Brownian motion of a charged particle in a magnetic field

J. I. Jiménez-Aquino, M. Romero-Bastida
Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa,
Apartado Postal 55-534, D.F., México,
e-mail: rbm@xanum.uam.mx, ines@xanum.uam.mx

Recibido el 31 de enero de 2006; aceptado el 18 de abril de 2006

In this work we show that Taylor's description of Brownian motion in a magnetic field is equivalent to the situation in which the constant magnetic field is allowed to point along any direction. This can be achieved by means of a rotation of the Langevin equation given in the space of coordinates $\mathbf{r}$, to another space of coordinates $\mathbf{r}'$, where the description of the problem is quite similar to that studied by Taylor. We use the over-damping approximation to show why, at equilibrium, the oscillatory behavior inherent in the system is not reflected in the diffusion process across the magnetic field, a fact not studied by Taylor. We also use the large-time approximation to study the effects of colored noise (small correlation time) on the diffusion processes across and along the magnetic field.

Keywords: (MSD) Mean Square Displacement.

En este trabajo mostramos que la descripción teórica de Taylor del movimiento Browniano en un campo magnético, es equivalente a la descripción teórica en la que el campo magnético constante apunta en una dirección arbitraria. Esto de logra mediante una rotación de la ecuación de Langevin, dada en el espacio de coordenadas $\mathbf{r}$ a otro espacio de coordenadas $\mathbf{r}'$, donde la descripción teórica del problema es totalmente similar al problema de Taylor. Usamos la aproximación de sobre-amortiguamiento, para mostrar porqué en el equilibrio, el comportamiento oscilatorio, inherente en el sistema, no se ve reflejado en el proceso de difusión transversal al campo magnético, situación que no fue estudiado por Taylor. Usamos también la aproximación en el límite de tiempos largos para estudiar los efectos del ruido de color (tiempos de correlación pequeños), en los procesos de difusión transversal y a lo largo del campo magnético.

Descriptores: (DCP) Desplazamiento Cuadrático Promedio.

PACS: 05.40.-a; 02.50.-r

1. Introduction

The stochastic diffusion of a plasma across a magnetic field arising from the fluctuations of the electric field was solved by Taylor in 1961 using a theoretical Langevin description [1]. In that work, a situation is considered in which the density gradient of charged particles exists only in the direction perpendicular to the external magnetic field. Due to this physical situation, the diffusion process is described by two coupled stochastic differential equations associated with two components of the ion velocity vector. The diffusion process is then characterized through the Mean Square Displacement (MSD) across the magnetic field. In the next year, the same problem on the Brownian motion in a magnetic field was solved by Kurşunoğlu [2] by making an extension to the Chandrasekhar [3] treatment of ordinary Brownian motion in terms of the probability distribution function associated with the magnitude of the velocity. In his work, Kurşunoğlu assumed that the surrounding medium in which the charged particles are diffused is the electromagnetic field which is envisaged by a set of classical harmonic oscillators. By assuming the existence of a fluctuating electric field in the plasma, the charged particles in the field can be assumed to undergo a large number of collisions per second with the oscillators. Therefore, the diffusion is described as a stochastic process where some kind of dynamical friction proportional to the velocity of the particle acts. Consequently, the usual expressions for the diffusion processes across and along the external magnetic field were obtained. Almost forty years later, the problem has again become of interest to other scientists, cf. [4-7]. In particular, in Ref. 4 the full description of the Brownian motion in the magnetic field is given through the transition probability densities for the velocity-space, phase-space, and the Smoluchowsky configuration-space. In Refs. 1, 2, and 4 the theoretical developments were pursued by assuming that the constant magnetic field vector explicitly points along the z-axis, that is $\mathbf{B} = (0, 0, B_z)$. Our purpose in this work is to study three theoretical extensions to Taylor's proposal. The first one consists in showing that Taylor's problem is equivalent to a situation in which the constant magnetic field is allowed to point along any direction, that is $\mathbf{B} = (B_x, B_y, B_z)$. This can easily be achieved by means of a rotation of this magnetic field along the z' axis of the transformed space of coordinates (x', y', z'). The second one, although it would seems to be obvious, was not considered by Taylor and consists in the following: a charged particle in a constant magnetic field is a strictly rotational phenomenon and the conditions under which the Brownian motion in a magnetic field have been studied (small fluctuations of the electric field) also correspond to a rotational phenomenon. The question then is why, at equilibrium (large time-limit), the rotational effects of the charged Brownian particle are not reflected in the diffusion process across the magnetic field as effectively shown by Taylor and Kurşunoğlu [1, 2]? The answer to this question can easily be clarified if we focus on the Langevin equation in the large-friction force approximation, also known as the over-damping problem. The third case is the following: because the large-time limit or diffu-

sive regime is the regime of interest, we take advantage of the results of the over-damped problem to study the effects of colored noise on the MSD across and along the magnetic field for a small correlation time. Two appendices are included to support our theoretical results, and conclusions are given at the end of the work.

2. The Langevin equation for a plasma

The Brownian motion of a charged particle can be described by an equation similar to the Langevin equation associated with the velocity column vector $\mathbf{v}$, which reads as [2]

$$m\dot{\mathbf{v}} = -\alpha\mathbf{v} + \frac{e}{c}\mathbf{v}\times\mathbf{B} + e\mathbf{E}(t)\,, \tag{1}$$

where $-\alpha\mathbf{v}$, with α positive, is the friction force added to the Lorentz force for a charged particle in an electromagnetic field, and refers to a dynamical friction between the charges; it is not independent of the fluctuating force $\mathbf{E}(t)$, and it actually gives rise to the stochastic fluctuations of this electric field. The above equation can also be written as

$$\dot{\mathbf{v}} = -\beta\,\mathbf{v} + W\mathbf{v} + \widetilde{\mathbf{E}}(t)\,, \tag{2}$$

where $\beta = \alpha/m$, W is a real antisymmetric matrix given by

$$W = \begin{pmatrix} 0 & \omega_{_3} & -\omega_{_2} \\ -\omega_{_3} & 0 & \omega_{_1} \\ \omega_{_2} & -\omega_{_1} & 0 \end{pmatrix}\,, \tag{3}$$

whose elements, defined as $\omega_i = eB_i/mc$, are known as the Larmor frequency, B_i being each component of the magnetic field vector $\mathbf{B}$ where the subindex i may take on the values $1, 2, 3$ representing the coordinates x, y, z respectively. The fluctuating electric field $\widetilde{\mathbf{E}}(t) = (e/m)\mathbf{E}(t)$ satisfies the properties of Gaussian white noise with zero mean value and correlation function

$$\langle \widetilde{E}_i(t)\widetilde{E}_j(t')\rangle = 2q\,\delta_{ij}\,\delta(t-t')\,, \tag{4}$$

q being the noise intensity.

If we make the change of variable $\mathbf{v}' = R^{^T}\mathbf{v}$, where $R^{^T}$ is the transpose of the rotation matrix R given in Appendix A, then Eq. (2) is becomes

$$\dot{\mathbf{v}}\,' = -\beta\,\mathbf{v}' + W'\mathbf{v}' + \widetilde{\mathbf{E}}'(t)\,, \tag{5}$$

where $\widetilde{\mathbf{E}}'(t) = R^{^T}\widetilde{\mathbf{E}}(t)$ and $W' = R^{^T}WR$ is another antisymmetric matrix such that $\mathfrak{R}(t) = \mathrm{e}^{W't}$ is in general an orthogonal rotation matrix satisfying $\mathfrak{R}^{^T}(t) = \mathfrak{R}^{-1}(t)$, *i.e.* the transpose is its inverse and therefore $\mathfrak{R}^{-1}(t) = \mathrm{e}^{-W't}$. Such matrices, as shown in Appendices A and B, are given by

$$W' = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,\quad \mathfrak{R}(t) = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{6}$$

such that $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = e^2\,B^2/m^2\,c^2$, with $B^2 = B_1^2 + B_2^2 + B_3^2$ being the square modulus of the external magnetic field $\mathbf{B}$. The second term of Eq. (5) can also be written as the cross product $W'\mathbf{v}' = (e/mc)\mathbf{v}'\times\mathbf{B}'$, where $\mathbf{B}'$ can be visualized as an external magnetic field given by $\mathbf{B}' = B\,\hat{\mathbf{k}}'$, $\hat{\mathbf{k}}'$ being the unitary vector along the z' axis. Therefore, in the transformed space of coordinates, the external magnetic field points, in a natural way, along the z' axis. The Brownian motion of a plasma will be better described through the Langevin equation (5), whose solution may be written as

$$\mathbf{v}'(t) = \mathrm{e}^{-\beta\,t}\,\mathfrak{R}(t)\,\mathbf{v}'(0) + \mathfrak{R}(t)\int_0^t \mathrm{e}^{-\beta(t-s)}\,\mathfrak{R}^T(s)\,\widetilde{\mathbf{E}}'(s)\,ds\,. \tag{7}$$

To calculate the correlation function for the velocity $\mathbf{v}'(t)$ at two different times, we have to impose the initial condition on velocity $\mathbf{v}(0)$. We assume that it is determined by the Maxwell distribution function

$$P_1(\mathbf{v}(0)) = \left(\frac{m}{2\pi k_B T}\right)^{3/2}\exp\left[-\frac{m}{2k_{_B}T}\,\mathbf{v}(0)\cdot\mathbf{v}(0)\right]\,, \tag{8}$$

in which $\mathbf{v}(0)\cdot\mathbf{v}(0)$ denotes a scalar product, k_B is the Boltzmann constant, and T is the temperature of the surrounding medium. It is also assumed that there is no correlation between $\mathbf{v}(0)$ values and $\widetilde{\mathbf{E}}(t)$ for any $t \geq 0$. Because it will be necessary to average over $\mathbf{v}'(0)$ using the Maxwell distribution, as well as averaging stochastically with respect to the noise $\widetilde{\mathbf{E}}'(t)$, we use the notation $\{\cdots\}$ for the averages over $\mathbf{v}'(0)$. In this case, due to the transformations of $\mathbf{v}'(t)$ and $\widetilde{\mathbf{E}}'(t)$, it can also be shown that there is no correlation between $\mathbf{v}'(0)$ and $\widetilde{\mathbf{E}}'(t)$. Also

$$\langle\mathbf{v}'(t)\rangle = \mathrm{e}^{-\beta\,t}\,\mathfrak{R}(t)\,\mathbf{v}'(0) \quad\text{and}\quad \{\langle\mathbf{v}'(t)\rangle\} = 0\,. \tag{9}$$

The correlation function at two different times for the components of velocity $\mathbf{v}'(t)$ reads as

$$\{\langle v_i'(t_1)v_j'(t_2)\rangle\} = \mathrm{e}^{-\beta\,(t_1+t_2)}\,\mathfrak{R}_{ik}(t_1)\mathfrak{R}_{jl}(t_2)\,\{v_k'(0)v_l'(0)\} + \mathfrak{R}_{ik}(t_1)\mathfrak{R}_{jl}(t_2)\langle h_k(t_1)\,h_l(t_2)\rangle, \tag{10}$$

where the stochastic average is

$$\langle h_k(t_1)\,h_l(t_2)\rangle = \int_0^{t_1}\int_0^{t_2}\mathrm{e}^{-\beta(t_1+t_2-t_1'-t_2')} \times(\mathfrak{R}^T)_{km}(t_1')(\mathfrak{R}^T)_{ln}(t_2')\,\langle\widetilde{E}_m'(t_1')\widetilde{E}_n'(t_2')\rangle\,dt_1'\,dt_2'. \tag{11}$$

Using the Maxwell distribution (8), it can be shown that

$$\{v_k'(0)v_l'(0)\} = \frac{k_{_B}\,T}{m}\,\delta_{kl} \tag{12}$$

and, using the correlation function (4), we can also show that

$$\langle\widetilde{E}_m'(t_1')\widetilde{E}_n'(t_2')\rangle = 2q\,\delta_{mn}\,\delta(t_1'-t_2')\,. \tag{13}$$

Therefore, the stochastic average (11) reduces to

$$\langle h_k(t_1)\, h_l(t_2)\rangle = 2q \int_0^{t_1}\int_0^{t_2} \mathrm{e}^{-\beta(t_1+t_2-t_1'-t_2')} \times(\mathfrak{R}^T)_{kn}(t_1')(\mathfrak{R}^T)_{ln}(t_2')\ \delta(t_1'-t_2')\, dt_1'\, dt_2'. \quad (14)$$

So, according to Eqs. (12) and (14), the correlation function (10) is then

$$\{\langle v_i'(t_1)v_j'(t_2)\rangle\} = \frac{k_{_B}\,T}{m}\,\mathrm{e}^{-\beta\,(t_1+t_2)}\ \mathfrak{R}_{ik}(t_1)\mathfrak{R}_{jk}(t_2) + \frac{q}{\beta}\,\mathfrak{R}_{ik}(t_1)\mathfrak{R}_{jk}(t_2)\big[\mathrm{e}^{-\beta\,|t_1-t_2|} - \mathrm{e}^{-\beta\,(t_1+t_2)}\big]. \quad (15)$$

The constant q can be calculated from the fluctuation-dissipation relation [8], which is related to the average kinetic energy at equilibrium. This average can be calculated from Eq. (15) at the time $t_1 = t_2 = t$, yielding

$$\frac{1}{2}m\,\{\langle v'^{\,2}(t)\rangle\} = \frac{3k_{_B}\,T}{2}\,\mathrm{e}^{-2\beta\,t} + \frac{3q\,m}{2\beta}\,\big[1-e^{-2\beta\,t}\big], \quad (16)$$

where $v'^{\,2}(t)$ is the square modulus of velocity $\mathbf{v}'(t)$. As time goes to infinity, the Brownian particle attains thermal equilibrium with the surrounding medium in which it is immersed. Consequently, this average kinetic energy should be $3k_{_B}T/2$. Eq.(15) agrees with this value if and only if $q = \beta\, k_{_B}T/m$, which is precisely the fluctuation-dissipation relation. Once this relation is used in (16), it is seen that, for all time,

$$\frac{1}{2}m\,\{\langle v'^{\,2}(t)\rangle\} = \frac{3}{2}\,\frac{k_{_B}\,T}{m}\,, \quad (17)$$

which exhibits one aspect of the stationarity of the process. Therefore, in the transformed space of velocities, the plasma diffusion also satisfies the same fluctuation-dissipation relation as that of the usual Brownian motion.

2.1. The Mean-Square Displacement

In the transformed space of velocities $\mathbf{v}'$, it is clear that $\mathbf{v}'(t) = d\mathbf{r}'(t)/dt$, where $\mathbf{r}'(t) = R^{^T}\,\mathbf{r}(t)$. So, if at time $t = 0$ the particle is at $\mathbf{r}'(0)$, then the MSD for the vector $\mathbf{r}'(t)$ is given by

$$\langle[\mathbf{r}'(t)-\mathbf{r}'(0)]^2\rangle \equiv \sum_{i=1}^{3}\langle[r_i'(t)-r_i'(0)]^2\rangle = \sum_{i=1}^{3}\int_0^t\int_0^t \{\langle v_i'(t_1)v_i'(t_2)\rangle\}\, dt_1\, dt_2. \quad (18)$$

To calculate this quantity, we must calculate the MSD for each component of vector $\mathbf{r}'(t)$. If we define the components of this vector as $r_1'(t) \equiv x'(t)$, $r_2'(t) \equiv y'(t)$ and $r_3'(t) \equiv z'(t)$, then the MSD for each component and their cross correlations can be calculated using Eq. (15). For the $x'(t)$ component we have, after some algebra, that

$$\langle(\Delta x')^2\rangle \equiv \langle[x'(t)-x'(0)]^2\rangle = \frac{q}{\beta}\bigg[\int_0^t\int_0^t \cos\omega(t_1-t_2)\,\mathrm{e}^{-\beta\,|t_1-t_2|}\,dt_1\,dt_2\bigg], \quad (19)$$

which can be readily evaluated, yielding:

$$\langle(\Delta x')^2\rangle = 2\bigg(\frac{D}{1+(\omega/\beta)^2}\bigg)\,t - \frac{q}{\beta\,\Lambda_1^2}(1-\mathrm{e}^{-\Lambda_1\,t}) - \frac{q}{\beta\,\Lambda_2^2}(1-\mathrm{e}^{-\Lambda_2\,t}) \quad (20)$$

where $D{=}q/\beta^2{=}k_{_B}T/m\beta$, which is consistent with the value of the Einstein coefficient, $\Lambda_1{=}\beta{-}\mathrm{i}\omega$, and $\Lambda_2{=}\beta{+}\mathrm{i}\omega$. The same expression is obtained for the $y'(t)$ component, that is $\langle(\Delta y')^2\rangle{\equiv}\langle[y'(t)-y'(0)]^2\rangle{=}\langle(\Delta x')^2\rangle$. The MSD for the $z'(t)$ component is simply

$$\langle(\Delta z')^2\rangle \equiv \langle[z'(t)-z'(0)]^2\rangle = \frac{q}{\beta}\bigg[\int_0^t\int_0^t \mathrm{e}^{-\beta\,|t_1-t_2|}\,dt_1\,dt_2\bigg], \quad (21)$$

which leads to the following result

$$\langle(\Delta z')^2\rangle = 2\,D\,t - \frac{2q}{\beta^3}(1-\mathrm{e}^{-\beta\,t})\,, \quad (22)$$

which is also an expected result. On the other hand, it can be shown that cross correlation functions

$$\langle(\Delta x')(\Delta y')\rangle = \langle(\Delta y')(\Delta x')\rangle = 0,$$
$$\langle(\Delta x')(\Delta z')\rangle = \langle(\Delta z')(\Delta x')\rangle = 0,$$

and

$$\langle(\Delta y')(\Delta z')\rangle = \langle(\Delta z')(\Delta y')\rangle = 0.$$

From the expressions given in Eqs. (20) and (22), we can make the analysis for short and large times. For short times, such that $\beta t \ll 1$ and $\omega t \ll 1$, we get the following expressions for the three components:

$$\langle(\Delta x')^2\rangle = \langle(\Delta y')^2\rangle = \langle(\Delta z')^2\rangle = \frac{q}{\beta}\,t^2; \quad (23)$$

therefore,

$$\langle[\mathbf{r}'(t){-}\mathbf{r}'(0)]^2\rangle{=}\sum_{l=1}^{3}\langle[r_i'(t){-}r_i'(0)]^2{=}3\bigg(\frac{k_{_B}\,T}{m}\bigg)\,t^2. \quad (24)$$

This result shows that, in this regime of approximation, the MSD is the same for the three components and proportional to t^2, which corresponds to the behavior of a free particle. This means that, in this limit of approximation, the plasma is not sensitive to the surrounding medium in which it is immersed. For large times, such that $\beta t \gg 1$, $\omega t \gg 1$

and also $\omega < \beta$ we have, for the x' and y' components, the following expression:

$$\langle(\Delta x')^2\rangle = \langle(\Delta y')^2\rangle = 2\left(\frac{D}{1+(\omega/\beta)^2}\right)t\,, \qquad (25)$$

and for the z' component,

$$\langle(\Delta z')^2\rangle = 2\,Dt\,. \qquad (26)$$

Finally, the MSD will be

$$\langle[\mathbf{r}'(t)-\mathbf{r}'(0)]^2\rangle = \left[\left(\frac{2}{1+(\omega/\beta)^2}\right)+1\right]2D\,t\,. \qquad (27)$$

The expressions given by Eqs. (25) and (26) are the MSD across and along the magnetic field respectively, and they are proportional to t. Therefore, in this regime of approximation, which corresponds to the diffusive regime, the plasma enters in contact with the surrounding medium through the incessant collisions with the oscillators. We can also see, from Eqs. (25) and (26), that the MSD's are related by

$$\langle(\Delta x')^2\rangle{=}\langle(\Delta y')^2\rangle{=}\frac{\langle(\Delta z')^2\rangle}{1+(\omega/\beta)^2}{=}\frac{\beta^2\langle(\Delta z')^2\rangle}{\beta^2+\omega^2}\,, \qquad (28)$$

and the maximum value of the MSD across the magnetic field occurs at $\beta=\omega$, that is

$$\langle(\Delta x')^2\rangle_{\max} = \langle(\Delta y')^2\rangle_{\max} = \left(\frac{k_{_B}T}{eB}\right)ct\,, \qquad (29)$$

where T is the temperature of the diffusing particle at equilibrium.

As can be seen, for intermediate times the MSD across the magnetic field, given by Eq. (20), contains the rotational effects of the system through the factors $\mathrm{e}^{-\Lambda_1 t} = (\cos\omega t + \mathrm{i}\sin\omega t)\,\mathrm{e}^{-\beta t}$ and $\mathrm{e}^{-\Lambda_2 t} = (\cos\omega t - \mathrm{i}\sin\omega t)\,\mathrm{e}^{-\beta t}$, which clearly disappear in the large time limit. For this reason the expression given by (25) does not contain these rotational effects.

2.2. The over-damped problem

Expressions (25) and (26) describe the plasma diffusion at equilibrium and they do not properly exhibit the rotational character of the system. This physical situation can be understood if we pay attention to the solution to the Langevin equation in the large frictional force approximation. In this limiting case, the time derivative of Eq.(5) can be neglected, resulting in the following approximation:

$$\mathbf{v}' = \widetilde{W}'^{-1}\,\widetilde{\mathbf{E}}'(t)\,, \qquad (30)$$

where $\widetilde{W}'^{-1}$ is defined as the inverse of the matrix $\widetilde{W}' = \beta I - W'$, such that

$$\widetilde{W}'^{-1} = \begin{pmatrix} \frac{\beta}{\beta^2+\omega^2} & \frac{\omega}{\beta^2+\omega^2} & 0 \\ \frac{-\omega}{\beta^2+\omega^2} & \frac{\beta}{\beta^2+\omega^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{pmatrix}. \qquad (31)$$

The large friction-force approximation (30) is well known as the over-damped problem, in which W'^{-1} can be called a "diffusion coefficient matrix". It is now clear that, in the dynamical evolution of Eq. (30), there is no rotational effect. Here, the MSD can simply be calculated by solving Eq. (30) in terms of the vector $\mathbf{r}'(t)$. In this case,

$$\langle[\mathbf{r}'(t)-\mathbf{r}'(0)]^2\rangle = \sum_{i=1}^{3}\langle[r_i'(t)-r_i'(0)]^2]\rangle$$
$$= 2q\sum_{i,k=1}^{3}(\widetilde{W}'^{-1})_{ik}^2\int_0^t\int_0^t\delta(t_1-t_2)\,dt_1\,dt_2. \qquad (32)$$

So, for each component we conclude that

$$\langle(\Delta x')^2\rangle = \langle(\Delta y')^2\rangle = 2\left(\frac{D}{1+(\omega/\beta)^2}\right)t\,, \qquad (33)$$

and

$$\langle(\Delta z')^2\rangle = 2\,D\,t\,. \qquad (34)$$

The rest of the cross correlation functions are equal to zero.

2.3. The colored noise problem

For this problem, the fluctuating electric field $\widetilde{\mathbf{E}}(t)$ satisfies the properties of Gaussian colored noise with zero mean value and correlation function [13, 14]

$$\langle\widetilde{E}_i(t)\widetilde{E}_j(t')\rangle = \frac{q}{\tau}\,\delta_{ij}\,\mathrm{e}^{-|t-t'|/\tau}\,, \qquad (35)$$

with τ the correlation time of noise.

Because the large-time limit is the time interval of interest, we study the colored noise problem in the over-damped approximation given by Eq.(30). In this case, the MSD is now written as

$$\langle[\mathbf{r}'(t)-\mathbf{r}'(0)]^2\rangle = \sum_{i=1}^{3}\langle[r_i'(t)-r_i'(0)]^2]\rangle$$
$$= \frac{q}{\tau}\sum_{i,k=1}^{3}(\widetilde{W}'^{-1})_{ik}^2\int_0^t\int_0^t\mathrm{e}^{-|t-t'|/\tau}\,dt_1\,dt_2\,. \qquad (36)$$

By evaluating the integral we have, for the x' component

$$\langle(\Delta x')^2\rangle = 2\left(\frac{D}{1+(\omega/\beta)^2}\right)\left[t-\tau\left(\mathrm{e}^{-t/\tau}-1\right)\right], \qquad (37)$$

and the same expression for the y' component. The MSD for the z' component is

$$\langle(\Delta z')^2\rangle = 2\,D\left[t-\tau\left(\mathrm{e}^{-t/\tau}-1\right)\right]. \qquad (38)$$

The rest of the cross correlation functions are equal to zero. In the limit of the small correlation time such that $t \gg \tau$, we get

$$\langle(\Delta x')^2\rangle = \langle(\Delta y')^2\rangle = 2\left(\frac{D}{1+(\omega/\beta)^2}\right)(t+\tau)\,, \qquad (39)$$

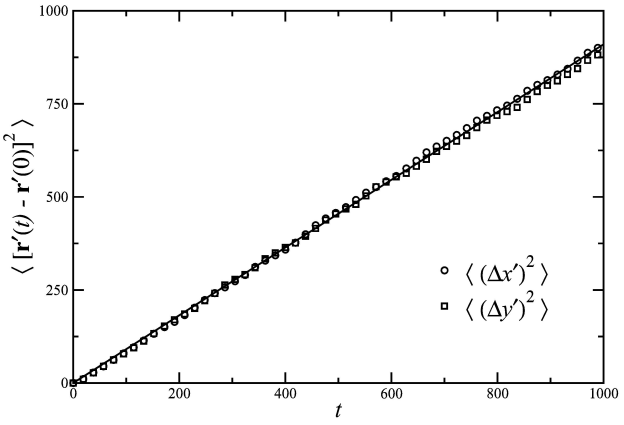

FIGURE 1. MSD across the magnetic field for each of the charged particle position components in the transformed space. The theoretical result given by Eq. (33) is the dashed line, and the simulation results are the circles and squares.

and

$$\langle (\Delta z')^2 \rangle = 2\, D(t+\tau)\,. \tag{40}$$

Obviously, for $\tau = 0$ the expressions (39) and (40) reduce to the white noise case.

2.4. Numerical simulation of the over-damped problem

In order to corroborate the plasma diffusion across the magnetic field, we perform a numerical study of the problem in the over-damping case for white noise. If we rescale the position and time variables as $\mathbf{r}' \to \mathbf{r}'/l_m$ and $t \to \omega t$, where $l_m = (mc^2/B^2)^{1/3}$ and ω is the aforementioned Larmor frequency, we obtain a dimensionless expression for Eq.(5), in which we take $\omega = 30$. These equations were written as a multivariate Langevin equation for which a suitable integration algorithm was proposed in Ref. 9. The simulation parameters were taken as $\Delta t = 0.001$ for the integration timestep, $\beta = 100.0$ and $q = 5\,000$, which guarantee that we are simulating the over-damping problem, and the position variables represent a driftless Wiener process [10]. The results of the theoretical expression (33) agree with the simulation results, as can be appreciated in Fig. 1.

3. Conclusions

By means of a rotation of the Langevin equation given by Eq. (2), we have shown that the Brownian in a constant magnetic field, allowed to point along any direction, is equivalent to that studied by Taylor, as effectively shown by Eq. (5). In our case, the two x' and y' velocity components are coupled and are very similar to those proposed for Taylor, except by the expression of the Larmor frequency, which in our case is $\omega = eB/mc$, where $B^2 = B_1^2 + B_2^2 + B_3^2$. This is the frequency with which the charged particle rotates around the z'-axis of the transformed space of coordinates before it reaches a state of equilibrium state as time goes to infinity. The z' velocity component satisfies the usual Brownian motion equation and is independent of the other two equations. So, for a large-time limit, the MSD across the magnetic field (25) is similar to that obtained by Taylor and Kurşunoğlu, except for the Larmor frequency ω. It reduces to the Taylor's and Kurşunoğlu's results if $B_1 = B_2 = 0$, for which $\omega = eB_3/mc$. According to this result, the diffusion process across the magnetic field defines an effective diffusion constant given by $D_e = D/[1 + (\omega/\beta)^2]$.

We have also shown that, in the large-time limit, the diffusion process is essentially equivalent to the over-damped problem, which yields to the Langevin approximation (30). The solution to this equation is clearly not rotational, as shown by Eq.(32), and therefore the plasma diffusion across the magnetic field is described without any oscillatory behavior, as can be corroborated in the numerical simulation results displayed in Fig. 1.

Finally, we take advantage of the over-damped approximation to calculate the effect of colored noise on the diffusion processes across and along the magnetic field for small τ, yielding to the approximations (39) and (40). The non-markovian contribution to the MSD's in this approximation is a time translation for small τ values.

Acknowledgments

The authors wish to thank R. M. Velasco for her valuable comments and suggestions. Financial support from the Consejo Nacional de Ciencia y Tecnología (CONACYT) México is also acknowledged. acknowledgments

A Appendix: Transformation of matrix W

Given the antisymmetric matrix

$$W = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{A.1}$$

it can be transformed into another antisymmetric matrix W', which defines a new reorientation of the magnetic field $\mathbf{B}$. The transformation can be achieved by means of a rotation matrix *R* which is composed by the unitary eigenvectors of matrix W [11]. Therefore, the rotation matrix will be

$$R = \begin{pmatrix} \frac{-\omega_1\omega_3}{\omega\sqrt{\omega_1^2+\omega_2^2}} & \frac{\omega_2}{\sqrt{\omega_1^2+\omega_2^2}} & \frac{\omega_1}{\omega} \\ \frac{-\omega_2\omega_3}{\omega\sqrt{\omega_1^2+\omega_2^2}} & \frac{-\omega_1}{\sqrt{\omega_1^2+\omega_2^2}} & \frac{\omega_2}{\omega} \\ \frac{\omega_1^2+\omega_2^2}{\omega\sqrt{\omega_1^2+\omega_2^2}} & 0 & \frac{\omega_3}{\omega} \end{pmatrix}, \tag{A.2}$$

where the first and second column are the real and imaginary part of one of the two complex eigenvectors of *W*. So, the following transformation $R^T WR$ leads to

$$W' = R^T WR = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ and R^T is the transpose of matrix R, such that $R^T R = I$.

B Appendix: The rotation matrix $\mathfrak{R}(t)$

To show that the relation $\mathrm{e}^{W't} = \mathfrak{R}(t)$ is a rotation matrix, we follow the proposal of Ref. 12. For this purpose, we define the rotation angle $\phi = \omega t$ and the matrices

$$M_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \mathrm{i}M_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(B.1)}$$

such that A is real and antisymmetric and therefore $W't = \mathrm{i}\phi M_z$. Using the property of the exponential, we have

$$\mathrm{e}^{W't} = \mathrm{e}^{\mathrm{i}\phi M_z} = I + \mathrm{i}\phi M_z + \frac{(\mathrm{i}\phi M_z)^2}{2!} + \frac{(\mathrm{i}\phi M_z)^3}{3!} + \cdots \quad \text{(B.2)}$$

where I is the unit matrix. Collecting both the odd and even terms and taking into account that

$$M_z^{2n} = S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_z^{2n+1} = M_z\,, \quad \text{(B.3)}$$

where S is a symmetric matrix and $\mathrm{i}M_z^{2n+1} = A$, it can be shown that Eq.(B2) reduces to

$$\mathrm{e}^{W't} = I + (\cos\phi - 1)S + \sin\phi\, A; \quad \text{(B.4)}$$

therefore, by defining $\mathfrak{R}(t) = \mathrm{e}^{W't}$, we finally have that

$$\mathfrak{R}(t) = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(B.5)}$$

is an orthogonal rotation matrix because $\mathfrak{R}(t)\mathfrak{R}^T(t) = I$.

1. J.B. Taylor, *Phys. Rev. Lett.* **6** (1961) 262.
2. B. Kurşunoğlu, *Ann. Phys.* **17** (1962) 259.
3. S. Chandrasekhar, *Rev. Mod. Phys.* **15** (1943) 1.
4. R. Czopnik and P. Garbazewsky, *Phys. Rev. E* **63** (2001) 021105.
5. T.P. Simões and R.E. Lagos, *Physica A* **355** (2005) 274.
6. H. Hayakawa, *Physica D* bf 205 (2005) 48.
7. L. Ferrari, *J. Chem. Phys.* **118** (2003) 11092.
8. N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
9. D.T. Guillespie, *Am. J. Phys.* **64** (1996) 1246.
10. D.T. Guillespie, *Phys. Rev. E* **54** (1996) 2084.
11. E. Piña, *Acta Mexicana de Ciencia y Tecnología* **VII** (1973) 55.
12. E. Piña, *Dinámica de Rotaciones* (Universidad Autónoma Metropolitana, México, 1996).
13. G.E. Uhlenbeck and L.S. Ornstein, *Phys. Rev.* **36** (1930) 823.
14. M.C. Wang and G.E. Uhlenbeck, *Rev. Mod. Phys.* **17** (1945) 323.