On the potential of an infinite dielectric cylinder and a line of charge: Green's function in an elliptic coordinate approach

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A two-dimensional Laplace equation is separable in elliptic coordinates and leads to a Chebyshev-like differential equation for both angular and radial variables. In the case of the angular variable $\eta\,(-1 \leq \eta \leq 1)$, the solutions are the well known first class Chebyshev polynomials. However, in the case of the radial variable $\xi\,(1 \leq \xi < \infty)$ it is necessary to construct another independent solution which, to our knowledge, has not been previously reported in the current literature nor in textbooks; this new solution can be constructed either by a Fröbenius series representation or by using the standard methods through the knowledge of the first solution (first-class Chebyshev polynomials). In any case, either must lead to the same result because of linear independence. Once we know these functions, the complete solution of a two-dimensional Laplace equation in this coordinate system can be constructed accordingly, and it could be used to study a variety of boundary-value electrostatic problems involving infinite dielectric or conducting cylinders and lines of charge of this shape, since with this information, the corresponding Green's function for the Laplace operator can also be readily obtained using the procedures outlined in standard textbooks on mathematical physics. These aspects are dealt with and discussed in the present work and some useful trends regarding applications of the results are also given in the case of an explicit example, namely, the case of a dielectric elliptic cylinder and an infinite line of charge.

Keywords: Elliptic coordinates; Green function; two-dimensional Laplace equation; Chebyshev functions.

La ecuación de Laplace en dos dimensiones es separable en coordenadas elípticas, y la separación de variables resulta en ecuaciones tipo Chebyshev para las dos coordenadas, radial (ξ) y angular (η). En el caso de la coordenada angular η, $(-1 \leq \eta \leq 1)$, las soluciones son los polinomios de Chebyshev de primera clase, los cuales están muy bien estudiados. Sin embargo, en el caso de la coordenada radial ξ $(1 \leq \xi < \infty)$, existe la necesidad de construir otra solución independiente, que (a nuestro conocimiento) no está reportada en libros de texto ni en artículos; esta nueva solución puede ser construida, ya sea en forma de una serie de Fröbenius o usando los métodos de integración que involucran el conocimiento de la primera solución. Cualquiera de estos dos métodos nos llevará al mismo resultado, debido a la independencia lineal de las soluciones. Una vez que conozcamos dichas funciones, la solución completa la ecuación de Laplace en dos dimensiones para este sistema de coordenadas puede ser construida, y dicha solución puede ser aplicada para estudiar una variedad de problemas de contorno que involucren cilindros dieléctricos o conductores infinitos o líneas de carga, pues con esta información, podemos obtener fácilmente la función de Green para el operador de Laplace usando el procedimiento de los libros de texto de métodos matemáticos. Estos aspectos se discuten en el presente trabajo, y se dan algunas indicaciones respecto a las aplicaciones de los resultados, incluyendo un ejemplo explícito: el caso de un cilindro elíptico dieléctrico y una linea infinita de carga.

Descriptores: Coordenadas elípticas; función de Green; ecuación de Laplace en dos dimensiones; funciones de Chebyshev.

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1. Introduction

Laplace equations play a fundamental role in potential theory since many two-dimensional boundary-value problems are of crucial importance for both physics and mathematics; this is the case, for instance, in electrostatics, fluid flow through obstacles, conformal mapping and so on [1].

The solution of this equation for a specific boundary-value problem in electrostatics can give information that is *a priori* unknown, namely, when an initially isolated conductor (charged or raised to a given potential) is perturbed by a charge distribution, the charge on the conductors surface after the perturbation redistributes to an unknown distribution, then the conventional solution for the potential as an integral involving the surface charge cannot be used; in those cases, the general solution of Laplace equation becomes an important tool to obtain the new potential.

In most electrostatic problems, a given charge distribution(s) is (are) usually involved and one must solve the Poisson equation instead, but in this case the general solution of the Laplace equation is still important since it can be used to construct an auxiliary function, the Green function, which allows one to find the particular solution of Poisson equation that satisfies all the boundary conditions.

The knowledge of the general solution of a two-dimensional Laplace equation involves its separability in a given coordinate system; it is separable, for instance, in rectangular, polar, parabolic, elliptic and other common coordinate systems [2]. In the specific case of elliptic coordinates, its separation leads to a Chebyshev-type ordinary differential equation for both the angular (η) and the radial (ξ) coordinates. The solution associated with the angular variable are the well known first-class Chebyshev polynomials, but in the case of the radial one, they are no longer useful because this coordinate is defined in [1,∞) and clearly the polynomials diverge at infinity, a matter that could not be desirable from physical grounds, as we shall see explicitly.

The latter fact implies that we need to find a different solution which must behaved properly in this interval; once such a solution is known, the Green's function associated with the Laplace operator in this coordinate system can be readily constructed. The knowledge of both the general solution and the Green's function for the Laplace operator can used to solve a variety of electrostatic boundary-value problems that involve infinite conductors and infinite charged lines in elliptic coordinates [3].

The aim of this work is to stress the importance inherent in the knowledge of the general solution of the Laplace equation and the ample possibilities of applications in boundary-value electrostatic problems. For the sake of clarity, this work has been structured as follows: In Sec. 2, we obtain the general solution of a two-dimensional Laplace equation in elliptic coordinates; a representation of the Green function in these coordinates is constructed in Sec. 3; an explicit example which involves the application of the later result is presented in Sec. 4; and finally, some interesting limiting situations of this example are discussed in Sec. 5.

2. General solution of a two-dimensional Laplace equation in elliptic coordinates

Several fields in physics and mathematics involve boundary-value problems in which elliptic coordinates arise; these are the cases of fluid flow with obstacles, electrostatics or conformal mapping, to mention just a few. In all of them a solution of Laplace equations, restricted to particular boundary conditions, is needed. A previous step to getting a particular solution of any linear partial differential equation is the knowledge of its general solution, which, after imposing the proper boundary conditions, provides the desired solution to the problem. This section is devoted to finding the general solution of the two-dimensional Laplace equation in elliptic coordinates, since it can be useful to study a variety of boundary-value problems of this symmetry.

Confocal elliptic coordinates (ξ, η) are defined as[4]:

$$\begin{aligned} x &= a\xi\eta, & \xi &\in [1,\infty), \\ y &= a(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}, & \eta &\in [-1,1], \end{aligned} \tag{1}$$

where $2a$ is the interfocal distance. The family of curves generated by this coordinate system is as follows:

$\xi = \text{const},\ -1 \le \eta \le 1,$ ellipses with foci on the x-axis

$\eta = \text{const},\ 1 \le \xi < \infty,$ confocal hyperboles.

The corresponding scale factors are given as

$$h_\xi = \left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right]^{1/2} = a\left[\frac{\xi^2-\eta^2}{\xi^2-1}\right]^{1/2}$$

and

$$h_\eta = \left[\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right]^{1/2} = a\left[\frac{\xi^2-\eta^2}{1-\eta^2}\right]^{1/2},$$

from which the Laplace operator, defined as

$$\nabla^2 = \frac{1}{h_\xi h_\eta}\left\{\frac{\partial}{\partial \xi}\left[\frac{h_\eta}{h_\xi}\frac{\partial}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\frac{h_\xi}{h_\eta}\frac{\partial}{\partial \eta}\right]\right\}$$

can be readily calculated to give

$$\nabla^2 = \frac{1}{a^2(\xi^2-\eta^2)} \times \left\{(\xi^2-1)\frac{\partial^2}{\partial \xi^2} + \xi\frac{\partial}{\partial \xi} + (1-\eta^2)\frac{\partial^2}{\partial \eta^2} - \eta\frac{\partial}{\partial \eta}\right\} \tag{2}$$

With this, the Laplace equation for the electrostatic potential $\Psi(\xi,\eta)$, can then be written as

$$\frac{1}{a^2(\xi^2-\eta^2)}\left\{(\xi^2-1)\frac{\partial^2}{\partial \xi^2}+\xi\frac{\partial}{\partial \xi}+(1-\eta^2)\frac{\partial^2}{\partial \eta^2}-\eta\frac{\partial}{\partial \eta}\right\} \times\Psi(\xi,\eta)=0,$$

or

$$\left\{(\xi^2-1)\frac{\partial^2}{\partial \xi^2} + \xi\frac{\partial}{\partial \xi} + (1-\eta^2)\frac{\partial^2}{\partial \eta^2} - \eta\frac{\partial}{\partial \eta}\right\} \times\Psi(\xi,\eta) = 0. \tag{3}$$

By inspection, one can see immediately that the last equation is separable since, if we assume that

$$\psi(\xi,\eta) = S(\xi)H(\eta),$$

two ordinary second order differential equations can be obtained:

$$\left\{(1-\eta^2)\frac{d^2}{d\eta^2} - \eta\frac{d}{d\eta} + \gamma\right\} H(\eta) = 0\,, \tag{4}$$

and

$$\left\{(\xi^2-1)\frac{d^2}{d\xi^2} + \xi\frac{d}{d\xi} - \gamma\right\} S(\xi) = 0\,, \tag{5}$$

where γ is the separation constant. The equation for the angular coordinate η has two regular singular points at $\eta = \pm 1$, so if seek a well behaved solution, γ is restricted to the values $\gamma = m^2$ with $m = 0, 1, 2\ldots$ With this restriction on the separation constant, Eqs. (4,5) can be identified as being of Chebyshev type[4].

Interestingly enough, although the solutions to both equations are the first-class Chebyshev polynomials, the corresponding solution to this type of equation in the range $[1,\infty)$

is not, to our knowledge, previously reported in the literature[5], but this is a matter that can be important when a second linearly independent solution is needed. This second solution can be constructed either by a Fröbenius series representation or, equivalently, as an integral representation involving the first solution, *i.e.*, the first class Chebyshev polynomials[4].

With all this, the most general solution to the Laplace equation in elliptic coordinates can then be written as

$$\Psi(\xi,\eta)=\sum_{m=0}^{\infty}\{A_mT_m(\xi)+B_mS_m(\xi)\}\times\{C_mT_m(\eta)+D_mS_m(\eta)\}\,, \quad (6)$$

where A_m, B_m, C_m, D_m are constants to be determined once specific boundary conditions are imposed on $\Psi(\xi,\eta)$.

3. Two-dimensional Green function in elliptic coordinates

The knowledge of the two linearly independent solutions T_m, S_m provide the necessary tools for the construction of the Green function associated with the Laplace's operator. The two-dimensional Green function satisfies the inhomogeneous equation

$$\nabla^2 G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\,, \quad (7)$$

which, in elliptic coordinates, is transformed to [4]

$$\nabla^2 G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\frac{1}{h_\xi h_\eta}\delta(\xi-\xi')\delta(\eta-\eta'). \quad (8)$$

Let us recall that the polynomials $\{T_m(\eta)\}$ are orthogonal in [-1,1] and indeed constitute a basis set with respect to which any function can be represented in this interval. Then, by the closure relation, one can say that

$$\delta(\eta-\eta') = \sum_{m=0}^{\infty} b_m \frac{T_m(\eta)T_m(\eta')}{(1-\eta^2)^{1/2}}, \quad (9)$$

where b_m is a constant related to the normalization of T_m, that is[4]

$$\int_{-1}^{1}\frac{T_m(\eta)T_{m'}(\eta)}{(1-\eta^2)^{1/2}}d\eta = a_m\delta_{m,m'},$$

from which

$$b_m = \frac{1}{a_m} = \begin{cases} \frac{1}{\pi}, & m=0 \\ \frac{2}{\pi}, & m\neq 0 \end{cases}. \quad (10)$$

Furthermore, the basis $\{T_m(\eta)\}$ can be also useful in making a representation of $G(\boldsymbol{\rho},\boldsymbol{\rho}')$ of the form

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = G(\xi,\eta;\xi',\eta') = \sum_m g_m(\xi,\xi',\eta')T_m(\eta),$$

where the coefficients g_m can be interpreted as the coordinates of vector $G(\xi,\eta;\xi',\eta')$ in the infinite dimensional space generated by the basis vectors $\{T_m(\eta)\}$.

Combining Eqs. (2,8,9) and using the linear independence property of $\{T_m(\eta)\}$, the coefficients g_m must satisfy the following differential equation:

$$\left\{(\xi^2-1)\frac{d^2}{d\xi^2}+\xi\frac{d}{d\xi}-m^2\right\}g_m(\xi,\xi',\eta') = -(\xi^2-1)^{1/2}\delta(\xi-\xi')b_mT_m(\eta') \quad (11)$$

As the reader must be aware, when $\xi \neq \xi'$, this equation is similar to Eq. (5), that is, a Chebyshev-type equation whose linearly independent solutions are $T_m(\xi), S_m(\xi)$, as mentioned in the previous section.

In order to complete the construction of the Green function, we shall follow the method described, for instance, in Refs. 4 and 5; that is, since the point $\xi=\xi'$ represents an inhomogeneity, the function g_m must have the following properties:

$$g_m^- = g_m^+ \text{ at } \xi=\xi' \text{ (continuity)}$$

$$\left.\frac{dg_m^+}{d\xi}\right|_{\xi'} - \left.\frac{dg_m^-}{d\xi}\right|_{\xi'} = -\frac{b_mT_m(\eta')}{(\xi'^2-1)^{1/2}} \text{ (discontinuous derivative)}$$

$$g_m(\xi,\xi',\eta') = g_m(\xi',\xi,\eta') \text{ (symmetry)}, \quad (12)$$

where g_m^-, g_m^+ are the linearly independent solutions to Eq. (12) for $1<\xi<\xi'$ and $\xi'<\xi<\infty$, respectively. Hence, if we choose

$$\begin{array}{c} g_m^- = CT_m(\xi),\ 1<\xi<\xi' \\ g_m^+ = DS_m(\xi),\ \xi'<\xi<\infty \end{array},$$

where C, D are constant to be determined through the first two properties of g_m. After performing the required algebra to find C and D, the complete Green function can be then written as

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = G(\xi,\eta;\xi',\eta') = \sum_{m=0}^{\infty}\frac{T_m(\eta)T_m(\eta')}{a_m}f_m(\xi,\xi'), \quad (13)$$

where

$$f_m(\xi,\xi') = \begin{cases} S_m(\xi')T_m(\xi), & 1<\xi<\xi' \\ S_m(\xi)T_m(\xi'), & \xi'<\xi<\infty \end{cases} \quad (14)$$

Finally, to close this section, it is worth mentioning that for a more detailed description of both the general procedure andthe behavior of the Fröbenius series representation of S_m, the reader is encouraged to review the work of Pérez-Enríquez *et al.*[5].

4. Infinite dielectric elliptic cylinder and charged line

In this section we shall apply some of the previous results to a specific boundary-value electrostatic problem, namely, the case of an infinite dielectric elliptic cylinder in the presence of an infinite line of charge. Because of the symmetry in any plane $z = const$, normal to both the cylinder and the line, this three-dimensional problem can be reduced to a two-dimensional one, namely, to that related to the coordinates (x, y) or their counterpart, the elliptic coordinates (ξ, η). Let us assume that the line of charge is located at the point $(\xi^{'}, \eta')$ and that the surface of the dielectric cylinder is represented by $\xi = \xi_o = const$; the electrostatic potential can be then written as:

$$\Psi_i(\boldsymbol{\rho}) = \sum_{m=0}^{\infty} \{A_m T_m(\xi) + B_m S_m(\xi)\} T_m(\eta) \qquad \xi < \xi_o < \xi^{'}, \tag{15}$$

which again, appealing to the properties of the set $\{T_m(\eta)\}$, can be envisioned as a representation of the electrostatic potential in terms of these basis vectors and the quantities in parentheses would mean its coordinates in the infinite-dimensional space spanned by them, and

$$\Psi_o(\boldsymbol{\rho}) = \Psi_\lambda(\boldsymbol{\rho}) + \sum_{m=0}^{\infty} \{C_m T_m(\xi) + D_m S_m(\xi)\} T_m(\eta) \qquad \xi_o < \xi < \xi^{'}, \tag{16}$$

where $T_m(\xi), S_m(\xi)$ are the two independent solutions to the Chebyshev equation defined in the previous sections; A_m, B_m, C_m, D_m are constants to be determined through the boundary conditions that the potential must satisfy at $\xi = \xi_o$; and $\Psi_\lambda(\boldsymbol{\rho})$ is the potential due to the charged line, which is of the form

$$\Psi_\lambda(\boldsymbol{\rho}) = -\frac{\lambda}{2\pi\varepsilon_o} \ln|\boldsymbol{\rho}-\boldsymbol{\rho}'| = \frac{\lambda}{\varepsilon_o} G(\boldsymbol{\rho},\boldsymbol{\rho}'). \tag{17}$$

The potential for $\xi > \xi^{'}$, $\Psi_>(\boldsymbol{\rho})$, can be readily found, once $\Psi_o(\boldsymbol{\rho})$ has been uniquely specified, since $\Psi_o(\boldsymbol{\rho}) = \Psi_>(\boldsymbol{\rho})$ at $\xi = \xi'$, to ensure the continuity of the electrostatic potential in all space.

As we mentioned before, the constants appearing in Eqs. (15) and (16) can be found by imposing the proper boundary conditions, that is, the continuity of the tangential component of the electric field and the normal component of the electric displacement at $\xi = \xi_o$, respectively, a matter that can be expressed as

$$-\frac{1}{h_\eta}\frac{\partial \Psi_i}{\partial \eta}\bigg|_{\xi_o} = -\frac{1}{h_\eta}\frac{\partial \Psi_o}{\partial \eta}\bigg|_{\xi_o},$$

and

$$-\frac{\varepsilon}{h_\xi}\frac{\partial \Psi_i}{\partial \xi}\bigg|_{\xi_o} = -\frac{\varepsilon_o}{h_\xi}\frac{\partial \Psi_o}{\partial \xi}\bigg|_{\xi_o},$$

where $\varepsilon, \varepsilon_o$ are the permittivities of cylinder and vacuum, respectively.

In doing so, the following set of equations can be obtained:

$$(C_m - A_m)T_m(\xi_o) + (D_m - B_m)S_m(\xi_o) + \frac{\lambda}{\varepsilon_o a_m} T_m(\eta')\mathrm{f}_m(\xi_o, \xi') = 0, \tag{18}$$

and

$$(\varepsilon_o C_m - \varepsilon A_m)T'_m(\xi_o) + (\varepsilon_o D_m - \varepsilon B_m)S'_m(\xi_o) + \frac{\lambda}{a_m} T_m(\eta')\mathrm{f}^{'}_m(\xi_o, \xi') = 0, \tag{19}$$

where we have used the fact that both sets of functions, $\{T_m(\eta)\}$and $\{T'_m(\eta)\}$, are linearly independent. The primes in the functions mean the derivative of them with respect to their arguments.

As the reader must be aware, we have two equations and four unknowns, that is, this system is underdetermined so that, in order to get a unique solution of this problem, some physical considerations must be included. The first of them involves the behavior of the response of the material media (cylinder) as the line moves farther away, *i.e.*, as $\xi' \to \infty$. In this a case, the response of the dielectric must vanish for all values of ξ. A closer inspection of Eq. (16) leads to the conclusion that $C_m = 0$ for a proper behavior of the electric field in this limit, otherwise, when $\xi \gg 1$, the field will diverge, a matter that represents an unphysical situation. Moreover, regarding the field inside the cylinder ($1 \leq \xi < \xi_o$), it must also be well-behaved as $\xi \to 1$; in this case one can note that this happens for both inner solutions so that $B_m = 0$ or $A_m = 0$. If the latter is true, a closer inspection of Eqs. (18) and (19) leads to the conclusion that the thus solution obtained is the trivial one since, as $K \to 1$ ($K = \varepsilon/\varepsilon_0$ is the dielectric constant), the potential diverges (the determinant of the corresponding system of equations becomes zero in this limit), a matter that is not of physical interest, and then $B_m = 0$.

With these considerations, Eqs. (18) and (19) can provide a unique solution since we now have two equations with two unknowns. The resulting equations can be written as

$$A_m T_m(\xi_o) - D_m S_m(\xi_o) = \frac{\lambda}{\varepsilon_o a_m} T_m(\eta')\mathrm{f}_m(\xi_o, \xi'), \tag{20}$$

and

$$\varepsilon A_m T'_m(\xi_o) - \varepsilon_o D_m S'_m(\xi_o) = \frac{\lambda}{a_m} T_m(\eta')\mathrm{f}^{'}_m(\xi_o, \xi'), \tag{21}$$

whose solution is given by

$$A_m = \frac{\lambda}{\varepsilon_o a_m} \frac{T_m(\eta')S_m(\xi')}{[1+(\xi_o^2-1)^{\frac{1}{2}}(K-1)S_m(\xi_o)T'_m(\xi_o)]}, \tag{22}$$

and

$$D_m = \frac{\lambda(1-K)(\xi_o^2-1)^{\frac{1}{2}}}{\varepsilon_o a_m} \times \frac{T_m(\eta')S_m(\xi')T_m(\xi_o)T'_m(\xi_o)}{[1+(\xi_o^2-1)^{\frac{1}{2}}(K-1)S_m(\xi_o)T'_m(\xi_o)]}, \quad (23)$$

where we used the fact that the Wronskian of T_m and S_m is given by $W(T_m(\xi_o), S_m(\xi_o)) = -1/(\xi_o^2-1)^{\frac{1}{2}}$. With these constants the electrostatic potential is uniquely defined and can be used to calculate the electric field, the polarization in the cylinder, among other physical magnitudes of interest. To check out the consistency of the solution we shall examine, in the next section, some limiting situations.

5. Some interesting limiting situations

The proper behaviour of the solution obtained in the last section can be verified by the analysis of some limiting situations. In all four cases considered, we corroborate the fact that our approach is consistent enough.

i) $K \to 1$, the material medium becomes the vacuum; in this case, it can be seen that

$$A_m \to \frac{\lambda}{\varepsilon_o a_m} T_m(\eta')S_m(\xi') \text{ and } D_m \to 0\ , \quad (24)$$

which means that

$$\Psi_i \to \Psi_o \quad (25)$$

as it must be, since no material medium is already present in this limit.

ii) $\xi' \to \infty$, the charged line is placed far away from the elliptic cylinder; in this case

$$A_m, D_m \to 0, \quad (26)$$

which means that

$$\Psi_i \to 0 \text{ and } \Psi_o \to \Psi_\lambda, \quad (27)$$

a matter that is consistent with the fact that the electric field becomes zero at the position of the cylinder, *i.e.*, no response of this material can arise, as expected.

iii) $K \to \infty$, the material becomes a conductor; in this case we have

$$A_m \to 0 \text{ and}$$

$$D_m \to -\frac{\lambda}{\varepsilon_o a_m}\frac{T_m(\eta')S_m(\xi')T_m(\xi_o)}{S_m(\xi_o)}; \quad (28)$$

that is, the potential inside the ellipse is zero and the external one is given by

$$\Psi_o(\boldsymbol{\rho}) = \frac{\lambda}{\varepsilon_o}\sum_{m=0}^{\infty}\frac{1}{a_m}\left[T_m(\xi) - \frac{S_m(\xi)T_m(\xi_o)}{S_m(\xi_o)}\right] \times T_m(\eta')S_m(\xi')T_m(\eta) \quad (29)$$

as it can be noted, $\Psi_o = 0$ at $\xi = \xi_o$, consistent with the problem of a grounded conducting elliptic cylinder and a charged line.

iv) $a \to 0\ (\xi_o \to \infty), K \to \infty$, a circular conducting cylinder and a charged line. In this case, since $\xi_o < \xi < \xi^{'}$, the behavior of the potential given by Eq. (29) is dictated by the asymptotic behavior of T_m and S_m when $\xi_o, \xi, \xi^{'} \to \infty$ which is of the form

$$T_m(x) \sim x^m \text{ and } S_m(x) \sim x^{-m}\ , x \gg 1. \quad (30)$$

Now, let us recall that, in this coordinate system, if r_1 and r_2 are the distances of one point of the ellipse to the foci, then

$$r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$$

and

$$r_2 = \sqrt{r^2 + a^2 + 2ar\cos\theta},$$

where (r, θ) are the usual polar coordinates. Then, when $\xi_o \to \infty$, $a \to 0$ since, by definition

$$\xi_o = \frac{r_1 + r_2}{2a} \simeq \frac{R_o}{a}$$

(and $\xi^{'} \simeq R^{'}/a$) where $R_o\ (R^{'})$ is a typical size of the ellipse when $a \ll 1$. The above expressions for r_1, r_2 also allow us to say that

$$\eta = \frac{r_1 - r_2}{2a} \simeq \cos\theta$$

when $a \ll 1$ and thus, the potential given by Eq. (29), can be rewritten as

$$\Psi_o(\boldsymbol{\rho}) \simeq \frac{\lambda}{\varepsilon_o}\sum_{m=0}^{\infty}\frac{1}{a_m}\left[\xi^m - \frac{\xi^{-m}\xi_o^m}{\xi_o^{-m}}\right] \times T_m(\cos\theta')(\xi^{'})^{-m}T_m(\cos\theta); \quad (31)$$

with the aid of the behavior of ξ, ξ_o, or ξ' mentioned before, we have that

$$\Psi_o(\boldsymbol{\rho}) \simeq \frac{\lambda}{\varepsilon_o}\sum_{m=0}^{\infty}\frac{1}{a_m}\left[\left(\frac{R}{a}\right)^m - \left(\frac{R}{a}\right)^{-m}\left(\frac{R_o}{a}\right)^{2m}\right] \times \cos m\theta'\left(\frac{R'}{a}\right)^{-m}\cos m\theta \quad (32)$$

or

$$\Psi_o(\boldsymbol{\rho}) \simeq \frac{\lambda}{\varepsilon_o}\sum_{m=0}^{\infty}\frac{1}{a_m}\left[\left(\frac{R}{R'}\right)^m - \left(\frac{R''}{R}\right)^m\right] \times \cos m\theta'\cos m\theta, \quad (33)$$

where $R'' = R_o^2/R'$. Note that $\Psi_o = 0$ when $R = R_o$, so that this expression can be recognized as the potential of a charged line placed at R' with charge λ per unit length and its image placed at R'' with charge per unit length $-\lambda$, which is equivalent to the solution of the problem of a grounded circular cylinder of radius R_o and a charged line placed at R' rendered by the method of electrostatic images. As a reference, the reader is aware to confirm this fact by reviewing the expansion of the logarithmic term as is done in Gradshteyn & Ryzhik[6].

6. Conclusions

The general solution of the Laplace equation and its corresponding Green's function in elliptic coordinates were obtained in this work. The reported expressions for these functions can be used to study an interesting class of two-dimensional boundary-value electrostatic problems. In this context, we can mention, for instance, the boundary-value problem of an elliptic conducting cylinder, with a given potential or surface charge, and a charged line or in general, as presented here the case of a dielectric elliptic cylinder and an infinite charged lines and all of its limiting situations that constitutes the previous cases and more.

As a collateral result, a second, linearly independent, solution the Chebyshev differential equation in $[1, \infty)$ needed to be constructed using the Fröbenius method, or another standard procedure for such a cases, a matter which allowed us to construct the Green's function associated with the Laplace operator in this coordinate system. The formalism followed here to obtain these results can be extended to any two-dimensional coordinate system in which the Laplace equation is separable, or at least partially separable. In particular, calculations of the electric field for charge distributions with elliptic geometry such as those worked on by Furman[3] can be treated in a natural way using the formalism introduced here.

Moreover, the knowledge of Green's function would be useful for obtaininga representation of the Coulomb potential in two dimensions in this coordinate system[7], a matter that opens the possibility of some practical applications to problems of physical interest which involve this potential[8]. Work is in progress to apply some of the results to specific systems and will be published elsewhere.

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Appendix

The Chebyshev's differential equation outside the $[-1, 1]$ interval can be expressed as follows:

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + \xi\frac{dH}{d\xi} - n^2 H(\xi) = 0. \tag{A.1}$$

In order to find a solution, we will use the Frobenius method; in doing so, we suggest the following form for the function:

$$H(\xi) = \sum_{l=0}^{\infty} a_{-l}\xi^{k-l}. \tag{A.2}$$

Making the substitution of this expression and its derivatives in Eq. (A.1), we will have

$$\sum_{l=0}^{\infty}\left[(k-l)^2 - n^2\right] a_{-l}\xi^{k-l} - \sum_{l=2}^{\infty}(k+l+2)(k+l+1)a_{-l+2}\xi^{k-l} = 0, \tag{A.3}$$

and from it, the recurrence relations for the coefficients will appear:

$$a_0(k^2 - n^2) = 0 \tag{A.4}$$

$$a_{-1}\left[(k-1)^2 - n^2\right] = 0 \tag{A.5}$$

$$a_{-l}\left[(k-l)^2 - n^2\right] = (k+l+2)(k+l+1)a_{-l+2} \quad \text{for } l \geq 2. \tag{A.6}$$

From the secular equation (A.4), we shall find the values of k if we assume that

$$a_0 \neq 0, a_1 = 0 \text{ then } k^2 - n^2 = 0 \text{ or } k = \pm n. \tag{A.7}$$

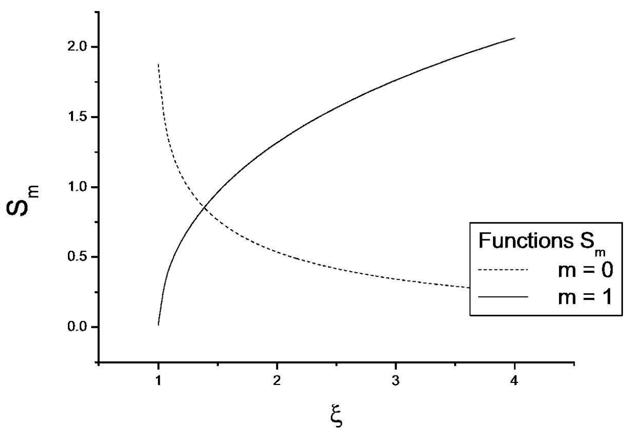

FIGURE 1. Graph for the Chebyshev functions of the second kind for $n = 0, 1$.

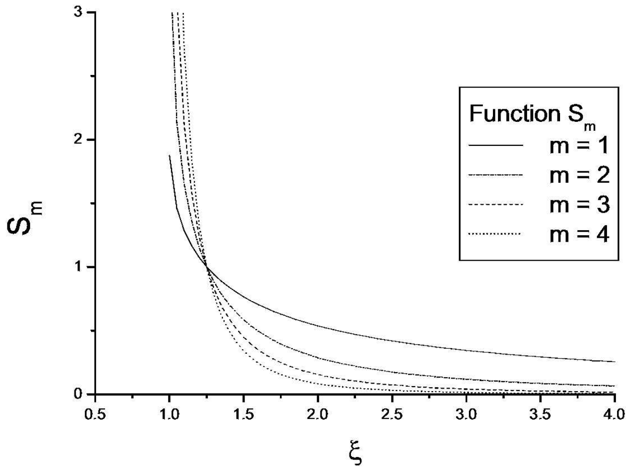

FIGURE 2. Graph for the Chebyshev functions of the second kind for $n = 1, 2, 3, 4$.

But we require the function $H(\xi)$ to vanish as $\xi \to \infty$; thus we must consider the value of k to be strictly negative, *i.e.*

$$k = -n, \text{ with } n > 0 \tag{A.8}$$

Using Eqs. (A.4-A.6), we can derive a compact expression for the coefficients:

$$a_{-2l} = \frac{n \cdot \prod_{s=l+1}^{2l-1}(n+s)}{2^{2l} l!} a_0, \text{ with } l = 1, 2, 3, \cdots \tag{A.9}$$

Functions of well-defined parity will then be built with the aid of these coefficients:

$$H_n^{\pm}(\xi) = a_0 \xi^{-n} \left\{ 1 + n \cdot \sum_{l=1}^{\infty} \frac{\prod_{s=l+1}^{2l-1}(n+s)}{2^{2l} l!} \xi^{-2l} \right\}, \tag{A.10}$$

where $+$ stands for n even and $-$ for n odd. In the special case where $n = 0$, the function $H_0(\xi)$ is the solution to the differential equation

$$(\xi^2 - 1)^{1/2} \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} \frac{dH_0(\xi)}{d\xi} \right] = 0, \tag{A.11}$$

where we have put it in self-adjoint form; this can be solved by direct integration and yields the function

$$H_0(\xi) = C \ln\left(\xi + \sqrt{\xi^2 - 1}\right) \tag{A.12}$$

We then call this set of functions the Chebyshev functions of the second class that are a solution to Eq. (A.1), and are defined by

$$S_n(\xi) = \begin{cases} a_0 \ln\left(\xi + \sqrt{\xi^2 - 1}\right), & \text{for } n = 0 \\ a_0 \xi^{-n} \left[1 + n \cdot \sum_{l=1}^{\infty} \frac{\Gamma(n+2l)(2\xi)^{-2l}}{\Gamma(n+l+1)\Gamma(l+1)} \right], & \text{for } n \geq 1 \end{cases} \tag{A.13}$$

In Figs. A.1 and A.2 we show graphs of those functions for values of the index $n = 0, 1, 2, 3, 4$.

By using Gauss' test (see Arfken p. 245) and Eq. (A.6) we can easily show that the series (A.10) converges at $\xi = 1$.

Finally, we consider it necessary to point out that this method for obtain Chebyshev functions of the second kind is not unique; an alternative way to built those functions would involve the direct evaluation of the Wronskian and the Chebyshev polynomials of the first kind, as discussed by Arfken for the Legendre polynomials[4]. Both representations are compatible when calculated for $\xi > 1 + \varepsilon$, but the series form of the functions $S_n(\xi)$ is easier to implement in a numerical calculation as that of Green's function on elliptic coordinates.

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