Simple deductions of the integral representations of the relativistic Faraday and Ampère-Maxwell laws and the relativistic transformation laws of the electromagnetic field

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By using simple concepts of special relativity and the differential representations of the Faraday and Ampère-Maxwell laws, we deduce their Gelman-Monsivais integral representation. The relativistic transformation laws of the electromagnetic field are also obtained without using tensorial analysis or covariant concepts.

Keywords: Special relativity; Maxwell equations; integral representation.

Usando conceptos simples de la relatividad especial y las representaciones diferenciales de las ecuaciones de Faraday y Ampère-Maxwell, se deducen las representaciones integrales de Gelman-Monsivais de estas últimas. Se obtienen al mismo tiempo las leyes de las transformaciones relativistas del campo electromagnético sin utilizar análisis tensorial o conceptos covariantes.

Descriptores: Relatividad especial; ecuaciones de Maxwell; representación integral.

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1. Introduction

It is very common to try to relate electromagnetism to special relativity. However, both theories require a high level of mathematical concepts to be related and it is very difficult to accomplish this in elementary courses. A counter-example of this consists in connecting the Faraday and Ampère-Maxwell laws with the relativistic transformation laws of the electromagnetic field by just using vectorial analysis and some elementary notions of special relativity. Indeed, in 1989, Ares de Parga and Rosales [1] have noted that Faraday's induction law required extension in order to apply it to situations in which circuits undergo deformation as they move and in which the motion may be relativistic. In 1990, Gelman [3] supplied a new integral expression for Faraday's law valid in a reference frame in which all parts of a circuit move at relativistic velocities. Recently, within the same order of ideas, Monsivais [4] obtained a new integral expression for the Ampère-Maxwell law. In both articles, the relativistic electromagnetic field transformations were used in order to obtain the integral representations. The purpose of this article is to obtain the Gelman-Monsivais integral representations [3, 4]. We also deduce the relativistic transformation laws of the electromagnetic field by using simple concepts of relativity and vectorial analysis.

The paper is organized as follows. By using the Faraday and Ampère-Maxwell laws in their differential representations and the Lorentz contraction, in Secs. 2 and 3, we will partially deduce the Gelman [3] integral representation of Faraday's law and the Monsivais [4] integral representation of Ampère-Maxwell's law. Section 4 will be advocated to deduce the exact integral representations of both laws and we will obtain the relativistic electromagnetic field transformations by using plane waves. Some final conclusions are presented in Sec. 5.

2. Faraday's law

Faraday's law is normally exposed in a partial way. Indeed, for simple expositions of electromagnetism, Faraday's law is presented as follows. Firstly, the electromotive force ε is defined as: (Gaussian units are required)

$$\varepsilon = \oint_C \mathbf{E} \cdot d\mathbf{l}, \tag{1}$$

where C represents a closed circuit and $\mathbf{E}$ is the electric field. Secondly, the magnetic flux ϕ through the open surface S whose contour is C, is defined as:

$$\phi = \int_S \mathbf{B} \cdot d\mathbf{S}. \tag{2}$$

Finally, the integral representation of Faraday's law for a circuit at rest is:

$$\varepsilon = -\frac{1}{c}\frac{d\phi}{dt}. \tag{3}$$

Since the law is valid for any circuit at rest, by using Stokes's theorem it can be expressed in a differential representation as:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{4}$$

Equation (4) represents the differential Faraday's law. Eq. (3) represents the integral form of the Faraday's law. However, it is restricted to consideration of static surfaces and contours. If we want to generalize this law to time-dependent surfaces and contours, we need to make some changes in the definitions of these quantities in order to maintain invariant the integral representation of the Faraday's law. Moreover, some authors [5] obtained an inexact relativistic transformation law of the electromagnetic field by changing the regular definition of the electromotive force. We shall propose an integral representation of the Faraday's law that permits us to obtain an exact relativistic transformation law of the electromagnetic field. Therefore, we firstly propose the electromotive force as follows:

$$\varepsilon = \oint\limits_{C^*(t)} G\mathbf{E}' \cdot d\mathbf{l}', \tag{5}$$

where G is an unknown function that we need to determinate and $\mathbf{E}'$and $d\mathbf{l}'$ represent the electric field and the differential vector length of the circuit, both measured in the proper frame. Since the circuit is moving or suffering a deformation, there exists a reference frame where the differential vector length must be at rest and in this frame which we shall call the proper frame, $\mathbf{E}'$ and $d\mathbf{l}'$ must be measured with the particularity that the differential time in the laboratory frame must vanish: that is, $dt = 0$. The contour C in the integral has been substituted by $C^*(t)$ in order to indicate the motion of the circuit and to distinguish it from a regular integral. Indeed, it must to be noted that the integral represented by Eq. (5) should be interpreted as the limit of an infinite sum of quantities that has to be measured in each step [4]; that is,

$$\varepsilon = \lim_{\Delta\mathbf{l}' \to 0} \sum_{\Delta\mathbf{l}'} G\mathbf{E}' \cdot \Delta\mathbf{l}', \tag{6}$$

where the sum is found by dividing the contour into n sections, Δl. Each section possesses a velocity, and as a consequence of this we can define n reference frames where each section of the contour is at rest. In each frame, the section of the contour measures $\Delta l'$. To each $\Delta l'$ corresponds a vector $\Delta\mathbf{l}'$. This means that G must be a function which depends on the relative motion between the frame where the differential length of the circuit $\Delta\mathbf{l}'$ is at rest and the laboratory frame; that is, G is a function of the velocity of the circuit, $G = G(\mathbf{v})$.

With these considerations, Faraday's law may be expressed as

$$\varepsilon = \oint\limits_{C^*(t)} G\mathbf{E}' \cdot d\mathbf{l}' = -\frac{1}{c}\frac{d}{dt} \int\limits_{S(t)} \mathbf{B} \cdot d\mathbf{S}, \tag{7}$$

where $C^*(t)$ represents the special contour defined in Eq. (5) associated with $C(t)$, the contour of the moving surface, $S(t)$.

Let us now consider a closed circuit which is moving at a speed $\mathbf{v}$ with respect to the laboratory frame. By using identity [2],

$$\frac{d}{dt}\int\limits_{S(t)} \mathbf{B} \cdot d\mathbf{S} = \left[\int\limits_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int\limits_{S(t)} (\mathbf{v} \cdot \nabla)\mathbf{B} \cdot d\mathbf{S} \right] \tag{8}$$

we arrive at the following:

$$\oint\limits_{C'^*(t)} G\mathbf{E}' \cdot d\mathbf{l}' = -\frac{1}{c} \left[\int\limits_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int\limits_{S(t)} (\mathbf{v} \cdot \nabla)\mathbf{B} \cdot d\mathbf{S} \right]. \tag{9}$$

It must be noted that, while the magnetic field is measured in the laboratory frame for both right hand terms of Eq. (9), the electric field, in the left hand term, is measured in the moving frames. The magnetic integral in Eq. (9) shows an extra term due to the motion of the circuit. By using the following vector identity,

$$(\mathbf{v} \cdot \nabla)\mathbf{B} = \nabla \times (\mathbf{B} \times \mathbf{v}) + \mathbf{v}(\nabla \cdot \mathbf{B}), \tag{10}$$

the magnetic Gauss law ($\nabla.\mathbf{B} = 0$) and Eq. (9), we arrive at

$$\oint\limits_{C^*(t)} G\mathbf{E}' \cdot d\mathbf{l}' = -\frac{1}{c} \left[\int\limits_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int\limits_{S(t)} \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{S} \right]. \tag{11}$$

By using the differential representation of the Faraday's law, we substitute $(1/c)(\partial\mathbf{B}/\partial t)$ by $\nabla \times E$ and we get

$$\oint\limits_{C^*(t)} G\mathbf{E}' \cdot d\mathbf{l}' = \int\limits_{S(t)} \nabla \times \mathbf{E} \cdot d\mathbf{S} - \frac{1}{c} \int\limits_{S(t)} \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{S}$$

Then, the application of Stokes's theorem gives us

$$\oint\limits_{C^*(t)} G\mathbf{E}' \cdot d\mathbf{l}' = \oint\limits_{C(t)} \mathbf{E} \cdot d\mathbf{l} - \frac{1}{c} \oint\limits_{C(t)} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l}. \tag{12}$$

Since Eq. (12) is valid for any circuit, and considering that the circuit is moving at a constant velocity $\mathbf{v}$ in the direction of the x-axis, that is, $\mathbf{v} = v\widehat{\mathbf{i}}$, we can assert that

$$GE'_{x'}dx' = E_x dx, \tag{13}$$

$$GE'_{y'}dy' = (E_y - \frac{1}{c}B_z v)dy$$

and

$$GE'_{z'}dz' = (E_z + \frac{1}{c}B_y v)dz.$$

On the other hand, the Lorentz contraction is obtained from the general transformation laws of the coordinates; that is:

$$dx' = \gamma(dx - vdt) \quad dy' = dy \quad dz' = dz, \tag{14}$$

where $\gamma = 1/\sqrt{1 - (\mathbf{V}^2/C^2)}$. Since we have mentioned that $\mathbf{dl}'$ must be measured for $dt = 0$, we arrive at

$$dx' = \gamma dx, \quad dy' = dy \text{ and } \quad dz' = dz. \tag{15}$$

Therefore, we finally obtain

$$GE'_x = \frac{1}{\gamma}E_x, \quad GE'_y = \left[E_y - \frac{v}{c}B_z\right] \tag{16}$$

and

$$GE'_z = \left[E_z + \frac{v}{c}B_y\right].$$

Because we do not know the value of the function G, we are not able to identify which is the exact integral form of the Faraday's law and the relativistic transformation law of the electric field. In Sec. 3, we will analyze Ampère-Maxwell's law and then we shall be able to give an integral representation of the Faraday and Ampère-Maxwell laws and the relativistic transformation laws of the electromagnetic field.

3. Maxwell-Ampère's Law

Maxwell-Ampère's law is expressed in its differential representation as:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{17}$$

For the case when the contour and the surface are at rest, we can be sure that:

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \frac{1}{c}\int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S} + \frac{4\pi}{c}\int_S \mathbf{J}\cdot d\mathbf{S}. \tag{18}$$

By means of the same concepts described in Sect. 2, we can propose the integral representation of the law as:

$$\oint_{C^*(t)} G_B \mathbf{B}'\cdot d\mathbf{l}' = \frac{1}{c}\frac{d}{dt}\int_{S(t)} \mathbf{E}\cdot d\mathbf{S} + \frac{4\pi}{c}\int_{S(t)} \mathbf{J_r}\cdot d\mathbf{S}, \tag{19}$$

where $\mathbf{B}'$ and $d\mathbf{l}'$ represent the magnetic field and the differential length vector, respectively, in the same way we have considered $\mathbf{E}'$ and $d\mathbf{l}'$ in Eq.(7). We are obliged to suppose that $\mathbf{J}_r$ does not represent the regular current vector $\mathbf{J}$; that is $\mathbf{J}_r \neq \mathbf{J}$. But taking into account Eq. (18), $\mathbf{J}_r$ must coincide with $\mathbf{J}$ when the surface is at rest. By developing Eq. (19) with the same identity used in Sect. 2 [2], Eq. (8), we arrive at

$$\oint_{C^*(t)} G_B \mathbf{B}'\cdot d\mathbf{l}' = \frac{1}{c}\int_{S(t)} \frac{\partial}{\partial t}\mathbf{E}\cdot d\mathbf{S} + \frac{1}{c}\int_{S(t)} (\mathbf{v}.\nabla)\mathbf{E}\cdot d\mathbf{S} + \frac{4\pi}{c}\int_{S(t)} \mathbf{J_r}\cdot d\mathbf{S}. \tag{20}$$

By using the identity represented by Eq. (10), we get

$$\oint_{C^*(t)} G_B \mathbf{B}'\cdot d\mathbf{l}' = \frac{1}{c}\int_{S(t)} \frac{\partial}{\partial t}\mathbf{E}\cdot d\mathbf{S} + \frac{1}{c}\int_{S(t)} \left[(\nabla\cdot\mathbf{E})\mathbf{v} - \nabla\times(\mathbf{v}\times\mathbf{E})\right]\cdot d\mathbf{S} + \frac{4\pi}{c}\int_{S(t)} \mathbf{J_r}\cdot d\mathbf{S}. \tag{21}$$

By using the electric Gauss law ($\nabla\cdot\mathbf{E} = 4\pi\rho$), the differential form of Ampère-Maxwell's law, Eq. (17), and Stokes's theorem, we get

$$\oint_{C^*(t)} G_B \mathbf{B}'.d\mathbf{l}' = \oint_{C(t)} \mathbf{B}\cdot d\mathbf{l} - \frac{1}{c}\oint_{C(t)} (\mathbf{v}\times\mathbf{E})\cdot d\mathbf{l} + \frac{4\pi}{c}\int_{S(t)} (\mathbf{J_r} + \rho\mathbf{v} - \mathbf{J}).d\mathbf{S}. \tag{22}$$

Since this is valid for any moving contour with a constant velocity in the x-axis direction, $\mathbf{v} = v\widehat{\mathbf{i}}$ and because the transformation of the electric field does not include a dependence in the charge density or charge current, we can conclude that

$$G_B B'_x = \frac{1}{\gamma}B_x$$

$$G_B B'_y = \left[B_y + \frac{v}{c}E_z\right]$$

$$G_B B'_z = \left[B_z - \frac{v}{c}E_y\right] \quad \text{and} \quad \mathbf{J} = \mathbf{J_r} + \rho\mathbf{v}. \tag{23}$$

It should be pointed out that it may be possible to consider that $\mathbf{J} = \rho\mathbf{V}$ in some specific situations, where $\mathbf{V}$ represents the velocity of the charges, and we can assert that $\mathbf{J_r} = \rho(\mathbf{V} - \mathbf{v})$ as is considered by Monsivais [4]. This is not true in general since in some situations we can have $\mathbf{J} \neq \overrightarrow{0}$ and at the same time $\rho = 0$. This happens, for example, when we have a current with positive and negative charges moving in opposite directions. In this situation, the charge density can vanish and the total current fails to do so. Another example of this is when a current travels inside a conductor etc.... Therefore, we prefer to just keep the result as:

$$\mathbf{J}_r = \mathbf{J} - \rho\mathbf{v}. \tag{24}$$

Then we can conclude that the integral representation of Ampère-Maxwell's law may be written in a partial form as:

$$\oint_{C^*(t)} G_B \mathbf{B}'\cdot d\mathbf{l}' = \frac{1}{c}\frac{d}{dt}\int_{S(t)} \mathbf{E}\cdot d\mathbf{S} + \frac{4\pi}{c}\int_{S(t)} (\mathbf{J} - \rho\mathbf{v})\cdot d\mathbf{S}. \tag{25}$$

4. Integral form of the Faraday and Ampère-Maxwell laws and the relativistic electromagnetic field transformations

In order to know the integral form of the Faraday and Ampère-Maxwell laws and the relativistic transformation laws of the electromagnetic field, we just need to know the value of G and G_B. To calculate these functions, we will require other physical aspects of electromagnetism. Indeed, let us consider a linear polarized plane wave moving in the x-axis direction, and let us suppose that the electric field oscillates in the y-axis. Then the magnetic field oscillates in the z-axis. Therefore the wave can be described by

$$\mathbf{E} = E\widehat{\mathbf{j}}\exp(ikx-wt) \quad \text{and} \quad \mathbf{B} = B\widehat{\mathbf{k}}\exp(ikx-wt). \quad (26)$$

By using Eqs. (16) and (23), it is easy to confirm that the electric and magnetic fields measured in a moving frame with respect the laboratory frame and whose velocity is constant in the x-axis, $\mathbf{v} = v\widehat{\mathbf{i}}$, are

$$\mathbf{E}' \text{ is parallel to } \widehat{\mathbf{j}} \quad \text{and} \quad \mathbf{B}' \text{ is parallel to } \widehat{\mathbf{k}}. \quad (27)$$

Therefore, we obtain a linear polarized plane wave in the moving frame, and we know that for a linear polarized plane wave in any reference frame,

$$\mathbf{B}' = \widehat{\mathbf{i}}\times\mathbf{E}'. \quad (28)$$

By using Eq. (16) and (23) in Eq. (28), we have

$$\frac{1}{G_B}\left[\left(1-\frac{v}{c}\right)E\right] = \frac{1}{G}\left[E\left(1-\frac{v}{c}\right)\right] \Rightarrow G = G_B. \quad (29)$$

In order to evaluate G, let us consider an infinite charged line with constant linear charge density λ. Consider a frame at rest with the charge. Applying the electric Gauss law, we obtain a radial electric field,

$$E = 2\frac{\lambda}{r} \qquad \text{and} \qquad B = 0. \quad (30)$$

If we calculate the magnetic field in a reference frame moving along the x-axis with a speed v, and applying Ampère's law, we obtain an azimuthal magnetic field,

$$B' = \frac{2\lambda' v}{cr}, \quad (31)$$

where λ' represents the charge density in the moving frame. Since the total charge is conserved, we can assert that

$$\lambda dl = \lambda' dl' \Rightarrow \lambda' = \gamma\lambda, \quad (32)$$

where we have used $dl' = (1/\gamma)dl$, instead of $dx' = \gamma dx$ as in Eq. (15). The reason is the following: starting from Eq.(14), and considering $dt = 0$, we arrive at $dx' = \gamma dx$. On the other hand, in order to calculate the magnetic field B', it has been necessary to consider the charge current as $\lambda' v$. This means that λ' has been considered at $dt' = 0$. Therefore, in this case, the identity $dl' = (1/\gamma)dl$ is the correct relation to be used in Eq. (32). So we obtain:

$$B' = \gamma\frac{v}{c}E. \quad (33)$$

By comparing Eq. (23) with Eq. (33), we can conclude that

$$G = \frac{1}{\gamma}. \quad (34)$$

Equation (23) has been obtained by assuming that the motion of a circuit was constant along the x-axis. Since the choice of the x-axis was arbitrary, we can immediately generalize for any direction and we invite the reader, by using few geometrical aspects, to show that the general transformations for any electromagnetic field can be expressed as follows:

$$\overrightarrow{E'} = \gamma(\overrightarrow{E} + \overrightarrow{\beta}\times\overrightarrow{B}) - \frac{\gamma^2}{\gamma+1}\overrightarrow{\beta}(\overrightarrow{\beta}.\overrightarrow{E})$$

$$\overrightarrow{B'} = \gamma(\overrightarrow{B} - \overrightarrow{\beta}\times\overrightarrow{E}) - \frac{\gamma^2}{\gamma+1}\overrightarrow{\beta}(\overrightarrow{\beta}.\overrightarrow{B}), \quad (35)$$

where $\overrightarrow{\beta} = \overrightarrow{v}/c$. Then, we can express the integral representation of Faraday's law as:

$$\oint_{C^*(t)} \frac{1}{\gamma}\mathbf{E}'\cdot d\mathbf{l}' = -\frac{1}{c}\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot d\mathbf{S}, \quad (36)$$

and the integral representation of Ampère-Maxwell's law as:

$$\oint_{C^*(t)} \frac{1}{\gamma}\mathbf{B}'\cdot d\mathbf{l}' = \frac{1}{c}\frac{d}{dt}\int_{S(t)} \mathbf{E}\cdot d\mathbf{S} + \frac{4\pi}{c}\int_{S(t)} (\mathbf{J}-\rho\mathbf{v})\cdot d\mathbf{S}. \quad (37)$$

These last two equations coincide with the results obtained by Gelman [3] and Monsivais [4].

5. Conclusion

We have deduced from the differential representations of the Faraday and Ampère-Maxwell laws their corresponding integral representations and at the same time we obtained the relativistic transformation laws of the electromagnetic field as described by Eqs. (35), (36) and (37). The most important aspects of these results are their relativistic characteristics. Although the integral representations have been obtained before by Gelman [3] and later by Monsivais [4] by using both the relativistic transformation laws of the electromagnetic field, in this paper the relativistic transformation laws of the electromagnetic field have been deduced in the process for obtaining the integral representations. Another interesting aspect is that we have used the electric Gauss law ($\nabla\cdot\mathbf{E} = 4\pi\rho$) and the magnetic Gauss law ($\nabla\cdot\mathbf{B} = 0$). Therefore, following Monsivais's ideas [4], we can conclude that the information that we can obtain by using the integral representations must be richer than that obtained just by using the differential representations of the Faraday and Ampère-Maxwell laws.

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