The exactly solvable self-gravitating fermion cluster in two dimensions

J. Sañudo
Departamento de Física, Universidad de Extremadura,
06071 Badajoz, Spain,
Tel: 34924289525, FAX: 34924289651,
e-mail: jsr@unex.es

A.F. Pacheco
Facultad de Ciencias and BIFI, Universidad de Zaragoza,
50009 Zaragoza, Spain,
Tel: 34976761135, FAX: 34976761140,
e-mail: amalio@unizar.es

Recibido el 23 de junio de 2006; aceptado el 9 de agosto de 2006

The mathematical model of a two-dimensional self-gravitating cluster formed by degenerate fermions, is solved analytically. The fermions interact with each other through a logarithmic potential. The radius of this system is shown to be constant, not depending on the total number of fermions that constitute the cluster.

Keywords: Thomas-Fermi model; white dwarfs.

En este artículo resolvemos analíticamente el modelo matemático de un cúmulo autogravitante de fermiones degenerados, en dos dimensiones. Los fermiones interactúan entre ellos mediante un potencial logarítmico. El radio resultante para el cúmulo no depende del número total de fermiones que lo integran.

Descriptores: Modelo de Thomas Fermi; enana blanca.

PACS: 31.5.Bs; 97.20.Rp

1. Introduction

White dwarfs are stars that support themselves against gravity by the pressure of degenerate electrons [1]. Their density profile is modelled by a 3/2 polytrope, *i.e.* by a self-gravitating spherical cluster formed by degenerate fermions. This polytropic model leads to a second-order non-linear differential equation that is easily solved numerically[2].

It is amusing and instructive to see that in two dimensions, where the Poisson equation is obeyed by a logarithmic potential, the corresponding equation for the fermion cluster is analytically solvable. It is the purpose of this paper to describe this mathematical model that is simple enough to be given as a graduate exercise.

In Sec. 2, after writing the three equations of the model and the boundary conditions, we obtain the solution for the fermion density. In Sec. 3, the gravitational potential is calculated. The two energy terms of the cluster are computed in Sec. 4. Finally, in Sec. 5 we present a brief discussion comparing this system with the system in three dimensions.

It is worth stressing the great similarity existing between this problem –in three dimensions- and the description of heavy atoms by means of the Thomas-Fermi model. The reason is that, in both systems, one is dealing with forces of the type r^{-2} between degenerate fermions. In two dimensions, the Thomas-Fermi has also received attention due to its exact solubility and academic potential [3,4]. In three dimensions, the equilibrium gravitational structure of polytropic cylinders has also been studied [5].

2. The equations and the density profile

This model is expressed by three equations, and circular symmetry is assumed. First we have the Poisson equation for the Newtonian gravity in two dimensions, namely[3,4]

$$\vec{\nabla}_r^2 \phi(r) = 2\pi G m n(r), \tag{1}$$

where ϕ is the gravitational potential, n is the particle density, G is the gravitational constant and m is the particle mass. The distance to the centre is r and the mass density is the product $mn(r)$.

Second, we have the equation of hydrostatic equilibrium in the cluster,

$$\frac{p_F^2(r)}{2m} + m\phi(r) = C, \tag{2}$$

where $p_F(r)$ is the Fermi momentum at a distance r from the centre and C is a constant. Thus, Eq. (2) says that the maximum energy that a particle can have in the cluster is independent of r. This equation is ordinarily expressed as a balance between the gravity force acting towards the centre and the Fermi pressure gradient acting outwards [1].

Finally, we have the equation of state of the system [3]

$$p_F(r) = \left[2\pi\hbar^2 n(r)\right]^{1/2} \tag{3}$$

that relates the Fermi momentum of the electrons to the particle density. This is for two dimensions and for a spin 1/2 particle (in three dimensions the equation of state for the degenerate Fermi gas can be found, for example in Refs. 1 and 2). Alternatively, the equation of state could be written as a relation between pressure and density. Recall that we are dealing with a zero temperature system. The presence of the Planck constant in the equation of state indicates the quantum nature of this cluster.

Thus, we have three equations for three unknown variables ϕ, n and p_F, which depend on r. Eliminating p_F between the second and third, and inserting the result into the first, we obtain a differential equation for $n(r)$:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dn(r)}{dr}\right] = -\frac{2Gm^3}{\hbar^2}n(r). \tag{4}$$

At this point, it is convenient to work with a dimensionless distance, x,

$$r = bx, \tag{5}$$

where b is a characteristic length scale. Choosing

$$b = \frac{\hbar}{\sqrt{2Gm^3}}, \tag{6}$$

and adopting the convention that a dot on a variable indicates derivation with respect to x, Eq. (4) becomes

$$\ddot{n}(x) + \frac{\dot{n}(x)}{x} + n(x) = 0. \tag{7}$$

With respect to the two conditions to be fulfilled by $n(x)$, the first one is obvious:

$$\dot{n}(x = 0) = 0. \tag{8}$$

This is necessary to avoid a singularity at the origin as required by Eq. (7). The second condition comes from the condition that, if the total number of fermions in the cluster is N, then

$$N = \int n(r)d\vec{r} = 2\pi b^2 \int_0^X n(x)xdx, \tag{9}$$

where R (or $X = R/b$) is the distance where n vanishes, or the radius of our cluster.

It is convenient to work with a non-dimensional density; $\hat{n}(x)$defined as

$$n(x) = n_0\hat{n}(x), \qquad n_0 \equiv n(0). \tag{10}$$

In the new notation, the two boundary conditions are expressed by:

$$\hat{n}(0) = 1, \qquad \dot{\hat{n}}(0) = 0, \tag{11}$$

and coming back to Eq. (7), it now reads as

$$\ddot{\hat{n}}(x) + \frac{\dot{\hat{n}}(x)}{x} + \hat{n}(x) = 0. \tag{12}$$

This equation is Bessel's equation of order zero[6]. With respect to the second condition in Eq. (9), we have:

$$\frac{N}{2\pi b^2 n_0} = \int_0^X \hat{n}(x)xdx = -\int_0^X \left(\frac{d}{dx}\left[x\frac{d\hat{n}(x)}{dx}\right]\right)dx$$
$$= -\left[x\frac{d\hat{n}(x)}{dx}\right]_0^X = -X\dot{\hat{n}}(X). \tag{13}$$

Now, the general solution of Eq. (12) is

$$\hat{n}(x) = AJ_0(x) + BY_0(x), \tag{14}$$

J_0 and Y_0 are Bessel functions of the first and second species respectively [6], and A and B arbitrary constants. Due to the fact that

$$J_0(x = 0) = 1, \qquad Y_0(x \to 0) \to \infty, \tag{15}$$

and bearing in mind Eq. (11), the solution of the problem is

$$\hat{n}(x) = J_0(x). \tag{16}$$

This function has its first node at 2.40483; therefore

$$\text{X} = 2.40483\cdots, \tag{17}$$

is the dimensionless radius of the cluster. Another property of these functions is

$$\dot{J}_0(x) = -J_1(x),$$
$$J_1(X = 2.40483\cdots) = 0.5191\cdots. \tag{18}$$

Thus from Eq. (13), we calculate

$$n_0 = \frac{N}{2\pi b^2 X J_1(X)}. \tag{19}$$

This completes the computation of the density.

3. The gravitational potential

In this section, we shall identify the form of the gravitational potential ϕ for distances $x \geq X$, and $0 \leq x \leq X$. At $x = X$, we shall require ϕ and its first derivative to be continuous. As usual, ϕ will be fixed except for an arbitrary additive constant. This constant is recognized in the C of Eq. (2). Therefore, we shall henceforth assume that

$$C = 0. \tag{20}$$

This is equivalent to saying that ϕ = 0 at $x = X$, that is, where the density vanishes.

For $x \geq X$, the Poisson equation (1) indicates that

$$\frac{1}{x}\frac{d}{dx}\left[x\frac{d\phi(x)}{dx}\right] = 0. \tag{21}$$

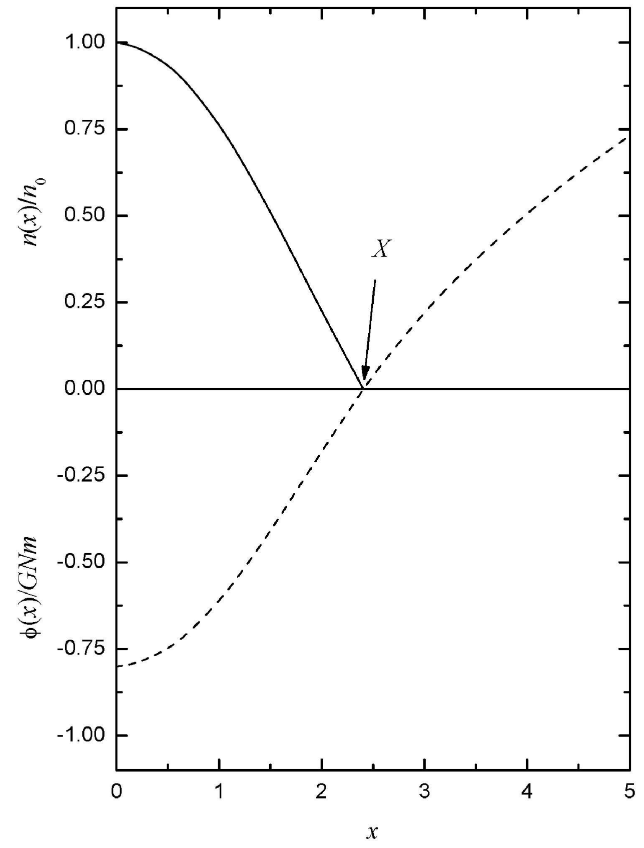

FIGURE 1. Profiles of density (continuous line) and potential (dashed line) vs. distance to the origin. The length unit, *b*, is given in Eq. (6). The dimensionless radius of the cluster, X, is given in Eq. (17) and the unit of density,n_0, is given in Eq. (19).

The solution of Eq. (21) that verifies $\phi\,(X)$ = 0 is

$$\phi(x) = Kln\left(\frac{x}{X}\right), \tag{22}$$

and the gravity field is

$$\vec{E}(x) = -\frac{K}{x}\vec{e}_x, \tag{23}$$

where K is a constant that will be fixed later, and the unitary vector $\vec{e}_x$ points outward.

For $0 \leq$ x $\leq$ X, using Eqs. (2) and (3), we obtain $\phi\,(x)$. The field is given by

$$\vec{E}(x) = -\frac{d}{dx}\left[\frac{C}{m} - \frac{\pi\hbar^2}{m^2}n(x)\right]\vec{e}_x$$
$$= -\frac{GNm}{X}\left[\frac{J_1(x)}{J_1(X)}\right]\vec{e}_x. \tag{24}$$

Thus, imposing the continuity of the field at $x = X$, we identify the value of K

$$K = GNm. \tag{25}$$

Thus, $\phi(x)$ can be expressed in a compact form as follows:

$$\phi(x) = GNm\hat{\phi}(x),$$

$$\hat{\phi}(x) = -\frac{J_0(x)}{XJ_1(X)}, \qquad 0 \leq x \leq X,$$
$$\hat{\phi}(x) = ln\left(\frac{x}{X}\right), \qquad x \geq X. \tag{26}$$

The universal solutions obtained for $n(x)/n_0 \equiv \hat{n}(x)$ and $\phi(x)/GNm \equiv \hat{\phi}(x)$ have been plotted in Fig. 1.

4. The energies

In this system, there are two global energy terms. One is the kinetic energy, E_k, of the degenerate fermions, and the other, the potential energy, V, that derives from the particle-particle gravitational attraction. Using the properties of Bessel functions[6], we find

$$\int_0^X xJ_0^2(x)dx = \left[\frac{x^2}{2}\dot{J}_0^2(x) + \frac{x^2}{2}J_0^2(x)\right]_0^X$$
$$= \frac{X^2}{2}\dot{J}_0^2(X) = \frac{X^2}{2}J_1^2(X). \tag{27}$$

Therefore the kinetic energy of the cluster is

$$E_k = \int_0^R \frac{1}{2}\left[\frac{p_F^2(r)}{2m}\right]n(r)d\vec{r} = \frac{\pi^2\hbar^2}{m}\int_0^R n^2(r)rdr$$
$$= \frac{GN^2m^2}{2X^2J_1^2(X)}\int_0^X J_0^2(x)xdx = \frac{GN^2m^2}{4}. \tag{28}$$

The factor 1/2 in the kinetic energy is specific to the degenerate fermions in two dimensions [3,4,7]. And with respect to the potential energy, we find

$$V = \frac{1}{2}\int_0^R \phi(r)mn(r)d\vec{r} = -\frac{\pi^2\hbar^2}{m}\int_0^R n^2(r)rdr$$
$$= -E_K = -\frac{GN^2m^2}{4}. \tag{29}$$

Thus, in two dimensions, the total energy of the degenerate cluster is zero.

5. Conclusions

We have studied the properties of a self-gravitating cluster formed by massive degenerate fermion, in two dimensions. We have solved this system using the terminology of the Thomas-Fermi model [1,8] for heavy atoms. The density profile and the dependence of the gravitational potential with respect to the distance are explicitly obtained. These functions are expressed in terms of Bessel functions. The kinetic and potential energies of the cluster have been calculated.

To conclude, it may be worthwhile making a comparison of this two-dimensional, self-gravitating fermion cluster with its equivalent in three dimensions. The most glaring difference lies in the scale length parameter b which varies as $N^{-1/3}$ in three dimensions, whereas in the two dimensional model is N-independent. In three dimensions the energy terms of the cluster (kinetic and potential) scale as $N^{7/3}$, while in two dimensions they scale as N^2. With respect to the density at the centre, in two dimensions it scales linearly with N, while in three dimensions it scales with N^2.

Acknowledgements

This work was supported in part by the Spanish DGICYT under Project FIS2005-06237.

1. See, for example, S.L. Shapiro, and S.A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (New York: Wiley, 1983).
2. See, for example, S. Eliezer, A. Ghatak, and H. Hora, *An Introduction to Equations of State* (Cambridge: Cambridge U.P., 1986).
3. R.K. Bhaduri, S. Das Gupta, and S.L. Lee, *Am. J. Phys.* **58** (1990) 983.
4. G.F. Kventsel and J. Katriel, *Phys. Rev. A* **24** (1981) 2299.
5. J. Ostriker, *Ap. J.* **140** (1964) 1056.
6. See, for example, M. Abramowitz and I.A. Stegun (Editors) *Handbook of Mathematical Functions* (New York: Dover Publications, 1970)
7. R. Pino, *Phys. Rev. B* **58** (1998) 4644.
8. M. Membrado, A.F. Pacheco, and J. Sañudo, *Astron. Astrophys.* **217** (1989) 92.