The nonlinear pendulum: formulas for the large amplitude period

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A simple and precise formula for the period of a nonlinear pendulum is obtained using the Linear Delta Expansion, a powerful non–perturbative technique which has been applied in the past to problems in different areas of physics. Our result is based on a systematic approach which allows us to obtain a new series for the elliptic integrals, in terms of which the exact solution of our problem is cast. A further improvement of the LDE result is then obtained by using Padé approximants. Finally we make a comparison with other approximations in the literature for the period of the pendulum, valid either at small or at large angles.

Keywords: Perturbation theory; linear delta expansion; pendulum.

Hemos obtenido una fórmula simple y precisa para el período de oscilación de un péndulo no-lineal utilizando la Expansión Delta Lineal, una técnica no-perturbativa que ha sido utilizada en el pasado en areas distintas de la física. Nuestro resultado se ha obtenido utilizando un método sistemático que permite obtener una serie para las integrales elípticas, por medio de la cual se expresa la solución exacta. Los resultados son mejorados también utilizando los aproximantes de Padé. Finalmente, hacemos también una comparación con otras aproximaciones en la literatura para el periodo del péndulo, tanto para ángulos pequeños como para grandes.

Descriptores: Téoria de perturbaciones; expansión delta lineal; péndulo.

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1. Introduction

This paper is focussed on the study of the period of oscillation of a simple pendulum; this problem has been considered in depth in the past by many authors, and a variety of approximations have been found for calculating its period with precision [1–8]. The simple pendulum is also a standard topic in many textbooks at different levels, both undergraduate and graduate, and the natural ground for the discussion of nonlinear effects; as a matter of fact, it is the first example of nonlinear problem that many students discuss in their classes. A good understanding of this example, both of the physical and of the mathematical aspects of the problem, is therefore very valuable in helping the student to later understand more complicated physical problems and to dominate the different methods used to deal with nonlinear problems.

Our purpose in the present paper is to illustrate a simple and effective procedure to calculate the period of oscillation of a pendulum. The method that we propose here can also be used to familiarize the student with the ideas of perturbation theory and of "non–perturbative" techniques; it is based on the ideas of the Linear Delta Expansion (LDE), which has been very effective in dealing with a very large class of problems, ranging from classical and quantum mechanics to quantum field theory (a very partial list is given by [9–11] and references therein).

The first encounter of a student with these topics is probably in the classes of quantum mechanics, where the application of the techniques to higher orders is normally complicated by the presence of operators. Classical mechanics provides a useful ground for introducing the idea of perturbation theory and non–perturbative methods, retaining the general features of the methods but without many of the complications arising from quantum mechanics.

The paper is organized as follows: in Sec. 2 we set up the problem and derive the fundamental equations; in Sec. 3 we describe a method to evaluate the formulas obtained in Sec. 2 and we compare our formulas with the formulas in the literature; finally in Sec. 4 we draw our conclusions.

2. The simple pendulum

A pendulum of mass m and length l oscillating in the (constant) gravitational field of the Earth obeys the nonlinear differential equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \tag{1}$$

where θ is the angle that the pendulum forms with the vertical direction, and $g \approx 9.81\ m/s^2$ is the acceleration due to gravity. A pendulum initially at rest with a certain angle θ_0 will perform oscillations of fixed amplitude θ_0 (provided that friction is not present) and of given period T once it is released. Although in the laboratory it is not possible to eliminate friction completely, thus observing damped oscillations of the pendulum, in our discussion we assume that it can be neglected and consider the idealized problem of a conservative pendulum.

Under this restriction we write the energy of the pendulum as

$$E = \frac{1}{2} m\, l^2\, \dot{\theta}^2 + m\, g\, l\, (1 - \cos\theta)\ . \tag{2}$$

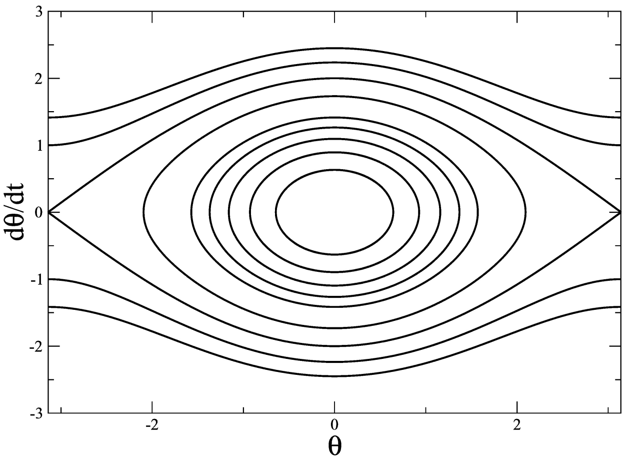

FIGURE 1. Phase space portrait for the nonlinear pendulum.

It is customary to discuss Eqs. (1) and (2) in the limit of small oscillations, where the trigonometric functions can be replaced be the leading order contributions stemming from their Taylor series; in this a case, one recovers the equation for the simple harmonic oscillator,

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0\,, \tag{3}$$

with a period independent of the amplitude, $T_0 = 2\pi\sqrt{\frac{l}{g}}$.

When larger amplitudes are considered, the period turns out to depend upon the amplitude, a typical outcome when nonlinearities are present; the precise form of such dependence can be calculated using the equation for the conservation of energy, Eq. (2), and writing

$$T = \int_{-\theta_0}^{+\theta_0} \frac{\sqrt{2m}\,l}{\sqrt{E - m\,g\,l\,(1-\cos\theta)}}\,d\theta\,. \tag{4}$$

Given that the potential energy calculated at the inversion points equals the total energy, we can also write

$$T = \frac{T_0}{\pi}\int_{-\pi/2}^{+\pi/2} \frac{dx}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 x}}, \tag{5}$$

which can be cast directly in terms of the elliptic integral of first kind as

$$T = \frac{2T_0}{\pi}\,K\left(\sin^2\frac{\theta_0}{2}\right)\,. \tag{6}$$

Equation (6) is the exact expression of the period of oscillation of the simple pendulum, which is found in standard textbooks of classical mechanics. In Fig. 1 we display the phase space for the pendulum.

3. The method

Although nowadays the numerical evaluation of expressions such as that of Eq. (6) poses no problem, and a value can be obtained with the desired accuracy for given values of θ_0, analytical approximations to Eq. (6) can be very valuable in practice.

We will proceed to derive such approximations following two different strategies: first we will describe a perturbative method, which corresponds essentially to individuating a small parameter in the problem and then perform a Taylor expansion in that parameter; in a second stage, we shall develop a non-perturbative procedure which does not rely on the presence of a small parameter and show that it can be used to obtain a very precise formula valid also for the large angle oscillations of a pendulum.

The starting point of our discussion is the general expression for the elliptic integral of first kind, which for convenience we write as

$$K(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-mt^2)}}\,, \tag{7}$$

with $0 < m \leq 1$. Note that $K(m)$ diverges at $m = 1^i$. The parameter m is easily related to the amplitude of the oscillations by looking back at Eq. (6).

For $|m| < 1$, one can Taylor expand the integrand of Eq. (7) and obtain the series

$$K(m) = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 m^n\right\}\,. \tag{8}$$

Since the purpose of converting the integral into a series is to obtain an easily applicable expression, we do not want in practice to evaluate many terms of the series, but only the first few terms, depending on the precision that we wish to achieve. Besides, although in the present example it is a straightforward task to find the structure of a term of a given order in the perturbative series, this is usually not the case; for example, standard perturbation theory in quantum mechanics requires, for each order, the evaluation of matrix elements of the potential with respect to a given basis, and perturbation theory in quantum field theory requires the evaluation of Feynman diagrams for each perturbative order (typically the calculation of these diagrams becomes more and more involved as the perturbative order grows).

If we truncate the series of Eq. (8) and consider the terms up to order $O\left[m^3\right]$, we obtain

$$K(m) \approx \frac{\pi}{2}\left[1 + \frac{1}{4}\,m + \frac{9}{64}\,m^2 + \ldots\right]\,, \tag{9}$$

a formula that can be used to obtain accurate estimates of the period of the simple pendulum for small angles. Notice that only the term of order zero in Eq. (9) is independent of m, *i.e.* of the amplitude of the oscillations; this is the reason of the isochronism of the small amplitude oscillations of a pendulum.

We will now describe how a non–perturbative series can be obtained: loosely speaking we refer to a method as being "non–perturbative" when it does not correspond to an expansion in some small parameter. As anticipated, the method that we propose is inspired by the "Linear Delta Expansion" (LDE), which we briefly explain as follows.

We first generalize Eq. (7) to

$$K_\delta(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1+\lambda-\delta(\lambda+mt^2))}} \tag{10}$$

where δ is parameter which has been introduced to permit power counting. λ is an arbitrary parameter, which disappears in the limit $\delta \to 1$. Clearly, by setting $\delta = 1$ in Eq. (10), one recovers the original integral. On the other hand, one can define

$$\Delta(t) \equiv -\frac{\lambda + mt^2}{1+\lambda}\ , \tag{11}$$

and write:

$$K_\delta(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1+\lambda)}}\ \frac{1}{\sqrt{1+\delta\ \Delta(t)}}\ . \tag{12}$$

For $|\Delta(t)| < 1$, which is fulfilled for $\lambda > -1/2$, we can expand the last term as:

$$\frac{1}{\sqrt{1+\delta\ \Delta(t)}} = \sum_{k=0}^{\infty} \frac{\Gamma(1/2)}{k!\ \Gamma(1/2-k)}\ \delta^k\ \Delta^k(t) \tag{13}$$

and obtain

$$K_\delta(m) = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\Gamma(1/2)\Gamma(j+1/2)m^j}{j!^2(k-j)!\Gamma(1/2-k)} \delta^k (-1)^k \times \frac{\lambda^{k-j}}{(1+\lambda)^{k+1/2}} \tag{14}$$

after performing the integrals. Note that Eq. (14) provides a family of series which all converge to the elliptic integral $K(m)$ for $\lambda > -1/2$ and $\delta = 1$[ii]; as before, we are not interested in evaluating the series, but rather in obtaining a precise and simple approximation with few terms. For this reason we consider the partial sum truncated at $k = N$, $K_N(m)$; whereas the infinite series is independent of λ, the partial series displays a dependence upon λ, as a direct consequence of having neglected an infinite number of terms. For a fixed order, *i.e.* for a given value of the "cutoff" N, we minimize this unwanted effect by applying the Principle of Minimal Sensitivity (PMS) [12] to the partial sum:

$$\frac{d}{d\lambda} K_N(m,\lambda) = 0\ . \tag{15}$$

We can understand Eq.(15) in the following way: the exact solution, $K(m)$, is independent of λ and therefore it is a horizontal line when plotted versus λ; Eq. (15) selects the points where the curve $K_N(m,\lambda)$ is closer to be a horizontal line, namely its extrema.

The solution to this equation selects the value of λ for which the approximation is less sensitive to changes in λ itself; remarkably, this equation permits a real root only for odd N, and it is given by

$$\lambda_{PMS} = -\frac{m}{2}, \tag{16}$$

independently of the order N. For the leading order, $N = 1$, we find the simple expression

$$K_1^{(pms)}(m) = \frac{\pi/2}{\sqrt{1-m/2}}. \tag{17}$$

We observe that this expression, although calculated to the first order, is non–polynomial in the parameter m; this feature is a direct consequence of having imposed the PMS, which provided us with an m-dependent value of λ. We also remark that our optimized expansion has a singularity at $m = 2$, contrary to the exact function, which is singular at $m = 1$. Of course the perturbative result to the same order (and to any finite order) does not show any singularity at a finite m, being a polynomial:

$$K_1^{(pert)} = \frac{\pi}{2}\ \left[1 + \frac{m}{4}\right]\ . \tag{18}$$

When a larger number of terms is considered, the series corresponding to the optimal value of $\lambda = -m/2$ is seen to provide a faster convergence rate than the "perturbative" series with the same number of terms (see Fig. 2). For larger values of m, the rate of convergence is smaller, as can also be appreciated by looking at Table I, where we have changed m, keeping the number of terms in the trucated series fixed. In all cases we see that our non-perturbative (PMS) series performs much better than the corresponding perturbative series.

As we mentioned before, the PMS series that we have obtained does not have a singularity at $m = 1$; this behaviour clearly limits the accuracy of partial sum for values of m close to 1, *i.e.* for large angles of oscillation. In order to remove this problem and obtain a solution valid even for large angles, we resort to Padé approximants, which are based on finding rational approximation reproducing a given expression. Padé approximants are very useful in treating many problems in classical and quantum mechanics [10].

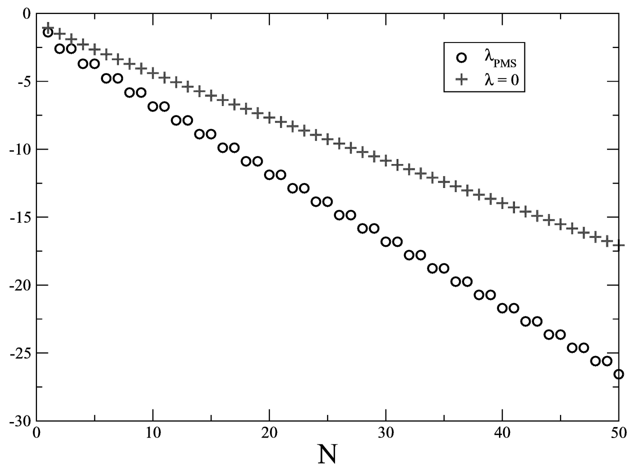

FIGURE 2. $\log_{10}|K(m) - K_N(m)|$ assuming $m = 1/2$ for a given number of terms in the series. The circles correspond to the PMS and the pluses to the perturbative series.

TABLE I. Comparison between the perturbative and PMS series calculated with ten terms as a function of m. The underlined digits are correct.

m	Perturbative	PMS	Exact
0	$\underline{1.5707963267948966}$	$\underline{1.5707963267948966}$	1.5707963267948966
0.1	$\underline{1.6124413487}197300$	$\underline{1.6124413487202193}$	1.6124413487202193
0.2	$\underline{1.6596235}974954030$	$\underline{1.659623598610}3125$	1.6596235986105280
0.3	$\underline{1.713889}3394309329$	$\underline{1.7138894481}207263$	1.7138894481787910
0.4	$\underline{1.77751}64193762719$	$\underline{1.7775193}674684559$	1.7775193714912533
0.5	$\underline{1.8540}343814512505$	$\underline{1.854074}5400483682$	1.8540746773013719
0.6	$\underline{1.949}2054685236691$	$\underline{1.94956}46343304475$	1.9495677498060258
0.7	$\underline{2.07}28570634903909$	$\underline{2.0753}069315791747$	2.0753631352924691
0.8	$\underline{2.2}422064609192961$	$\underline{2.25}62784055980924$	2.2572053268208536
0.9	$\underline{2}.4876684240301022$	$\underline{2.5}612167205241527$	2.5780921133481731

TABLE II. Maximum angle for which the different formulas for the period provide an error below 1%.

	$K_1(pert)$	$K_1(PMS)$	$K_{Parwani}$	K_{Pade}	$K_{Pade+Landen}$	K_{Cromer}
θ_{max}	$60°$	$75°$	$135°$	$166°$	$179°$	$152° < \theta < 180°$

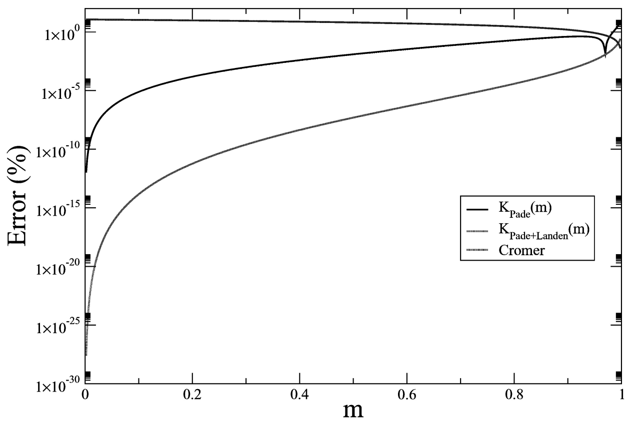

FIGURE 3. Error over $K(m)$ using the Padé formula, the Landen formula and the Formula of Cromer.

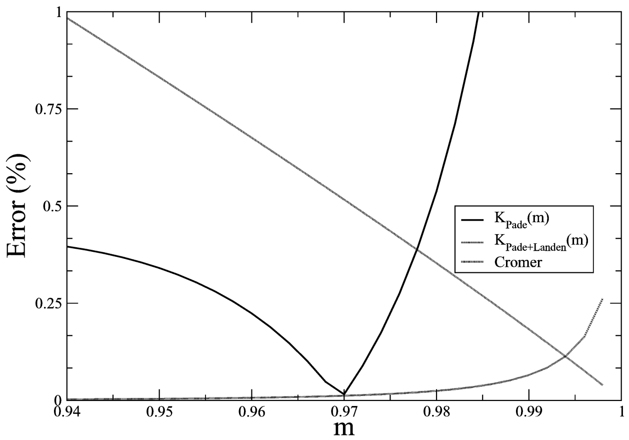

FIGURE 4. Amplification of Fig. 2 in the proximity of $m = 1$.

We proceed as follows: we consider the function

$$\Phi(m) \equiv \left[\frac{K(m)}{K_1^{pms}(m)}\right]^a = \left[\frac{2\sqrt{1-m/2}K(m)}{\pi}\right]^a \tag{19}$$

and calculate its Padé approximant $[2,1]$. For the moment being, a is an arbitrary parameter which will have to be fixed later.

In the present case, the Padé approximant [2,1] turns out to be

$$[\Phi(m)]_{[2,1]} = \frac{1 - m + \frac{3am^2}{64}}{1-m}. \tag{20}$$

This expression has the remarkable property of having the singularity at the correct point $m = 1$, independently of parameter a. We can now invert the equation defining Φ and write

$$K(m) = [\Phi(m)]^{1/a} \; K_1^{pms}(m)$$
$$\approx \left[\frac{1 - m + \frac{3am^2}{64}}{1-m}\right]^{1/a} \frac{\pi/2}{\sqrt{1-m/2}}. \tag{21}$$

We can follow different strategies to fix a; for example, the expression that we have found for $K(m)$ reproduces correctly the coefficients of the Taylor expansion in m up to order m^3. By setting $a = 35/6$, the terms going like m^4 and m^5 are also reproduced (notice that by promoting a to a function of m one could systematically reproduce the higher order terms, although we do not consider this improvement here). We have found numerically that the best results are obtained for $a \approx 13/2$, which gives us the remarkably simple formula

$$K_{pade}(m) = \left[\frac{1 - m + \frac{39}{128}m^2}{1 - m}\right]^{2/13} \frac{\pi/2}{\sqrt{1 - m/2}} . \quad (22)$$

This formula can still be drastically improved by using the Landen transformation [13], which relates the values taken by the elliptic integral at different points:

$$K(m) = \frac{1}{1 + \sqrt{m}} K\left(\frac{4\sqrt{m}}{(1 + \sqrt{m})^2}\right) \quad (23)$$

and

$$K(m) = \frac{2(1 - \sqrt{1 - m})}{m} K\left(\frac{(-2 + 2\sqrt{1 - m} + m)^2}{m^2}\right) . \quad (24)$$

The first formula relates the value of the elliptic integral $K(m)$ with the value at a larger point

$$m_1(m) = \frac{4\sqrt{m}}{(1 + \sqrt{m})^2};$$

the second formula relates the value of the elliptic integral $K(m)$ with the value taken at a smaller point

$$m_2(m) = \frac{(-2 + 2\sqrt{1 - m} + m)^2}{m^2}.$$

Equation (24) can be applied to any approximate expression of $K(m)$ valid for small m, changing it into an expression which becomes valid for much larger values of m. Indeed a repeated application of the transformation makes it possible to obtain arbitrarily accurate values of $K(m)$ for any given value of m. We have applied the Landen transformation to our Eq. (22) and we have compared the error, defined as

$$\frac{|K(m) - K_{approx}(m)|}{K(m)} \times 100,$$

with the errors found using Eq. (22) and the equation of Cromer [7], valid for $m \to 1$, which is given by

$$K_{Cromer}(m) = \frac{1}{2} \log \frac{16}{1 - m}. \quad (25)$$

In Fig. 3 we have plotted the three errors observing that even the simple formula (22) provides an excellent approximation up to quite large values of m. Indeed, in Fig. 4 we have made an amplification of Fig. 3 close to $m = 1$ and seen that Eq. (22) provides an error smaller than 1% up to $m \approx 0.986$; the Landen improved formula, which however is as simple as the previous one, provides an error smaller than 1% up to $m \approx 0.99999$. We also notice that Eq. (25) performs better than our formula only for $m > 0.98$, corresponding to an angle $\theta \approx 164°$.

Another approximate formula for the period of the pendulum has also been derived recently by Parwani [6]:

$$K_{Parwani} = \frac{\pi}{2} \left(\frac{\frac{\sqrt{3}}{2}\theta}{\sin\frac{\sqrt{3}}{2}}\right)^{1/2} , \quad (26)$$

providing good approximations for the period of the pendulum up to moderate values of the amplitude. In Table II, we have calculated the value of the maximum angle for which a maximum error of 1% is obtained, using the different approximations. Using our formula (22), we are able to reach a maximum angle of $166°$, despite its simplicity, compared with the value of $135°$ reached by Parwani.

4. Conclusions

In this paper we have derived a simple formula for the large-angle oscillations of the nonlinear pendulum, which compares quite favorably with other expressions found in the literature and has the advantage of being based on a systematic approach (and therefore of being improvable to any desired level of accuracy) and of never involving special functions. In its present form our formula based, on the LDE and on the Padé approximants, reaches a precision which is sufficient for all practical applications.

The main goal of the paper, however, was not to derive another approximate formula for the period of the pendulum. Although this formula is valuable and useful in practice, we believe that far more valuable is the explanation to the reader to non-perturbative techniques such as the LDE or the Padé approximants; these techniques are normally used in attacking more difficult problems in quite diverse areas of Physics, from classical and quantum mechanics to quantum field theory, and we believe that the pendulum can be used as a first example where a student can become familiarized with notions which are usually encountered much later in his studies.

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i By looking back at Eq. (6) we see that physically this is related to the fact that $\theta = \pi$ is a point of unstable equilibrium.

ii The reader can convince himself that the “perturbative” series, Eq. (8), corresponds to Eq. (14) taking $\lambda = 0$.

1. W.P. Ganley, *Am. J. Phys.* **53** (1995) 73.
2. R.B. Kidd and S.L. Fogg, *Phys. Teach.* **40** (2003) 81.
3. M.I. Molina, *Phys. Teach.* **35** (1997) 489.
4. P. Amore and R.A. Sáenz, *Europhysics Letters* **70** (2005) 425.

5. B.Wu and P. Li, *Meccanica* **36** (2001) 167.
6. R. Parwani, *European Journal of Physics* **25** (2004) 37.
7. A. Cromer, *Americal Journal of Physics* **63** (1995) 112.
8. L.M. Burko, *European Journal of Physics* **24** (2002) 125.
9. A. Duncana and M. Moshe, *Phys. Lett. B* **125** (1998) 352; A.Okopińska, *Phys. Rev. D* **35** (1987) 1835; G. Krein, D.P. Menezes, and M.B. Pinto, *Phys. Lett. B* **370** (1996) 5; M.P. Blencowe and A.P. Korte, *Phys. Rev. B* **56** (1997) 9422; H.F. Jones, P. Parkin, and D. Winder, *Phys. Rev. D* **63** (2001) 125013; J.L. Kneur, M.B. Pinto, and R.O. Ramos, *Phys. Rev. Lett.* **89** (2002) 210403; A. Pelster, H. Kleinert, and M. Schanz, *Phys. Rev. E* **67** (2003) 016604.
10. G.A. Arteca, F.M. Fernández, and E.A. Castro, *Large order perturbation theory and summation methods in quantum mechanics* (Springer, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, 1990).
11. H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, 3rd edition (World Scientific Publishing, 2004).
12. P.M. Stevenson, *Phys. Rev. D* **23** (1981) 2916.
13. *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).