Continuous groups of transformations and time-dependent invariants

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Recibido el 6 de diciembre de 2006; aceptado el 17 de abril de 2007

In this paper we present a very simple derivation of the constants of motion for dynamical systems, which requires only an elementary knowledge of the theory of continuous groups. In addition, through the infinitesimal Lorenz transformations group, we obtain a clear interpretation of the invariant for the harmonic oscillator.

Keywords: Lie groups; Lorenz group; dynamical systems; Noether's theorem; infinitesimal transformations.

Se presenta un método sencillo para la derivación de las constantes de movimiento de sistemas dinámicos, la cual requiere solamente conocimientos elementales de la teoría de grupos continuos. Además, mediante las transformaciones infinitesimales de Lorentz, se obtiene una interpretación clara del invariante para un oscilador armónico.

Descriptores: Grupos de Lie; grupo de Lorentz; sistemas dinámicos; teorema de Noether; transformaciones infinitesimales.

PACS: 02.90.+p

1. Introduction

In recent years, many authors have investigated Ermakov's invariants for linear dynamical systems, such as the harmonic oscillator [3, 4, 6, 7], and some of them have generalized the applications to non-linear systems described by a more general Lagrangian [5, 8, 9]. However, a clear physical interpretation of the invariants of motion has not been given, although a general interpretation has been mentioned [9, 11]. On the other hand, Lutzky derived constants of motion from the eight-parameter symmetry group for the one-dimensional harmonic oscillator using Noether's theorem [3, 4] and suggested a physical interpretation. Besides the mathematical interest in finding invariants, from the physical point of view their relevance lies in the fact that in some cases the differential equations that describe the evolution of the systems are not easy to solve but they admit invariants, and the knowledge of certain functions of the coordinates and momenta which remain constant during motion can be of great help in simplifying the equations of motion, and can lead to their solution. In contrast to Lutzky derivation, in this paper we present, as an alternative method, a very simple way to obtain invariants using only elementary notions of the theory of continuous groups and Noether's theorem. We also give an interpretation of the invariants using the generator of the transformation, which is possible thanks to the fact that Noether's symmetry is a Lie point transformation that leaves the action invariant, up to an additive time constant that is a function of the group parameter [12, 13]. In this paper, Eq. (13) is used to find the physical interpretation of the invariants, and this equation is precisely the generator of the one-parameter infinitesimal transformation. It is important to note that, with the method presented here, in contrast to Ermakov's method for finding invariants, we do not need an auxiliary equation in order to obtain the invariants [5, 6].

The outline of the paper is as follows: in Sec. 2 we obtain the constant of motion and in Sec. 3, via the Lorentz transformation, we interpret the invariant for the case of a harmonic oscillator. Finally, in Sec. 4 some concluding remarks are given.

2. Continuous groups of transformations and Noether's invariant

Let us suppose that we have a one-parameter group of transformations written in the form

$$\bar{x}^j = f^j\left(x^i; a\right), \qquad i, j = 1, 2, ...n, \tag{1}$$

with $a = a_0$ for the identity, *i.e.*,

$$f^j\left(x^i; a_0\right) = x^j. \tag{2}$$

Let f^j be the solution of the system of ordinary differential equations

$$\frac{df^j}{da} = \xi^j\left(\bar{x}^i\right)\psi\left(a\right), \tag{3}$$

satisfying the initial conditions (2) [1, 2]. If we define parameter t by

$$t = \int_{a_0}^{a} \psi(a)\, da, \tag{4}$$

then $t = 0$ yields the identity, and Eq. (3) becomes

$$\frac{d\bar{x}^j}{dt} = \xi^j\left(\bar{x}^i\right). \tag{5}$$

Assuming the functions ξ^j to be regular in the domain of x^i, the integral of (5) can be written in the form

$$\bar{x}^j = x^j + \xi_a^j(x)\, t, \tag{6}$$

which can be considered to be an infinitesimal transformation of the group with a generator given by

$$G_a = \xi_a^j \frac{\partial}{\partial \bar{x}^j}. \tag{7}$$

In fact, Eq. (1) can also be thought as a coordinate transformation that satisfies

$$\bar{x}_1^j = \frac{\partial f}{\partial x^i} x_1^i, \tag{8}$$

where x_1^i denotes the differential dx^i.

Equations (1) and (8) define the extended group of transformations in the $2n$ variables x^i and x_1^i whose generator is given by

$$G_{(1)a} = \xi_a^j \frac{\partial}{\partial x^j} + \xi_{(1)a}^j \frac{\partial}{\partial x_1^j}, \tag{9}$$

with

$$\xi_{(1)a}^j\left(x^i, x_1^i\right) = \frac{\partial \xi_a^j}{\partial x^k} x_1^k. \tag{10}$$

For the case of two variables (x^1, x^2), the generator (9) reduces to

$$G_{(1)a} = \xi_a^1 \frac{\partial}{\partial x^1} + \xi_a^2 \frac{\partial}{\partial x^2} + \left[\frac{\partial \xi_a^2}{\partial x^1} + \left(\frac{\partial \xi_a^2}{\partial x^2} - \frac{\partial \xi_a^1}{\partial x^1}\right) x^{2'} - \left(x^{2'}\right)^2 \frac{\partial \xi_a^1}{\partial x^2}\right] \frac{\partial}{\partial x^{2'}}, \tag{11}$$

where the relations

$$x^{2'} = \frac{x_1^2}{x_1^1}, \quad \frac{\partial}{\partial x_1^1} = -x^{2'} \frac{\partial}{\partial x_1^1} \partial x^{2'} \quad \text{and} \quad \frac{\partial}{\partial x_1^2} = \frac{\partial}{\partial x_1^1} \partial x^{2'}$$

have been used.

Substituting

$$x^1 = ct, \quad x^2 = x, \quad \frac{\xi_a^1}{c} = \xi, \quad \xi_a^2 = \eta, \tag{12}$$

Eqs. (7) and (11) transform into

$$G_a = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} \tag{13}$$

and

$$G_{(1)a} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \left(\dot{\eta} - \dot{\xi}\dot{x}\right) \frac{\partial}{\partial \dot{x}}, \tag{14}$$

respectively, where $\dot{\eta}$ and $\dot{\xi}$ represent the total derivatives

$$\dot{\eta} = \frac{\partial \eta}{\partial t} + \dot{x} \frac{\partial \eta}{\partial x} \qquad \dot{\xi} = \frac{\partial \xi}{\partial t} + \dot{x} \frac{\partial \xi}{\partial x}. \tag{15}$$

If ξ and η can be chosen so that the application of the symbol of the group (14) to the Lagrangian $L(x, \dot{x}, t)$ gives

$$G_{(1)a} L = \dot{f} - \dot{\xi} L, \tag{16}$$

then the system described by the Lagrangian $L(x, \dot{x}, t)$ has a Noether symmetry associated with the group operator (13). The introduction of $\dot{\xi}$ in Eq. (16) is not artificial since it denotes derivation, with respect to the parameter, of the derivative of the transformed time with respect to the non-transformed time. When t is considered as a variable, $\dot{\xi}$ appears explicitly.

By means of the following relations

$$\xi \frac{\partial L}{\partial t} = \left(\frac{\partial L}{\partial t} - \frac{\partial L}{\partial x}\dot{x} - \frac{\partial L}{\partial \dot{x}}\ddot{x}\right)\xi,$$
$$\frac{\partial L}{\partial \dot{x}} \frac{d\eta}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\eta\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta,$$
$$\frac{\partial L}{\partial \dot{x}} \dot{x} \frac{d\xi}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\dot{x}\xi\right) - \frac{\partial L}{\partial \dot{x}}\ddot{x}\xi - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x}\xi, \tag{17}$$

Eq. (16) can be transformed into

$$\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right](\eta - \dot{x}\xi) = \frac{d}{dt}\left[(\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f\right]. \tag{18}$$

When the Lagrange equation is satisfied, the left hand side of Eq. (18) vanishes and it follows immediately that

$$\phi = (\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f \tag{19}$$

is a constant of motion.

Equation (19) represents an invariant whose physical meaning has direct dependence with the selection of the parameters of the equation of motion and that of the transformation as well. In the following section, we provide an interpretation of this invariant for a special selection of the Lagrangian and the functions ξ and η.

3. Lorentz infinitesimal transformations and the interpretation of the invariant in classical mechanics

Let v_n be the four-dimensional space-time with coordinates x^j, and f^j the Lorentz transformations $L_i^j(v(t))$, so that Eq. (1) gives

$$\bar{x}^j = L_i^j(\bar{v}(t))x^i. \tag{20}$$

In this expression, the velocity $\bar{v}$ is considered to be a function of the parameter t, such that for $t = 0$, $\bar{v} = 0$. Using Eq. (20), one can calculate the variation with respect to t of any function of the velocities and coordinates $F(x^j(v^a))$, which can be written as

$$\frac{\delta F}{\delta t}(x^j(v^a)) = e^a x_a F = e^a \xi_a^j \frac{\delta F}{\delta x^j} \tag{21}$$

with

$$\xi_a^j = \left(\frac{\partial L_i^j}{\partial v^a}\right)\Bigg|_{t=0} x^i, \qquad e^a = \left(\frac{\delta v^a}{\delta t}\right)\Bigg|_{t=0}, \tag{22}$$

where e^a are the vectors of the basis and the ξ_a^j are the components of the infinitesimal Lorentz transformation. Then, the components of the corresponding linear operator can be written as

$$A_1 = \left(\frac{x^2}{c^2} \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^2} \right)$$

$$A_2 = \left(\frac{x^3}{c^2} \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^3} \right)$$

$$A_3 = \left(\frac{x^4}{c^2} \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^4} \right)$$

so that

$$A = A_1 e^1 + A_2 e^2 + A_3 e^3 = \left[\frac{\bar{r} \cdot \bar{e}}{c^2} \frac{\partial}{\partial t} - t \bar{e} \cdot \bar{\nabla} \right] \qquad (23)$$

is the generator of the Lorentz transformation group, and the commutators $[A_1, A_2]$, $[A_1, A_3]$ and $[A_2, A_3]$ provide the 3-dimensional rotating group

$$\bar{B} = \frac{\bar{r}}{c^2} \times \bar{\nabla}. \qquad (24)$$

With these concepts, it is possible to give a physical interpretation of the invariant ϕ [Eq. (19)] for a special case. Let us consider a harmonic oscillator with Lagrangian

$$L = \frac{1}{2} \left[\dot{x}^2 - \omega^2 x^2 \right]; \qquad (25)$$

comparing Eqs. (13) and (23), we can establish the identities $\xi = x/c^2$ and $\eta = -t$, so that with the aid of Eq. (25), expression (19) yields

$$\phi = \frac{x}{m_0 c^2} \left[\frac{m_0 \dot{x}^2}{2} + \frac{k x^2}{2} + m_0 c^2 + m_0 \frac{\ddot{x} x}{4} \right], \qquad (26)$$

which means that the total energy is conserved at every point x. The quantity in parentheses contains all the forms of energy of the system: kinetic, potential and rest energy, as well as the work done by the system, which is obtained when the function f is determined. In our case, we have taken the value $f = (1/4)\ddot{\xi} x^2$, [4] and $\omega = \sqrt{k/m}$.

On the other hand, using (22), Eq. (16) can also be written as

$$\left(\frac{\partial L_i^j}{\partial v^a} \right) \Big|_{t=0} x^i \left(\frac{\partial L}{\partial x^j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^j} \right) + \left(\frac{\partial L_i^j}{\partial v^a} \right) \Big|_{t=0} \frac{d}{dt} \left(x^i \frac{\partial L}{\partial \dot{x}^j} \right) = 0, \qquad (27)$$

whose first term vanishes, because of Hamilton's variational principle, so it reduces to

$$\frac{d}{dt} \left[\left(\frac{\partial L_i^j}{\partial v^a} \right) \Big|_{t=0} x^i \frac{\partial L}{\partial \dot{x}^j} \right] = 0. \qquad (28)$$

With the aid of (22), the identification of ξ and η, and keeping in mind the fact that $\partial L / \partial \dot{x}^j = p_j$, Eq. (28) states the invariance under Lorentz infinitesimal transformations of $(x^i - v^i t)$.

4. Concluding remarks

In this paper, we have obtained an alternative derivation of the Noether invariants. Besides, using the infinitesimal Lorentz transformations (23) we have given a physical interpretation of the Noether invariant for the special case of the harmonic oscillator characterized by the Lagrangian (25). This result generalizes that obtained by Ray *et al.* [11] for the time translational invariance of isolated systems.

The physical interpretation of the invariants for different identifications of the parameters, which yields conservation laws as well, is still an open problem and an extension of Eq. (10) to three and four dimensions is needed, in order to analyze two and three dimensional motions. In this case, Eq. (24) will play an important role, because it provides the conservation law of angular momentum.

Acknowledgments

We wish to thank the CONACYT and DAAD for financial support.

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