Complete electromagnetic multipole expansion including toroidal moments

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The electromagnetic multipole expansion presented in this paper is complete on two accounts: *i)* It is valid for all points in space, and *ii)* it recognizes the existence of toroidal moments. The electromagnetic field due to alternating poloidal currents in a toroidal solenoid is evaluated exactly via the solution of the inhomogeneous vector Helmholtz equations, using the outgoing wave Green function technique and the Debye potentials for sources and fields. The physical meaning of the toroidal moments can be appreciated when they are compared with the familiar electric and magnetic moments; the analysis of the long-wavelength limit of the exact results also explains the previous neglect of the toroidal moments. The magnetostatic limit and the point source limit are also physically and didactically interesting.

Keywords: Electromagnetic radiation; multipole expansion; electric, magnetic and toroidal moments.

El desarrollo multipolar electromagnético que se presenta en este artículo es completo por dos razones: *i)* es válido para todos los puntos del espacio, y *ii)* reconoce la existencia de momentos toroidales. El campo electromagnético producido por corrientes poloidales alternas en un solenoide toroidal se evalúa exactamente a través de la solución de ecuaciones de Helmholtz inhomogéneas, usando la técnica de la función de Green de onda saliente y los potenciales de Debye para fuentes y campos. El significado físico de los momentos toroidales se destaca al compararlos con los momentos eléctricos y magnéticos familiares; el análisis del límite de longitud de onda grande de los resultados exactos también explica la ignorancia previa de los momentos toroidales. El límite magnetostático y el límite de fuente puntual son también interesantes físicamente y didácticamente.

Descriptores: Radiación electromagnética; desarrollo multipolar; momentos eléctricos, magnéticos y toroidales.

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1. Introduction

The standard presentations of the multipole expansions of electrostatic, magnetostatic and electromagnetic fields in the textbooks are usually incomplete since they are restricted to the regions outside localized sources [1-6]. Two articles may be cited from the didactic literature making up for this incompleteness; one of them deals with the complete vector spherical harmonic expansion for Maxwell's equations [7], and the other gives the multipole expansions outside and inside the sources for the electrostatic and magnetostatic cases [8].

Another gap in the textbooks is the absence of any mention about toroidal moments. It will soon be fifty years since the violation of parity was established in the weak interactions, and the introduction of the anapole moment by Zel'dovich in his note on "Electromagnetic interaction with parity violation" [9]. The Russian authors have been active investigating the multipole expansion in classical and quantum field theory, the electromagnetic fields of toroidal solenoids and correspondingly the existence and importance of toroidal moments [10-13], in the ensuing period. It is time that the study of such topics should be incorporated into the advanced courses in electrodynamics.

This paper presents a complete electromagnetic multipole expansion valid for all points in space, with emphasis on the presence of toroidal moments on the same footing as the familiar electric and magnetic moments. Section 2 provides the theoretical framework for the study, including as the starting point Maxwell's equations for harmonic time varying sources, and their transformation into inhomogeneous Helmholtz equations. The corresponding solutions are constructed by using the outgoing wave Green function and its spherical multipole expansion, as well as the Debye potentials in order to exhibit the longitudinal and transverse components of the respective vector sources and fields [14]. Section 3 is devoted to the construction of the magnetic field due to poloidal currents in a toroidal solenoid with a circular ring sector cross section in each meridian plane, including the multipole expansion, the Debye potentials, the dynamic toroidal multipole moments, the long wavelength limit, the magnetostatic limit, and the point source limit. Section 4 contains a discussion about the main new results and some of their consequences.

2. Maxwell's equations and Debye potentials

The charge density ρ, the current density $\vec{J}$, the electric intensity field $\vec{E}$ and the magnetic induction field $\vec{B}$ are related by Maxwell's equations [1-6]:

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1}$$

$$\nabla \times \vec{E} = \frac{i\omega}{c}\vec{B} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{E}, \tag{4}$$

corresponding to Gauss' law, Faraday's law, nonexistence of magnetic monopoles, and Ampére-Maxwell's law, for harmonic time variations $e^{-i\omega t}$ with frequency ω for the sources and fields. Here $\omega/c = k$ is identified as the wave number.

Equations (1-4) are a set of linear coupled equations for the fields. They can be decoupled, by taking the curl of Eqs. (2) and (4) and using the remaining Eqs. (1) and (3), to obtain the respective Helmholtz inhomogeneous equations for the electric and magnetic field:

$$(\nabla^2 + k^2)\vec{E}(\vec{r}) = 4\pi\nabla\rho - \frac{4\pi i\omega}{c^2}\vec{J}(\vec{r}) \tag{5}$$

$$(\nabla^2 + k^2)\vec{B}(\vec{r}) = -\frac{4\pi}{c}\nabla \times \vec{J}(\vec{r}). \tag{6}$$

Equation (6) shows that the magnetic induction field is determined by the transverse component of the current density. In contrast, Eq. (5) shows that the electric intensity field is determined by the gradient of the charge density and both longitudinal and transverse components of the current density.

The solutions to Eqs. (5), (6) can be constructed with the help of the Green function of the Helmholtz equation

$$(\nabla^2 + k^2)\mathcal{G}_+(\vec{r};\vec{r}') = -4\pi\delta(\vec{r} - \vec{r}'). \tag{7}$$

The outgoing wave Green function and its multipole expansion are given by

$$\mathcal{G}_+(\vec{r};\vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=0}^{l}(-)^m N_{lm}^2 P_l^m(\cos\theta')P_l^m(\cos\theta)(2-\delta_{m0})\cos m(\varphi - \varphi'). \tag{8}$$

The particular solutions to Eqs. (5) and (6) involve the integrations of the respective sources and the Green function

$$\vec{E}(\vec{r}) = -\int dv'\nabla'\rho(\vec{r}')\mathcal{G}_+(\vec{r};\vec{r}')$$
$$+ \frac{i\omega}{c^2}\int dv'\vec{J}(\vec{r}')\mathcal{G}_+(\vec{r};\vec{r}') \tag{9}$$

$$\vec{B}(\vec{r}) = \int dv'\nabla' \times \vec{J}(\vec{r}')\mathcal{G}_+(\vec{r};\vec{r}'). \tag{10}$$

Use of the multipole expansion of the Green function of Eq. (8) in Eqs. (9) and (10) leads to the corresponding general and exact expansion of the electromagnetic field. The particular application to the toroidal solenoid is made in the following section.

The Debye potentials, as has been pointed out by Gray [14], are useful for exhibiting the decomposition of the source and force fields into their longitudinal and transverse (toroidal and poloidal) components, including the relationships among them. The gradient of the charge density is a longitudinal field, since its curl is identically zero. The current density field may be written as

$$\vec{J}(\vec{r}) = \nabla i^L(\vec{r}) + \nabla \times [\vec{r}i^T(\vec{r})] + \nabla \times \left\{\nabla \times [\vec{r} \times i^P(\vec{r})]\right\}$$
$$= \nabla i^L(\vec{r}) - i\vec{l}i^T(\vec{r}) - i\nabla \times \vec{l}i^P(\vec{r}), \tag{11}$$

where the second line makes use of the angular momentum operator $\vec{l} = -i\vec{r} \times \nabla$. The continuity equation satisfied by the source densities,

$$\nabla \cdot \vec{J}(\vec{r}) - i\omega\rho(\vec{r}) = 0, \tag{12}$$

involves only the longitudinal component of the current density; in terms of the corresponding Debye potential, it becomes

$$\nabla^2 i^L = i\omega\rho, \tag{13}$$

showing that this potential and the charge density are related through Poisson's equation. On the other hand, the curl of the current density,

$$\nabla \times \vec{J}(\vec{r}) = -i\nabla \times \vec{l}i^T(\vec{r}) - i\nabla \times [(\nabla \times \vec{l})i^P(\vec{r})]$$
$$= -i\nabla \times \vec{l}i^T(\vec{r}) - i\vec{l}[-\nabla^2 i^P(\vec{r})] \tag{14}$$

shows that its toroidal and poloidal Debye potentials are $-\nabla^2 i^P(\vec{r})$ and $i^T(\vec{r})$, respectively. The second line of Eq. (14) is obtained by performing the triple vector product and using the orthogonality of the divergence and angular momentum operators, as well as the commutability of the Laplace and angular momentum operators.

The use of Eqs. (11) and (14) together with the symmetry properties of the Green function and the hermiticity properties of the gradient and angular momentum operators allow us to rewrite Eqs. (9) and (10) to exhibit the longitudinal and transverse components of the respective force fields,

$$\vec{E}(\vec{r}) = -\nabla \int dv'[\rho(\vec{r}') - \frac{i\omega}{c^2}\mathrm{i}^L(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') - i\vec{l}\int dv'[-\frac{i\omega}{c^2}\mathrm{i}^T(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}')$$

$$- i\nabla \times \vec{l} \int dv'[-\frac{i\omega}{c^2}\mathrm{i}^P(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') \qquad (15)$$

$$\vec{B}(\vec{r}) = -i\vec{l}\int dv'[-\nabla'^2\mathrm{i}^P(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') - i\nabla \times \vec{l} \int dv'\mathrm{i}^T(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}'). \qquad (16)$$

The corresponding Debye potentials can be immediately read off from these equations in analogy with Eq. (11):

$$e^L(\vec{r}) = \int dv'[\rho(\vec{r}') - \frac{i\omega}{c^2}\mathrm{i}^L(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') \qquad (17)$$

$$e^T(\vec{r}) = \int dv'[-\frac{i\omega}{c^2}\mathrm{i}^T(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') \qquad (18)$$

$$e^P(\vec{r}) = \int dv'[-\frac{i\omega}{c^2}\mathrm{i}^P(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') \qquad (19)$$

$$b^L(\vec{r}) = 0 \qquad (20)$$

$$b^T(\vec{r}) = \int dv'[-\nabla'^2\mathrm{i}^P(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}') \qquad (21)$$

$$b^P(\vec{r}) = \int dv'\mathrm{i}^T(\vec{r}')]\mathcal{G}_+(\vec{r};\vec{r}'). \qquad (22)$$

Equations (18) and (22) show that toroidal currents produce toroidal electric and poloidal magnetic fields, while Eqs. (19) and (21) show that poloidal currents produce poloidal electric and toroidal magnetic fields. Also Eq. (17) shows that the charge density and the longitudinal currents are the sources of the longitudinal electric field.

3. Electromagnetic field of toroidal solenoids

This section starts by defining the toroids with a circular ring sector cross section in each meridian plane, and the alternat-

ing poloidal currents in the corresponding solenoids. The curl of the current density is evaluated and used in Eq. (10) together with the multipole expansion of the Green function, Eq. (8), in order to obtain the multipole expansion of the magnetic induction field. The electric intensity field can be evaluated by the integration of Eq. (9), but here it is preferable to obtain it through Eq. (4). The toroidal character of the magnetic field and the poloidal character of the electric field are explicitly exhibited, obtaining the multipole expansions of the respective Debye potentials along the way. The analysis focuses on the completeness of the multipole expansion [14], including the so-called toroidal moments [5]. Then the limiting situations of long wavelengths, including the magnetostatic case and the point source case, are studied in particular.

3.1. The multipole expansion

The toroid with a circular ring sector cross section is defined by its inner spherical ring ($r = a, \theta_2 < \theta < \theta_1, \varphi$), its upper conical ring ($a < r < b, \vartheta = \theta_1, \varphi$), its outer spherical ring ($r = b, \vartheta_1 < \theta < \vartheta_2, \varphi$), and its lower conical ring ($b > r > a, \theta = \vartheta_2, \varphi$). The alternating poloidal $Ie^{-i\omega t}$ current in the toroidal solenoid with N turns has the density

$$\vec{J}(\vec{r},t) = \frac{NIe^{-i\omega t}}{2\pi r \sin\theta}\left\{\frac{\hat{r}}{r}\left[\delta(\theta - \theta_1) - \delta(\theta - \theta_2)\right]\left[\Theta(r - a) - \Theta(r - b)\right]\right.$$

$$\left. +\hat{\theta}\left[\delta(r - b) - \delta(r - a)\right]\left[\Theta(\theta - \theta_1) - \Theta(\theta - \theta_2)\right]\right\}, \quad (23)$$

where the Dirac delta functions define the coil elements along which the current flows and the Heaviside step functions define the extent of such elements. The curl of the current density,

$$\nabla \times \vec{J}(\vec{r}) = \frac{NI}{2\pi r}\hat{\varphi}\left\{\frac{d}{dr}\left[\delta(r - b) - \delta(r - a)\right]\frac{\Theta(\theta - \theta_1) - \Theta(\theta - \theta_2)}{\sin\theta}\right.$$

$$\left. -\frac{1}{r^2}\frac{d}{d\theta}\left[\frac{\delta(\theta - \theta_1) - \delta(\theta - \theta_2)}{\sin\theta}\right]\left[\Theta(\theta - \theta_1) - \Theta(\theta - \theta_1)\right]\right\} \quad (24)$$

is toroidal, *i.e.*, in the azimuthal direction, and invariant under rotations around the axis of the toroid.

As already anticipated, the complete multipole expansion of the magnetic field follows from the combinations of Eqs. (10), (24) and (8),

$$
\vec{B}(\vec{r}) = \frac{NI}{2\pi c} \int\limits_0^\infty \int\limits_0^\pi \int\limits_0^{2\pi} r'^2 dr' \sin\theta' d\vartheta' d\varphi' \frac{\hat{\varphi}'}{r'} \left\{ \frac{d}{dr'} \left[\delta(r'-b) - \delta(r'-a) \right] \frac{\Theta(\theta'-\theta_1) - \Theta(\theta'-\theta_2)}{\sin\theta'} \right.
$$

$$
\left. -\frac{1}{r'^2} \frac{d}{d\theta'} \left[\frac{\delta(\theta'-\theta_1) - \delta(\theta'-\theta_2)}{\sin\theta'} \right] \left[\Theta(r'-a) - \Theta(r'-b) \right] \right\}
$$

$$
4\pi i k \sum_{l=0}^\infty j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=0}^l (-)^m N_{lm}^2 P_l^m(\cos\theta') P_l^m(\cos\theta)(2-\delta_{m0})\cos m(\varphi'-\varphi) \tag{25}
$$

The azimuthal angle integration can be done by using $\hat{\varphi}' = \hat{\varphi}\cos(\varphi'-\varphi) - \hat{R}\sin(\varphi'-\varphi)$ and the orthogonality of the cosine and sine functions,

$$
\int\limits_0^{2\pi} d\varphi' \hat{\varphi}' \cos m(\varphi'-\varphi) = \hat{\varphi}\pi\delta_{m1}. \tag{26}
$$

It follows that the magnetic induction field is also azimuthal and invariant under axial rotations, *i.e.*, a toroidal field.

The polar angle integrations are also straightforward, and according to the selection rule of Eq. (26) are limited to the terms with m=1 in the sum of Eq. (25). The first integral,

$$
\int\limits_{\theta_1}^{\theta_2} d\theta' P_l^1(\cos\theta') = P_l(\cos\theta_1) - P_l(\cos\theta_2), \tag{27}
$$

follows immediately from the relation between the associated and ordinary Legendre polynomials,

$$
P_l^1(\cos\vartheta) = \sin\theta \frac{d}{d(\cos\theta)} P_l(\cos\theta). \tag{28}
$$

The second integral, involving the Dirac delta functions, can be done by parts:

$$
\int\limits_0^\pi d\theta' \frac{d}{d\theta'} \left[\sin\theta' P_l^1(\cos\theta') \right] \sin\theta' P_l^1(\cos\vartheta') = -\frac{1}{\sin\theta'} \frac{d}{d\theta'} \left[\sin\theta' P_l^1(\cos\theta') \right]\Big|_{\theta'=\theta_1} + \frac{1}{\sin\theta'} \frac{d}{d\theta'} \left[\sin\theta' P_l^1(\cos\theta') \right]\Big|_{\theta'=\theta_2}
$$

$$
= -l(l+1) \left[P_l(\cos\theta_1) - P_l(\cos\theta_2) \right]. \tag{29}
$$

The last line is obtained by again using Eq. (28) and also the differential equation for the ordinary Legendre polynomials:

$$
\frac{1}{\sin\theta} \frac{d}{d\theta} \sin\vartheta \frac{d}{d\theta} P_l(\cos\theta) = -l(l+1)P_l(\cos\theta).
$$

The common factor in the polar angle integrals, Eqs. (27) and (29), should be noted.

The integrations over the radial coordinate are also direct, but the distinction among the different locations of the field point must be made. The first integral over the Dirac delta functions is done by parts:

$$
\int\limits_0^\infty r' dr' \frac{d}{dr'} \left[\delta(r'-b) - \delta(r'-a) \right] j_l(kr_<) h_l^{(1)}(kr_>)
$$

$$
= \begin{cases} \left\{ -\frac{d}{dr'} \left[r' h_l^{(1)}(kr') \right]\Big|_a^b \right\} j_l(kr) \,, \ r > a \\[2mm] -\frac{d}{dr'} \left[r' h_l^{(1)}(kr') \right]\Big|_{r'=b} j_l(kr) + \frac{d}{dr'} \left[r' j_l(kr') \right]\Big|_{r'=a} h_l^{(1)}(kr) \,, \ r < a < b \\[2mm] \left\{ -\frac{d}{dr'} \left[r' j_l(kr') \right]\Big|_a^b \right\} h_l^{(1)}(kr) \,, \ b < r. \end{cases} \tag{30}
$$

The second integral can be explicitly written as

$$
\int_a^b \frac{d}{dr'} j_l(kr_<) h_l^{(1)}(kr_>) = \begin{cases} \int_a^b \frac{dr'}{r'} h_l^{(1)}(kr') j_l(kr) \ , \ \ r < a \\[2mm] \int_a^r \frac{dr'}{r'} j_l(kr') h_l^{(1)}(kr) + \int_r^b \frac{dr}{r} h_l^{(1)}(kr') j_l(kr) \ , \ \ a < r < b \\[2mm] \int_a^b \frac{dr'}{r'} j_l(kr') h_l^{(1)}(kr) \ , \ \ b < r. \end{cases} \tag{31}
$$

The substitution of the integrals of Eqs. (26)-(31) in Eq. (25) gives

$$
\vec{B}(\vec{r}) = \frac{NI}{c} 4\pi i k \hat{\varphi} \sum_{l=1}^{\infty} (-) N_{l1}^2 P_l^1(\cos\theta) \Big[P_l(\cos\theta_1) - P_l(\cos\theta_2) \Big]
$$

$$
\times \begin{cases} \left\{ -\frac{d}{dr'}\Big[r' h_l^{(1)}(kr')\Big]\Big|_a^b + l(l+1) \int_a^b \frac{d}{dr'} h_l^{(1)}(kr') \right\} j_l(kr) \ , \ r < a \\[2mm] \left\{ -\frac{d}{dr'}\Big[r' h_l^{(1)}(kr')\Big]\Big|_{r'=b} j_l(kr) + \frac{d}{dr'}\Big[r' j_l(kr')\Big]\Big|_{r'=a} h_l^{(1)}(kr') \right. \\[2mm] \left. + l(l+1)\Big[\int_a^r \frac{dr'}{r'} j_l(kr') h_l^{(1)}(kr) + \int_a^b \frac{dr'}{r'} h_l^{(1)}(kr') j_l(kr) \Big] \right\} \ , \ a < r < b \\[2mm] \left\{ -\frac{d}{dr'}\Big[r' j_l(kr')\Big] h_l^{(1)}(kr') \Big|_a^b + l(l+1) \int_a^b \frac{d}{dr'} j_l(kr') \right\} h_l^{(1)}(kr) \ , \ b < r. \end{cases} \tag{32}
$$

The radial factors in Eq. (32) can be simplified by integrating the differential equation for the spherical Bessel functions. The result is

$$
\vec{B}(\vec{r}) = \frac{NI}{c} 4\pi i k \hat{\varphi} \sum_{l=1}^{\infty} (-) N_{l1}^2 P_l^1(\cos\theta) \Big[P_l(\cos\theta_1) - P_l(\cos\theta_2) \Big]
$$

$$
\times \begin{cases} k^2 \int_a^b dr' r' h_l^{(1)}(kr') j_l(kr) \ , \ r < a \\[2mm] \left\{ \frac{d}{dr'}\Big[r' j_l(kr)\Big]\Big|_{r'=r} h_l^{(1)}(kr) - \frac{d}{dr'}\Big[r' h_l^{(1)}(kr')\Big]\Big|_{r'=r} j_l(kr) \right. \\[2mm] \left. + k^2 \int_a^r dr' r' j_l(kr') h_l^{(1)}(kr) + k^2 \int_r^b dr' r' h_l^{(1)}(kr') j_l(kr) \right\} \ , \ a < r < b \\[2mm] k^2 \int_a^b dr' r' j_l(kr') h_l^{(1)}(kr) \ , \ b < r. \end{cases} \tag{33}
$$

It is appropriate at this point to recognize that the standard multipole expansion of the electromagnetic field [5, 6, 14] is usually limited to the region outside the sources, corresponding to $b < r$ in Eqs. (30-33). Following Lambert [7], here we have also obtained the field in the inner region $r < a$, where there are no sources, and in the intermediate region $a < r < b$, where the sources are located. All of this has been done within one and the same calculation by simply distinguishing among the different locations of the field point. It is also pertinent to point out that, as is to be expected, the solutions in the source free regions, $r < a$ and $b < r$, are superpositions of solutions of the homogeneous Helmholtz equations, while the solutions in the region where the sources

are located, $a < r < b$, involve non-linear combinations of the spherical Bessel functions and their derivatives or integrals.

3.2. The Debye potentials

Equation (33) may be written in an explicitly toroidal form by using the representation of the product of the azimuthal unit vector and the associated Legendre polynomials,

$$
\hat{\varphi} P_l^1(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} (-i\vec{l}) Y_{l0}(\theta, \varphi), \tag{34}
$$

so that

$$\vec{B}(r,\theta,\varphi) = -i\vec{l}\sum_{l=1}^{\infty}\frac{NI}{c}4\pi ik(-)\sqrt{\frac{4\pi}{2l+1}}N_{l1}^2\left[P_l^1(\cos\theta)-P_l(\cos\theta_2)\right]Y_{l0}(\theta,\varphi)$$

$$\times\begin{cases} k^2\int_a^b dr' r' h_l^{(1)}(kr')j_l(kr) \ , \ r < a \\[6pt] \left\{\left[\frac{d}{dr}(rj_l(kr))+k^2\int_a^r dr' r' j_l(kr')\right]h_l^{(1)}(kr)\right. \\[6pt] \left.+\left[-\frac{d}{dr}(rh_l^{(1)}(kr))+k^2\int_r^b dr' r' h_l^{(1)}(kr')\right]j_l(kr)\right\} \ , \ a < r < b \\[6pt] k^2\int_a^b dr' r' j_l(kr') h_l^{(1)}(kr) \ , \ b < r. \end{cases} \tag{35}$$

Comparison with Eqs. (16) and (21) gives the identification of the summation in Eq. (35) as the multipole expansion of the Debye potential b^T. Notice that the Laplacian in Eq. (21), because of its hermiticity, may be made to operate on the Green function, giving, according to Helmholtz equation, Eq. (7), $-k^2$ times the Green function plus the point source density term; such a relationship is apparent in the radial factors of Eq. (35).

As already mentioned at the beginning of this section, the electric intensity field may be obtained by using Eq. (4), when the magnetic induction, Eq. (35), and the current density, Eq. (23), are known. The electric intensity is obviously poloidal as a result of the application of the curl to the toroidal magnetic induction and the poloidal character of the current density itself. The current density may be written in its multipole expansion form by using the corresponding representations of the Dirac delta and the Heaviside step functions in the polar angle, respectively, in Eq. (23). For the sake of space, the complete expressions $\vec{E}$ for and $\vec{J}$ are not written out explicitly.

3.3. The dynamic toroidal multipole moments

At this point it is more instructive to go on to write the magnetic and electric fields in the region outside the external sphere, $b < r$, in the form that allows the characterization of the multipole expansions of the electromagnetic field of toroidal solenoids,

$$\vec{B}(b<r,\theta,\varphi)=\sum_{l=1}^{\infty}\psi_{l0}\vec{l}h_l^{(1)}(kr)Y_{l0}(\theta,\varphi) \tag{36}$$

$$\vec{E}(b<r,\theta,\varphi)=\sum_{l=1}^{\infty}\psi_{l0}\frac{i}{k}\nabla\times\vec{l}h_l^{(1)}(kr)Y_{l0}(\theta,\varphi), \tag{37}$$

where

$$\psi_{l0}=-\frac{4\pi NIk^3}{c}\sqrt{\frac{2l+1}{4\pi}}\frac{1}{l(l+1)}$$

$$\times\left[P_l(\cos\theta_1)-P_l(\cos\theta_2)\right]\int_a^b dr' r' j_l(kr'). \tag{38}$$

Here the explicit value of the normalization constant N_{l1} has been incorporated.

For purposes of comparison, we transcribe next the corresponding general equations for the multipole expansions of the electromagnetic field [6, 14],

$$\vec{B}(\vec{r})=\sum_l\sum_m\left[\psi_{lm}^E\vec{l}h_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\right.$$

$$\left.+\psi_{lm}^M\frac{(-i)}{k}\nabla\times\vec{l}h_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\right] \tag{39}$$

$$\vec{E}(\vec{r})=\sum_l\sum_m\left[\psi_{lm}^E\left(\frac{i}{k}\right)\nabla\times lh_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\right.$$

$$\left.+\psi_{lm}^M\vec{l}h_l^{(1)}(kr)Y_{lm}(\theta,\varphi)\right], \tag{40}$$

where the dynamic multipole moments, in the terminology of [14], are given by

$$\psi_{lm}^M=\frac{4\pi ik^2}{cl(l+1)}\int dv' j_l(kr')Y_{lm}^*(\theta',\varphi')\vec{r}\cdot\nabla'\times\vec{J}(\vec{r}') \tag{41}$$

$$\psi_{lm}^E=-\frac{4\pi ik^2}{cl(l+1)}\int dv' j_l(kr')Y_{lm}^*(\theta',\varphi')$$

$$\times\left[ik\vec{r}'\cdot\vec{J}(\vec{r}')-c(2+\vec{r}'\cdot\nabla')\rho(\vec{r}')\right] \tag{42}$$

Notice that Eq. (5b) of Ref. 14, equivalent to our Eq. (42), is missing a factor of k on its rhs, as can be verified by comparison with Eq. (16-91) in Ref. 6.

The comparison of Eqs. (36-38) and (39-42) is direct. The poloidal component of the magnetic induction field is absent in Eq. (36) and the toroidal component of the electric intensity field is absent in Eq. (37), because the dynamic magnetic multipole moments in Eq. (41) vanish at the source level due to the poloidal character of the current, Eq. (23). The vanishing of the integrand in Eq. (41) follows immediately from the orthogonality of the radial vector and the curl of the current density, Eq.(24). On the other hand, the dynamic electric multipole moments, Eq. (42), are determined by the radial components of the current density and of the gradient of the charge density, as well as by the charge density itself. In the case under study, the charge density is absent, and the radial component of the current density is the

first term inside the curly brackets on the rhs of Eq. (23). The evaluation of the corresponding integral in Eq. (42) leads to the same value of Eq. (38). In conclusion, the multipole expansions of Eqs. (36,37,38) correspond to the terms in Eq. (42) involving the dynamic transverse electric multipole moments associated with the radial component of the poloidal current density, *i.e.* Eq. (42) with $\rho = 0$.

One might be tempted to add the superscript E in Eqs. (36-38) to complete the characterization of the multipole expansion of the electromagnetic field of the toroidal solenoid. However falling into such a temptation is tantamount to missing the existence of the toroidal moments. Following Ref. 10, the correct superscript to be added in Eqs. (36-38) is for toroidal, and Eq. (42) gives an exact relationship among the dynamic multipole moments of electric, toroidal and charge types.

$$\psi_{lm}^E(k) = \psi_{lm}^T(k) + \psi_{lm}^Q(k). \tag{43}$$

The distinction among these types of dynamic multipole moments and the connection among them can be traced back to the different ways of separating the source terms in the inhomogeneous Helmholtz equations, Eqs. (5,6). In Eqs. (41), (42) it is recognized that the source factors are the radial components of the respective sources in Eqs. (6) and (5). Let us consider first the unambiguous case of the magnetic moments. The corresponding source factor in the integrand of Eq. (41) may be written in terms of the decomposition of the current density of Eq. (11) in the alternative forms

$$\vec{r}' \cdot \left[\nabla' \times \vec{J}(\vec{r}') \right] = \left[\vec{r}' \times \nabla' \right] \cdot \vec{J}(\vec{r}')$$

$$= i\vec{l} \cdot \left[\nabla i^L - i\vec{l} i^T - i\nabla \times \vec{l} i^P \right] = \hat{\vec{l}}^2 i^T, \tag{44}$$

where in the first line the dot and cross are exchanged in the triple scalar product, and in the second line the orthogonality

of the operators is taken into account. Equation (44) shows that the dynamic magnetic multipole moments depend only on the toroidal component of the current. On the other hand, there is room for ambiguity in the case of the electric moments. In fact, the corresponding source factor taken from Eq. (5),

$$\vec{r}' \cdot \left[-\nabla'\rho(\vec{r}') + \frac{ik}{c}\vec{J}(\vec{r}') \right]$$

$$= -\vec{r}' \cdot \nabla' \left[\rho(\vec{r}') - \frac{ik}{c}i^L(\vec{r}') \right] + i\hat{\vec{l}}^2 i^P(\vec{r}') \tag{45}$$

involves both longitudinal and poloidal components of the source. Taking into account that the basis functions for the multipole expansions are eigenfunctions of the ∇^2 and $\hat{\vec{l}}^2$ operators, it can be recognized that Eq.(41) is in direct correspondence with Eq. (22), while Eq.(42) is related to both Eqs. (17) and (21). The standard terminology of Eqs. (39)-(42) of transverse electric and magnetic moments can be made more precise by making the distinction of Eq. (43), and recognizing that the magnetic moments arising from toroidal currents could be appropriately called poloidal moments, just as the moments arising from the poloidal currents are called toroidal moments.

3.4.   The long wavelength limit

When the sources are confined in a region with a radial extent that is small compared with the wavelength $\lambda = 2\pi/k$, it is usual to approximate the spherical Bessel functions in Eqs. (34-42) by the dominant terms of their power series expansions close to the origin. The result of such approximation is that the dynamic multipole moments, which are defined in terms of spherical Bessel functions as weight functions in the integrals of Eqs. (41) and (42), are connected to the static multipole moments defined in terms of the powers of the radial coordinate as weight functions,

$$\psi_{lm}^M(k \to 0) = \frac{4\pi i k^{l+2}}{cl(l+1)(2l+1)!!} \int dv' r'^l Y_{lm}^*(\theta', \varphi')\vec{r}' \cdot \nabla' \times \vec{J}(\vec{r}') = \frac{4\pi i k^{l+2}}{l(2l+1)!!} M_{lm} \tag{46}$$

$$\psi_{lm}^E(k \to 0) = -\frac{4\pi i k^{l+2}}{cl(l+1)(2l+1)!!} \int dv' r'^l Y_{lm}^*(\theta', \varphi') \left[ik\vec{r}' \cdot \vec{J}(\vec{r}') - c(2 + \vec{r} \cdot \nabla')\rho(\vec{r}') \right]$$

$$= -\frac{4\pi i k^{l+2}}{cl(2l+1)!!} \left[\frac{1}{l+1} \int dv' r'^l Y_{lm}^*(\theta', \varphi') ik\vec{r}' \cdot \vec{J}(\vec{r}') + c \int dv' r'^l Y_{lm}^*(\theta', \varphi')\rho(\vec{r}') \right] M_{lm}. \tag{47}$$

Equation (46) is the same as Eqs. (58) and (17) of Ref 14. The charge part of Eq. (47) is the same as Eqs. (59), (33) and (32) of Ref. 14, where the last integral is the familiar electrostatic multipole moment Q_{lm} [6, 8]. The part associated with the radial component of the current density in Eq. (47) is usually dropped in the long wavelength approximation, but this is not justified as illustrated in this paper

and Refs. 10 to 13. In the notation of Ref. 10, the dynamic multipole moments in Eqs. (41), (42) and (43) are normalized to give the time dependent form factors

$$M_{lm}(-k^2, t) = \frac{l(2l+1)!!}{4\pi i k^{l+2}} \psi_{lm}^M(k) e^{-i\omega t} \to M_{lm} e^{-i\omega t} \tag{48}$$

$$E_{lm}(-k^2,t) = \frac{cl(2l+1)!!}{4\pi k^{l+1}}\psi_{lm}^E(k)e^{-i\omega t} \qquad (49)$$

$$Q_{lm}(-k^2,t) = -\frac{l(2l+1)!!}{4\pi i k^{l+2}}\psi_{lm}^Q(k)e^{-i\omega t} \to Q_{lm}e^{-i\omega t} \quad (50)$$

$$T_{lm}(-k^2,t) = \frac{l(2l+1)!!}{4\pi k^{l+3}}\psi_{lm}^T(k)e^{-i\omega t}. \qquad (51)$$

In terms of these form factors, Eq.(43) becomes

$$E_{lm}(-k^2,t) = k^2 T_{lm}(-k^2,t) + \dot{Q}_{lm}(-k^2,t) \qquad (52)$$

which is the central result of Ref. 10.

The standard systems of electromagnetic sources and fields [5–7, 14] are described in terms of the charge form factors, Eq.(50), the magnetic form factors, Eqs.(48) and the transverse electric form factors, Eq. (49). In the long wavelength approximation the latter becomes

$$E_{lm}(-k^2,t) = \dot{Q}_{lm}(-k^2,t) \qquad (53)$$

as follows from Eq. (52).

However, in systems with poloidal currents like the toroidal solenoids, Eq. (53) does not hold. Instead of the transverse electric form factors, it is more general to use the toroidal form factors of Eq. (51), which are independent of the magnetic and the charge form factors.

In Secs. 3.3 and 3.4, we have considered the electromagnetic fields in the region $b < r$ in order to compare with the standard available results of Refs. 6 and 14. The treatment of Sec. 3.1 includes the fields for the remaining regions $r < b$, since we have the complete fields, as Lambert [7] pointed out. In Sec. 3.5, we study the magnetic induction field in the static limit $\omega \to 0$, $k \to 0$, or infinite wavelength limit. In Sec. 3.6, the point source limit corresponding also to an infinitely long wavelength limit, but for a finite frequency, is analyzed.

3.5. The magnetostatic limit

The limit of stationary currents with frequency $\omega \to 0$ in the toroidal solenoids corresponds to the magnetostatic case. It is clear that Eq. (4) becomes Ampere's law, Eqs. (6) and (7) become Poisson's equation, the Green function of Eq. (8) becomes the Coulomb potential, and the spherical Bessel functions are replaced by their power approximations.

We take up the problem at the level of Eq. (33) replacing,

$$j_l(kr_<)h_l^{(1)}(kr)|_{k\to 0} - \frac{ir_<^l}{k(2l+1)r_<^{l+1}}. \qquad (54)$$

Notice that the common factor ik before the sum in Eq. (33) when multiplied by the factor $(-i/k)$ in Eq. (54) gives one. For the sourceless regions, $r < a$ and $b < r$, the radial integrals are finite, but they are to be multiplied by the factor k^2 which vanishes in the static limit. In the notation of Secs. 3.3 and 3.4, the toroidal form factors are finite, but the toroidal moments vanish in the inner and outer

regions bounding the toroid. The magnetic induction correspondingly vanishes in both regions. For the region where the sources are located, the radial factor becomes

$$\left[\frac{d}{dr}(r^{l+1})\right]\frac{1}{r^{l+1}} - \left[\frac{d}{dr}\left(\frac{1}{r^l}\right)\right]r^l = \frac{2l+1}{r}. \qquad (55)$$

The coefficient in the numerator of Eq (55) will cancel the coefficient in the denominator of Eq. (54), and Eq. (33) takes the form

$$\vec{B}(a<r<b,\theta,\varphi) = \frac{NI}{cr}4\pi\hat{\varphi}\sum_{t=1}^{\infty}(-)\frac{2l+1}{4\pi}P_l^1(\cos\theta)$$

$$\times\left[P_l(\cos\theta_1) - P_l(\cos\theta_2)\right]$$

$$= \frac{2NI}{cr\sin\theta}\left[\Theta(\theta-\theta_1) - \Theta(\theta-\theta_2)\right] \quad (56)$$

In the first line, the explicit value of the normalization constant was substituted. In the last line, the sum is identified with the difference of the Heaviside step functions in the polar angle, which follow from the completeness of the orthonormal Legendre basis

$$\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\cos\theta)P_l(\cos\theta_i) = \delta(\cos\theta - \cos\theta_i)$$

$$= -\frac{\delta(\theta-\theta_i)}{\sin\theta} \qquad (57)$$

and its integration

$$\Theta(\theta-\theta_i) = \int_0^\theta d\theta' \delta(\theta'-\theta_i)$$

$$= -\sum_{l=0}^{\infty}\frac{2l+1}{2}\int_0^\theta d\theta' \sin\theta' P_l(\cos\theta')P_l(\cos\theta_i)$$

$$= -\sum_{l=1}^{\infty}\frac{2l+1}{2}\frac{1}{l(l+1)}\sin\theta P_l^1(\cos\theta)P_l(\cos\theta_i). \quad (58)$$

In conclusion, the magnetic induction vanishes outside the solenoid and is azimuthal and inversely proportional to the radial distance from the axis in the interior. Notice that the toroidal moments between the two spheres are all different from zero, and the summation of all the multipole components of the field was carried out in Eq. (56).

3.6. The point source limit

Equations (36), (37) for the electromagnetic field of toroidal solenoids correspond to the transverse electric fields of Eqs. (39), (40), with the identification $\psi^E = \psi^T$ following from Eq. (43), since the charge moments vanish in this case. Consequently, the polarization and angular distribution of the radiation of each of the multipole components of the field due

to the toroidal solenoid have the same characteristics as the radiation of the corresponding transverse electric multipole fields [6, 14]. The difference is in the amplitudes given by Eq. (38), in contrast with the usual case dominated by the charge moments, last term in Eqs. (47).

In the case of a solenoid with small dimensions compared to the wavelength, *i.e.* $kb \ll 1$, the toroidal moments in Eq. (38) become

$$\psi_{l0}^T(kb \ll 1) = -\frac{NI}{c} \frac{\sqrt{4\pi(2l+1)}}{l(l+1)}$$

$$\times \left[P_l(\cos\theta_1) - P_l(\cos\theta_2) \right] \frac{k[(kb)^{l+2} - (ka)^{l+2}]}{l+2} \quad (59)$$

Since the ratio of the moments of two consecutive multipoles is of the order $kb \ll 1$, the component with the lowest multipolarity is the dominant one. The most dominant of all is the toroidal dipole with $l = 1$.

Comparison of Eq. (54) with the corresponding charge multipole moment of Eq. (47) shows one extra factor of k for the toroidal moments. This is translated into an additional factor of ω^2 in the power radiated by a toroidal multipole relative to that of the corresponding electric multipole. Thus the power radiated by a toroidal solenoid, approximated as a toroidal dipole, goes as ω^6, in contrast with the well known ω^4 dependence of electric and magnetic dipoles.

Since the toroidal and electric multipoles of a given order have the same angular momentum and parity properties, the simultaneous presence of both types of multipoles gives a frequency dependence of the radiated power that is more complicated than the corresponding dependence for each individual type.

4. Discussion

The general elements for constructing a complete multipole expansion for the electromagnetic field produced by any localized distribution of charges and currents have been identified in Sec. 2. They include:

1) Maxwell's equations (1)-(4) and the corresponding inhomogeneous Helmholtz equations (5), (6), connecting the electric and magnetic fields and their sources;

2) the outgoing spherical wave Green function and its spherical multipole expansion, Eq. (8); and

3) the Debye potentials for the sources, Eqs. (11) and (13), and the fields, Eqs. (15), (16), allowing the immediate identifications of their respective longitudinal and transverse - toroidal and poloidal - components, Eqs. (17)-(19) and (20)-(22).

Specifically, the electric intensity field may have longitudinal, toroidal or poloidal components arising from a charge density distribution or a longitudinal current distribution, a toroidal current or a poloidal current, respectively; while the magnetic induction field is transverse, and its poloidal and toroidal components arise from toroidal and poloidal currents, respectively. The electric and magnetic moments studied in the standard textbooks arise from longitudinal and toroidal sources; the construction of the fields arising from poloidal currents in a solenoid presented in Section 3 of this work is a way to complete the study of the multipole expansion introducing the toroidal moments.

The complete multipole expansion of the electromagnetic field arising from alternating poloidal currents in toroidal solenoids has been explicitly constructed, Eq. (33). The expansion is complete in the sense of Ref. 7 that the field is described at all points in space $0 < r < \infty$, and also in the sense that it exhibits the existence of the toroidal multipole moments [10]. In fact, the poloidal currents in the toroidal solenoids possess vanishing magnetic and charge multipole moments; the toroidal moments are the moments of the poloidal currents, just as the magnetic (poloidal) moments are the moments of toroidal currents. The explicit, exact relationships among the transverse electric, toroidal and charge multipole moments and form factors are given through Eqs. (43) and (48-52).

The standard multipole expansion of the electromagnetic field is formulated in terms of the set of form factors $Q_{lm}(-k^2, t), M_{lm}(-k^2, t)$ and $E_{lm}(-k^2, t)$, or the corresponding dynamic multipole moments, in the long wavelength approximation, $E_{lm}(-k^2 \to 0, t) \to Q_{lm}(-k^2 \to 0, t)$. It is in this approximation that the presence of the toroidal form factors is lost. Reference 10 proposes the use of the alternative set of form factors $Q_{lm}(-k^2, t), M_{lm}(-k^2, t)$, and $T_{lm}(-k^2, t)$, which according to the discussion at the end of our Secs. 3.3 and 3.4 are respectively connected to the charge density and longitudinal component of the current, the toroidal component of the current, and the poloidal component of the current, Eqs. (44) and (45). The description of the electromagnetic field of the toroidal solenoids with poloidal currents requires these toroidal moments; for such systems the exact values of the form factors are $Q_{lm}(-k^2, t) = 0, M_{lm}(-k^2, t) = 0$ and $E_{lm}(-k^2, t) = k^2 T_{lm}(-k^2, t)$.

The long wavelength approximation was applied in two particular cases. In the magnetostatic case of Sec. 3.5, the multipole expansion of the magnetic induction field involves vanishing components of each multipolarity outside the solenoid, and the components in the interior of the solenoid add up to the field being inversely proportional to the radial distance from the axis. In particular it can be pointed out that the Zel'dovich anapole in subsection 3.6 corresponds to a point toroidal solenoid with a stationary poloidal current [9]. The toroidal multipole components of the radiation field were analyzed in the point source limit and compared with the standard transverse electric multipole components, recognizing the presence of an extra factor of ω^2 in the power radiated by the toroidal moments relative to that of the corresponding electric moments. Specifically, the toroidal dipole moment is the dominant dynamical approximation to

the anapole [15, 16]. The interested reader may find in the last references some illustrative works on the evaluations of toroidal dipole moments induced in nucleons, leptons, nuclei, atoms and molecules, by parity non-conserving weak interactions, and external electric fields.

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