be varied depending on the problem that is to be solved.

Next, some expressions are given to determine the coefficients of an inverse polynomial of a higher degree, which allows a reduction of the error and an increased of the interval of validity of the solution.

4. Coefficients for inverse polynomials of higher degree.

Using the method as described above, the coefficients of inverse, higher degree polynomials can be calculated; some of them are listed below. Expressions for b_4 to b_7 are the same as those reported by Dwight [9], whereas b_8 and b_9 are not reported in any other work we know [ii] .

Let $y(x)$ be given by

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots, \quad \text{with} \quad a_1 = 0. \tag{39}$$

Then, its inverse polynomial is given by

$$x = b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 + b_5 y^5 + b_6 y^6 + b_7 y^7 + \ldots, \tag{40}$$

where coefficients b_1,b_2 and b_3 are given by the relationships in (10), and [9]

$$b_4 = \frac{1}{a_1^7}\left(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3\right),$$

$$b_5 = \frac{1}{a_1^9}\left(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3\right),$$

$$b_6 = \frac{1}{a_1^{11}}(7a_1^3a_2a_5 + 7a_1^3a_3a_4 + 84a_1a_2^3a_3 - a_1^4a_6) - 28a_1^2a_2^2a_4 - 28a_1^2a_2a_3^2 - 42a_2^5,$$

$$b_7 = \frac{1}{a_1^{13}}(8a_1^4a_2a_6 + 8a_1^4a_3a_5 + 4a_1^4a_4^2 + 120a_1^2a_2^3a_4 + 180a_1^2a_2^2a_3^2 + 132a_2^6 - a_1^5a_7 - 36a_1^3a_2^2a_5 - 72a_1^3a_2a_3a_4 - 12a_1^3a_3^3 - 330a_1a_2^4a_3).$$

The two new coefficients are:

$$b_8 = \frac{1}{a_1^{15}}(9a_1^5a_2a_7 + 9a_1^5a_3a_6 + 9a_1^5a_4a_5 + 165a_1^3a_2^3a_5 + 495a_1^3a_2^2a_3a_4 + 165a_1^3a_2a_3^3 + 1287a_1a_2^5a_3 - a_1^6a_8 - 429a_2^7 - 45a_1^4a_2^2a_6 - 45a_1^4a_2a_4^2 - 90a_1^4a_2a_3a_5 - 45a_1^4a_3^2a_4 - 495a_1^2a_2^4a_4 - 990a_1^2a_2^3a_3^2),$$

$$b_9 = \frac{1}{a_1^{17}}(10a_1^6a_2a_8 + 10a_1^6a_3a_7 + 10a_1^6a_4a_6 + 5a_1^6a_5^2 + 55a_1^4a_3^4 + 220a_1^4a_2^3a_6 + 330a_1^4a_2^2a_4^2 + 660a_1^4a_2^2a_3a_5 + 660a_1^4a_2a_3^2a_4 + 5005a_1^2a_2^4a_3^2 + 2002a_1^2a_2^5a_4 + 1480a_2^8 - a_1^7a_9 - 55a_1^5a_2^2a_7 - 55a_1^5a_3^2a_5 - 110a_1^5a_2a_3a_6 - 110a_1^5a_2a_4a_5 - 55a_1^5a_3a_4^2 - 715a_1^3a_2^4a_5 - 2860a_1^3a_2^3a_3a_4 - 1430a_1^3a_2^2a_3^3 - 5005a_1a_2^6a_3),$$

As for the error terms, we only present the error terms for a third degree polynomial, and they are shown in Eq. (14); the error terms for higher degree can be calculated starting from the composition of polynomials, as was done in section 3; a detailed computation of the error terms would be too lengthy for this paper.

5. Conclusions

In this present work, we have evaluated the error of the inverse polynomial obtained through a method proposed by Arfken [1]. In addition, accepting a particular amount of error, we have shown how to obtain the interval of validity for the inverse approximating polynomial. We have shown how to apply the polynomial inversion method up to the third degree for solving five different problems not easily solved at the undergraduate level in physics and engineering. The first shows only how to do the inversion of the polynomial, and the second explains the procedure for finding the interval of validity for the inversion made in problem 1. The third example shows how to find one root of a third-degree equation. The fourth and fifth examples find the solutions for two basic physics problems where transcendental equations are found: one is for obtaining the semi width of a single-slit Fraunhofer diffraction pattern. The other is the calculation of the wavelength at which the radiation emitted by a black body has a maximum value. In addition, as a new result, we have obtained explicit expressions for two new coefficients for the inverse polynomial not reported in other references.

Once the expressions for the coefficients of the inverse polynomial and the expression of the error are obtained, the procedure of the inverted polynomial has the advantage of being very easy to apply, since it reduces to a direct evaluation, and in the case of extraction of roots, it also reduces to an evaluation, having the disadvantage that it is useful only in a vicinity around the origin; the radius of this vicinity depends on the magnitudes of the coefficients of the polynomial to invert. A useful advantage that the inverse polynomial offers is that it allows one to obtain an approximate analytic expression for transcendent inverse functions, which can be easily evaluated. It can be concluded that the procedure of inversion of polynomials is a good first approach to the solutions of some problems that involve polynomials or that can be approximated with them.

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i It is worth to recognize here that in many text books [1,9] the term **reversion** is used instead; the meaning, however, is the same as implied along this paper.

ii We tried to find the two additional references given by Dwight on page 15 of Ref. [9]; however we were not able to find them. Our sources state that the Philosophical Magazine was not published until 1921. The other reference is a book not found in México. In addition, *MATHEMATICA* [11] reports a general expression which is not completely defined because it is formally defined but not explicitly expressed; furthermore, *MATHEMATICA* does not account for the error of the inversion procedure.

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