The elastic rod

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The form of an elastic rod in equilibrium subject to a buckling by the action of two opposite forces at its ends is explicitly calculated and drawn. The full expression for the radius of curvature in the equation of the beam is considered. It is known that the differential equation describing the form of the rod, written in terms of the arc length and the angle that forms the tangent line to the curve with the horizontal axis of coordinates, is exactly the same one finds in describing the dynamics of great amplitude oscillations of a simple pendulum. This equation is solved exactly in terms of Jacobi's elliptic functions. The solutions are drawn by using in iterated form the addition formulas of those functions. Useful relations among the physical constants of the system and the geometric parameters of the rod are also obtained.

Keywords: Elastic rod; Jacobian functions; iterated drawing

Se calcula explícitamente y se dibuja la forma que toma el pandeo de una varilla elástica sujeta a la acción de dos fuerzas opuestas en sus extremos. Se considera la expresión completa del radio de curvatura en la ecuación de la vigueta. Se sabe que la ecuación diferencial que describe la forma de la varilla elástica, escrita en función de la longitud de arco y del ángulo que forma la línea tangente a la curva con el eje horizontal es exactamente la misma que se encuentra en la descripción de la dinámica de grandes oscilaciones del péndulo simple. Dicha ecuación se resuelve en términos de funciones elípticas de Jacobi. Las soluciones se dibujan mediante el uso iterado de las fórmulas de adición de esas funciones. Se encuentran también relaciones útiles entre las constantes físicas del problema y los parámetros geométricos de la varilla.

Descriptores: Varilla elástica; funciones jacobianas.

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1. Introduction

The problem of determining the bending form of an elastic rod in equilibrium, submitted to the action of two opposite forces at the ends of the rod, has been considered by several authors, such as Feynman [1], and Landau and Lifshitz [2]. Feynman writes the dynamic equation, without solving it, although comments he knows how to find the solutions numerically, he says "The solutions can also be expressed in terms of some functions, called the *Jacobian elliptic functions*, that someone else has already completed" [1]. Landau and Lifshitz reduce the problem to quadratures, get a first integration, and declare that the form of the rod can be obtained in terms of elliptic functions [2].

We compute explicitly the form of the rod, assuming in order to write the equation of the beam—the same as the previous authors—that its length is large compared with its cross section, but we assume the complete expression for the radius of curvature, that is only approximated in many books of engineering (see for example Ref. 3).

We describe the geometric form of the rod in terms of two coordinates x and y. Along the rod the arc length is defined by

$$ds^2 = dx^2 + dy^2 = dx^2\left(1 + y'(x)^2\right) , \tag{1}$$

where we assume at first the form of the bending rod is described by $y(x)$ and $y'(x)$ denotes the derivative of $y(x)$ with respect to x. In the following we use more frequently the coordinates x and y as functions of the arc length. The derivatives of these functions are related to the $y'(x)$. By using equation (1) we have

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + y'(x)^2}} . \tag{2}$$

Use of the chain rule produces

$$\frac{dy}{ds} = \frac{dy(x)}{dx}\frac{dx}{ds} = \frac{y'(x)}{\sqrt{1 + y'(x)^2}} . \tag{3}$$

The expressions (2) and (3) have been introduced since it is well known that the derivative $y'(x)$ is equal to the tangent function of the angle forming the geometric tangent to the line $y(x)$ with the horizontal axis of coordinates. Let us to call θ to that angle, namely

$$y'(x) = \tan\theta . \tag{4}$$

Substitution of this property in the previous equations is used to express the derivatives of the coordinates with respect to the arc length in terms of angle θ

$$\frac{dx}{ds} = \cos\theta\,, \quad \frac{dy}{ds} = \sin\theta\,. \tag{5}$$

Equation (1) tell us that these two quantities are the components of the unit tangent vector to the curve representing the bending rod

$$\mathbf{t}= \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} = \frac{1}{\sqrt{1+y'(x)^2}} \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{6}$$

This mathematical preamble guides us to compute the radius of curvature R of the rod that comes from the Frenet equation on any curve

$$\frac{d\mathbf{t}}{ds} = \frac{1}{R}\mathbf{n}\,, \tag{7}$$

where the inverse of the radius of curvature is the magnitude of this vector and **n** is a unit vector orthogonal to vector $\mathbf{t}$, called the normal vector to the curve.

We use this expression to compute the radius of curvature but it is useful to keep both expressions for the vector $\mathbf{t}$ in (6). Using again the chain rule to derive the middle form of the tangent vector it results

$$\frac{d\mathbf{t}}{ds} = \frac{-y''(x)}{[1+y'(x)^2]^2} \begin{pmatrix} y'(x) \\ -1 \end{pmatrix} = -\frac{d\theta}{ds} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}. \tag{8}$$

Therefore

$$\frac{1}{R} = \frac{-y''(x)}{[1+y'(x)^2]^{\frac{3}{2}}} = -\frac{d\theta}{ds}. \tag{9}$$

The rod is placed in horizontal position, along the x axis, before the forces are applied on it. Then, two equal forces are applied on both ends of the rod in the direction of the x axis, and the rod bends symmetrically, deflecting an amplitude $y(x)$ from the equilibrium position. This is zero at both ends of the rod. With this convention R is positive because the second derivative $y''(x)$ is negative; and also because the angle is maximum at the ends of the rod, decreases to zero at the center of the rod , and decreases with negative values to the minimum value at the right end of the rod. In any case the derivative of θ with respect to s is negative, that explain the minus sign in the previous equation to deduce a positive radius of curvature.

Note that some authors use the approximation $-y''(x)$ for the inverse of the radius of curvature that is only valid for very small deflections. This simplification is not used here.

The normal vector becomes

$$\mathbf{n} = \frac{1}{\sqrt{1+y'(x)^2}} \begin{pmatrix} y'(x) \\ -1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}. \tag{10}$$

The expression for the flexion moment M that acts on any point of the rod and the bending it produces at the same point is given by

$$M = \frac{YI}{R}\,, \tag{11}$$

where Y is the Young's modulus, and I is the inertia moment on the section area of the rod, with the origin placed at the center of mass of the cross section

$$I = \int y^2 dA$$

The torque on the point of height $y(x)$ is equal to the product of the force F and this height, that should be in equilibrium when it is equal to the bending moment

$$Fy(x) = \frac{YI}{R} = -YI\frac{d\theta}{ds}\,, \tag{12}$$

where we have used the equation (9) for the curvature $1/R$ of the rod. This equation will be used in the sequel to write $y(x)$ from the derivative of θ.

Deriving both sides of equation (12)

$$\frac{d^2\theta}{ds^2} = -\frac{F}{YI}\frac{dy}{ds}\,, \tag{13}$$

where we substitute the second of equations (5) to get

$$\frac{d^2\theta}{ds^2} = -\frac{F}{YI}\sin\theta\,. \tag{14}$$

We introduce the non-dimensional variable defined as

$$\tau = s\sqrt{\frac{F}{YI}}\,, \tag{15}$$

to write equation (14) in the form

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta\,. \tag{16}$$

This is the same equation that is found in describing the motion of a simple pendulum when it is written in non-dimensional form.

2. Parametric equations of the elastic curve. The Jacobi's elliptic functions

For large deformations it is necessary to solve the previous equation through the Jacobi functions. We develop in this section the basic theory to use these functions.

Multiplying both members of equation (16) by $d\theta/d\tau$ and integrating with respect to τ we obtain the equivalent to the energy conservation in the pendulum motion

$$\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 - \cos\theta = \text{const.} \tag{17}$$

It is convenient to express this equation in terms of half the angle as

$$\frac{1}{2}\frac{d\theta}{d\tau} = \pm\sqrt{k^2 - \sin^2\frac{\theta}{2}}\,, \tag{18}$$

where $k^2 > 0$ is the integration constant. In order to see clearly the relation to elliptic integrals as are generally presented in many books we introduce the variable ϕ which is defined by

$$\sin\phi = \frac{1}{k}\sin\frac{\theta}{2}\,, \tag{19}$$

that transform the differential equation (18) into

$$\frac{d\phi}{d\tau} = \pm\sqrt{1 - k^2\sin^2\phi}$$

which is integrated

$$\tau = \int \frac{d\phi}{\pm\sqrt{1-k^2\sin^2\phi}}\,.$$

This integral has the standard form of an elliptic integral of first class. The inverse of this integral defines the Jacobi function

$$\mathrm{sn}(\tau, k) = \sin\phi$$

or in terms of angle θ

$$\sin\frac{\theta}{2} = -k\,\mathrm{sn}(\tau,k)\,. \tag{20}$$

Minus sign was selected because θ has the opposite sign to s or τ.

From this equation and the identity of the jacobian functions

$$k^2\,\mathrm{sn}^2(\tau,k) + \mathrm{dn}^2(\tau,k) = 1$$

we have

$$\cos\frac{\theta}{2} = \mathrm{dn}(\tau,k)\,. \tag{21}$$

Sustitution of (20) in (18) and using the identity of the Jacobi functions

$$\mathrm{sn}^2(\tau,k) + \mathrm{cn}^2(\tau,k) = 1$$

it follows

$$\frac{1}{2}\frac{d\theta}{d\tau} = -k\sqrt{1 - \frac{1}{k^2}\sin^2\frac{\theta}{2}} = -k\,\mathrm{cn}(\tau,k)\,. \tag{22}$$

The minus sign is explained by the opposite sign between θ and τ. From this follows that the coordinate y in terms of the τ parameter as it was found in equation (12) is

$$y(x) = -\frac{YI}{F}\frac{d\theta}{ds}\,. \tag{23}$$

Therefore

$$y(\tau) = \sqrt{\frac{YI}{F}}2k\,\mathrm{cn}(\tau,k)\,. \tag{24}$$

To find the coordinate x as a function of τ we start from the first of the equations (5), and using the trigonometric identities, we found

$$\frac{dx}{ds} = \cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\,\mathrm{dn}^2(\tau,k) - 1\,, \tag{25}$$

where the solution (21) was used.

Integrating this equation with respect to τ we have

$$x(\tau) = \sqrt{\frac{YI}{F}}\left[2\int \mathrm{dn}^2(\tau,k)d\tau - \tau\right]. \tag{26}$$

This is an elliptic integral of second class defined by

$$\int_0^\tau \mathrm{dn}^2(w,k)dw = E(\tau,k)\,. \tag{27}$$

Therefore the horizontal coordinate (26) is expressed in the next form

$$x(\tau) = \sqrt{\frac{YI}{F}}[2E(\tau,k) - \tau]\,. \tag{28}$$

3. Geometric relations of the elastic curve

The maximum value of the coordinate y occurs [5] when the function cn(τ,k) takes the value 1, corresponding to the zero value of τ, then according to Eq. (24) this amplitude h is

$$h = 2k\sqrt{\frac{YI}{F}}\,, \tag{29}$$

which allows us to write the constant k as a function of the physical and geometric constants of the elastic

$$k = \frac{h}{2}\sqrt{\frac{F}{YI}}\,. \tag{30}$$

The coordinate y is zero at the ends of the rod, when the arc length is equal to one half the total length L of the rod. This occurs when function $\mathrm{cn}(\tau,k)$ becomes zero, that happens [5] when τ is equal to $K(k)$, the function known as the complete elliptic integral of first class. Using (15) we obtain

$$K(k) = \frac{L}{2}\sqrt{\frac{F}{YI}}\,. \tag{31}$$

For this same value of $\tau = K(k)$, coordinate x is equal to one half of the distance b between the two ends of the rod. From Eq. (28) we find the property

$$\frac{b}{2} = \sqrt{\frac{YI}{F}}[2E(k) - K(k)]\,, \tag{32}$$

where $E(k)$ is the complete elliptical integral of second class $E(k) = E(K(k),k)$.

From Eqs. (30) and (31) we found the parameter k as a function of the ratio of the lengths h and L

$$\frac{h}{L} = \frac{k}{K(k)}\,, \tag{33}$$

which shows parameter k depending only on the geometry of the bending rod.

In a similar way, from Eqs. (31) and (32), we obtain also the ratio of the lengths b and L as a function of k

$$\frac{b}{L} = 2\frac{E(k)}{K(k)} - 1\,, \tag{34}$$

that shows the three distances h, b and L are not independent.

Observing these equations we come to the conclusion that the force F, necessary to hold the equilibrium is a function of the geometry represented by two of these lengths; the moment of inertia I, which is purely geometric also, and of the Young modulus Y.

4. The form of the elastic curve. The iterative method

In this section we propose a method to draw the bending rod in equilibrium. The drawing will be written in terms of a length unit such that

$$2\sqrt{\frac{YI}{F}} = 1\,. \tag{35}$$

To proceed we require a discrete or stroboscopic point of view. To attain this objective we focus on the values of the elliptic functions computed at arc length values that are multiples of a small finite length Δs. We use an integer index j as a subindex to denote the functions evaluated at $j\Delta s$

$$x_j = x(j\Delta s)\,, \quad y_j = y(j\Delta s) \tag{36}$$

This couple of coordinates is computed by iteration from the previous j by using the addition theorems [4] for the elliptic functions. For the x coordinate we need

$$E(u+v) = E(u) + E(v) - k^2 \mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u+v)$$

and

$$\mathrm{sn}\,(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - k^2\mathrm{sn}^2 u\,\mathrm{sn}^2 v}\,.$$

The y coordinate requires the addition formula

$$\mathrm{cn}\,(u+v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{1 - k^2\mathrm{sn}^2 u\,\mathrm{sn}^2 v}\,.$$

We obtain the iterations

$$\begin{aligned} x_{j+1} &= x_j + E(\Delta s, k) \\ &\quad - \frac{(k^2 - y_j^2)SCD + S^2 y_j\sqrt{k^2 - y_j^2}\sqrt{1 - k^2 - y_j^2}}{1 - S^2(1 - y_j^2)} \\ &\quad - \frac{\Delta s}{2}\,, \end{aligned} \tag{37}$$

$$y_{j+1} = \frac{C y_j - SD\sqrt{k^2 - y_j^2}\sqrt{1 - k^2 + y_j^2}}{1 - S^2(k^2 - y_j^2)} \tag{38}$$

where we have introduced the following notation

$$S{=}\mathrm{sn}(\Delta s, k), \quad C{=}\mathrm{cn}(\Delta s, k), \quad D{=}\mathrm{dn}(\Delta s, k). \tag{39}$$

Quantities C and D were computed in terms of S according to

$$C = \sqrt{1 - S^2}\,, \quad D = \sqrt{1 - k^2 S^2} \tag{40}$$

and for small Δs we use also the first order approximations

$$S = \mathrm{sn}\,(\Delta s, k) = \Delta s\,, \quad E(\Delta s, k) = \Delta s \tag{41}$$

Three drawings are here presented for the values of k 0.3, 0.5, and 0.95, that were produced employing this algorithm (Fig. 1).

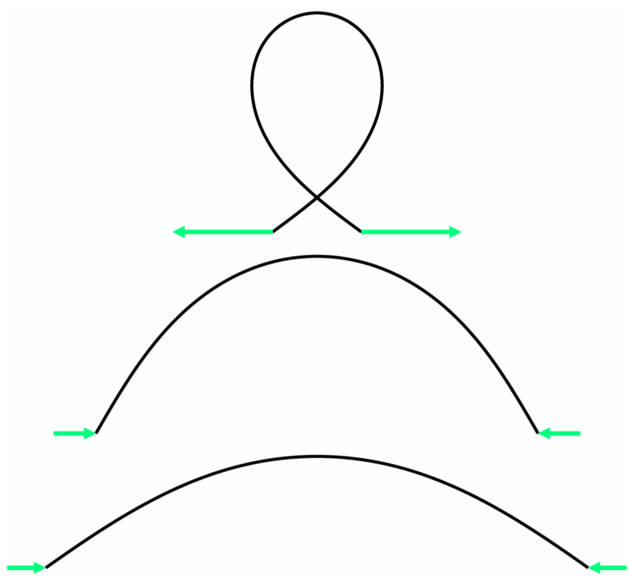

FIGURE 1. Drawings for values 0.3, 0.5, 0.95 of the constant k. The force is represented by the square of the function $K(k)$.

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