Approximation expressions for the large-angle period of a simple pendulum revisited

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This paper presents the experimental accuracy performance of each of the approximation expressions relative to the exact period for large amplitudes of a simple pendulum in the interval $0^\circ \leq \theta \leq 180^\circ$. The plots of the linearized exact period as a function of linearized formulae were carried out and relative errors in these expressions were investigated. In addition, this paper gives a clear idea how each formula approximates the exact period.

Keywords: Large amplitude; simple pendulum; approximation expressions; exact period; linearization; laboratory experiment.

Este artículo presenta la eficacia sobre la precisión experimental de cada una de las expresiones de aproximaciones con relación al periodo exacto de grandes amplitudes de un péndulo simple en el intervalo $0^\circ \leq \theta \leq 180^\circ$. Las curvas del periodo exacto linealizado en función de las ecuaciones de aproximaciones linealizadas han sido desarrolladas y los errores relativos de estas expresiones han sido estudiados. Además, este artículo da una idea clara de cómo cada fórmula se acerca del periodo exacto.

Descriptores: Grandes amplitudes; péndulo simple; expresiones de aproximaciones; periodo exacto; experiencia de laboratorio.

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1. Introduction

The pendulum is of great importance in science and education. It was used by Galileo as an accurate and simple timekeeper, and by Newton to prove the equivalence between gravitational and inertia mass, and until recently the simple pendulum was a pedagogical instrument used by teachers and students to measure the local acceleration of gravity and to study linear and nonlinear oscillations. In the limiting case of smallest amplitudes, for which the approximation $\sin\theta \approx \theta$ (θ in radians), the periodic oscillation for a simple pendulum is approximately harmonic, and is described by an expression given in most undergraduate textbooks [1]. However, for large amplitudes the periodic oscillations are described by a complicated formula which involves an elliptic integral or its expansion series or numerical analysis [2,3]. To help physics students and teachers resolve the problems of a simple pendulum for a large displacement angle, several works on approximation schemes have been developed by researchers to discuss large-angle oscillations [4-7].

To investigate the accuracy performance of each approximation expression in the text, we have introduced the technique of linearization, by which the exact period expression and the approximation formulae were put into a linear form and the linear form of the exact period was then plotted as a function of these linear formulae. The linearization method makes possible the validation and verification of the accuracy of the approximation expressions compared with the exact period.

2. Exact period expression

The nonlinear oscillations of a simple pendulum are described by the following differential equation[2]

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \tag{1}$$

where g is the acceleration of gravity, L is the length of the pendulum, and θ is the amplitude of the angular displacement. The oscillations of the pendulum are subject to the initial conditions

$$\theta(0) = \theta_0 \quad \text{and} \quad \left(\frac{d\theta}{dt}\right)_{t=0} = 0,$$

where θ_o is the amplitude of oscillation. The system oscillates between symmetric limits [- θ_o, + θ_o]. The periodic solution θ *(t)* to Eq. (1) depends on the amplitude θ_o. For θ = 0, we have T = T_0 and the period, T_0, of a linear oscillation of a simple pendulum is given by $T_0 = 2\pi/\omega$, where $\omega = \sqrt{g/L}$.

The motion of the nonlinear pendulum depends on amplitude θ of the angle of displacement, and the relationship between the amplitude and the corresponding period T describes the dynamic of this nonlinear motion.

The solution to Eq. (1) is expressed in terms of elliptic integrals, and could be solved either numerically or by using approximations. In most cases, there are no analytical solutions to this differential equation.

The simplest form of approximation is when the oscillations of a pendulum are small enough that sin θ can be approximated by θ, and Eq. (1) for nonlinear motion reduces to the linear equation

$\sin\theta \approx \theta$,for small θ

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{2}$$

Then the period of oscillation is given by

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \tag{3}$$

In this case, the period of oscillation depends on the length of the pendulum and the acceleration due to gravity, and is independent of the amplitude θ. However, the exact solution to the period of oscillations is given by [8]

$$T_{ex} \approx T_0 \frac{2}{\pi} K(k), \tag{4}$$

where the function K(k), $k = \sin^2(\theta_0/2)$, is known mathematically as an elliptic integral of the first kind. The values of elliptic integrals are tabulated for numerous values of k which can be found in mathematical tables [9]. The power expansion of Eq. (4) provides the series result [10]

$$T_{ex} \approx T_0 \left(1 + \frac{1}{4}k^{+}\frac{9}{64}k^2 + \frac{25}{256}k^3 + \ldots\right). \tag{5}$$

Further expanding the k =$\sin^2(\theta_0/2)$ terms in Eq. (5) in the power series gives us an expression for the exact period of the form

$$T_{ex} \approx T_0 \left(1 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4 + \frac{173}{737280}\theta^6 + \frac{22931}{1321205760}\theta^8 + \ldots\right). \tag{6}$$

3. Approximate formulae and validation

The approximation expressions for large-angle amplitude of a simple pendulum that we have investigated are

Bernoulli's expression[7]

$$T_B = T_0\left(1 + \frac{\theta^2}{16}\right) \tag{7}$$

Molina's expression[6]

$$T_M = T_0\left(\frac{\sin(\theta)}{\theta}\right)^{-3/8} \tag{8}$$

Kidd and Fogg's expression[5]

$$T_{KF} = T_0\left(\cos(\theta/2)\right)^{-0.5} \tag{9}$$

Parwani's expression[4]

$$T_P = T_0\left(\frac{\sin\left(\sqrt{3}\frac{\theta}{2}\right)}{\left(\sqrt{3}\frac{\theta}{2}\right)}\right)^{-0.5} \tag{10}$$

Lima and Arun's expression[7]

$$T_{log} = T_o\left(\frac{\ln\left(\cos\left(\frac{\theta}{2}\right)\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right) \tag{11}$$

To compare the performance of each approximation formula with the exact period for a large amplitude in the interval $0^\circ \leq \theta \leq 180^\circ$, we have proceeded to the linearization of the exact period found by the power expansion series and the approximation expressions. The linearizations of these equations are

$$\ln T_{ex} \approx \ln T_0 + b\ln\left(1 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4 + \frac{173}{737280}\theta^6 + \frac{22931}{1321205760}\theta^8 + \ldots\right) \tag{12}$$

$$\ln T_B = \ln T_0 + b\ln\left(1 + \frac{\theta^2}{16}\right) \tag{13}$$

$$\ln T_M = \ln T_0 + b\ln\left(\frac{\sin(\theta)}{\theta}\right)^{-3/8} \tag{14}$$

$$\ln T_{KF} = \ln T_0 + b\ln\left(\cos(\theta/2)\right)^{-0.5} \tag{15}$$

$$\ln T_P = \ln T_0 + b\ln\left(\frac{\sin\left(\sqrt{3}\frac{\theta}{2}\right)}{\left(\sqrt{3}\frac{\theta}{2}\right)}\right)^{-0.5} \tag{16}$$

$$\ln T_{\log} = \ln T_0 + b\ln\left(\frac{\ln\left(\cos\left(\frac{\theta}{2}\right)\right)}{1-\cos\left(\frac{\theta}{2}\right)}\right) \tag{17}$$

To validate these approximation expressions and to examine their accuracy compared to the exact period oscillation, the log values of these equations were determined, and graphs of these logs were plotted as a function of the log of the exact period. The curves of these plots represent straight lines of the form:

$\ln T_{ex} = \ln T_0 + b\ln\left(\text{Approximation expressions}\right)$, this corresponds to $y = bx + a$,

where $\ln T_0$ represents the intercept and b is the slope of the straight line. The *best* values of intercepts and slopes for the approximation expressions should be *zero* and *one*, respectively.

The equations obtained for straight lines representing the linearization of approximation expressions with their (R-squared) correlation coefficients in the ranges $0^\circ \leq \theta \leq 90^\circ$ and $0^\circ \leq \theta \leq 180^\circ$ are given in Table I. Two examples of the linear regression fittings for these approximations are presented to the reader. The first example was Eq. (25), Fig. 1, and the second one, Eq. (27), Fig. 2.

TABLE I. Equations for the straight lines with their R-squared in the ranges $0^\circ \leq \theta \leq 90^\circ$ and $0^\circ \leq \theta \leq 180^\circ$.

Eq. No	Equations	R-squared (%)	Angular amplitude Ranges (degrees)
18	ln T_{ex} =0.0022 + 0.8759 ln T_B	99.91	$0^\circ \leq \theta \leq 90^\circ$
19	ln T_{ex} =0.0364 + 0.5283 ln T_B	95.97	$0^\circ \leq \theta \leq 180^\circ$
20	ln T_{ex} =-0.0002 + 1.018 ln T_M	100	$0^\circ \leq \theta \leq 90^\circ$
21	ln T_{ex}=-0.1003 + 1.6795 ln T_M	78.40	$0^\circ \leq \theta \leq 180^\circ$
22	ln T_{ex}=-0.0007 + 1.0393 ln T_{KF}	99.99	$0^\circ \leq \theta \leq 90^\circ$
23	ln T_{ex}=0.1423 + 1.9968 ln T_{KF}	73.65	$0^\circ \leq \theta \leq 180^\circ$
24	ln T_{ex}=-0.0002 + 1.0123 ln T_{log}	100	$0^\circ \leq \theta \leq 90^\circ$
25	ln T_{ex}=-0.0559 + 1.3871 ln T_{log}	90.37	$0^\circ \leq \theta \leq 180^\circ$
26	ln T_{ex} =0.0003 + 0.9949 ln T_P	100	$0^\circ \leq \theta \leq 90^\circ$
27	ln T_{ex}=0.0044 + 0.9555 ln T_P	99.98	$0^\circ \leq \theta \leq 180^\circ$

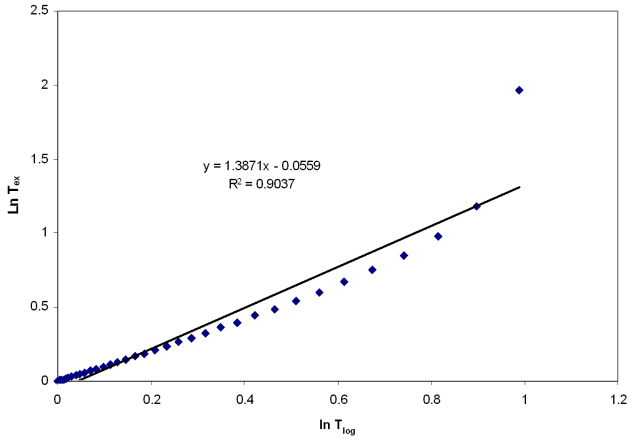

FIGURE 1. Fitting the linear regression of Lima and Arun approximation (ln T_{log}) Angular amplitude $0^\circ \leq \theta \leq 180^\circ$.

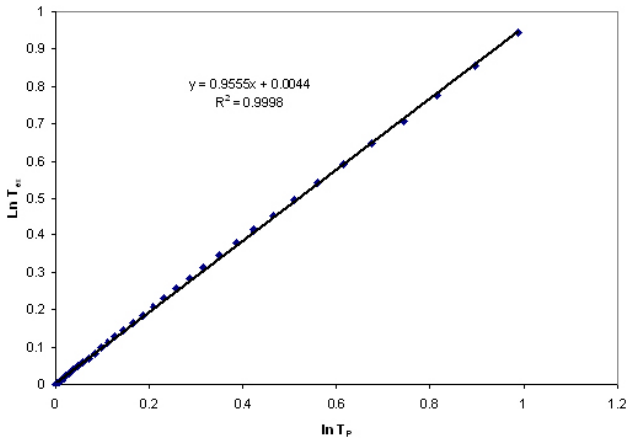

FIGURE 2. Fitting the linear regression of Parwani approximation (ln T_P) Angular amplitude $0^\circ \leq \theta \leq 180^\circ$.

From the analysis of the fitting results, we might give a precise idea to physics students and teachers about the degree of accuracy of each known approximation expression compared to the exact period oscillation of a simple pendulum.

4. Comparison between exact and approximate expressions

The reported errors of the known approximation expressions compared with the exact period oscillations were given for each formula [7].

- The formula of Bernoulli (T_B) shows errors of 0.1% (0.5%) for amplitudes above 41°(60°);
- The Kidd and Fogg approximation (T_{KF}) has errors greater than 0.1% for amplitudes $\theta \geq 57^\circ$ and reaches 0.8% for θ = 90°;
- The proposed expression by Molina (T_M) has an error greater than 0.1% for $\theta \geq 69^\circ$ and this error is 0.4% for angular displacement, θ = 90°;
- The approximation expression developed by Parwani [4] presents an accuracy of 1% for amplitudes as large as $\theta \approx 130^\circ$;
- The approximation expression proposed by Lima and Arun [7] has an error greater than 0.1% for amplitude greater than 74° and reaches 0.2% for θ = 86°. This formula was supposed to approximate the exact period better than existing approximations.

The quality of known approximations is determined by comparison to the exact value, which is obtained by the power expansion series. To increase the accuracy in the values of the exact period, we have taken into account all terms up to and including the eight-order term. The ratios of exact period (T_{ex} /T_0) and approximation expressions (T/T_0) are plotted in Fig. 3. As can be seen from this plot, all curves increase differently with angular amplitudes.

TABLE II. Slopes and relative errors of approximation expressions

Approx.	Slope $0 \leq \theta \leq 90°$	Slope $0 \leq \theta \leq 180°$	Relative error E% $0 \leq \theta \leq 90°$	Relative error E % $0 \leq \theta \leq 180°$
T_P	0.9961	0.9555	0.51	4.45
Tl_{og}	1.0123	1.3871	1.23	38.71
T_M	1.018	1.6795	1.8	67.95
T_{KF}	1.0393	1.9968	3.93	99.68
T_B	0.8759	0.5283	12.41	47.17

TABLE III. Intercepts and relative errors of approximation expressions.

Approx.	Intercepts $0 \leq \theta \leq 90°$	Intercepts $0 \leq \theta \leq 180°$	Relative error E % $0 \leq \theta \leq 90°$	Relative error E % $0 \leq \theta \leq 180°$
T_P	0.0003	0.0044	0.03	0.44
T_{log}	-0.0002	-0.0559	0.02	5.59
T_M	-0.0002	0.1003	0.02	10.03
T_{KF}	-0.0007	-0.1423	0.07	14.23
T_B	0.0022	0.0364	0.22	3.64

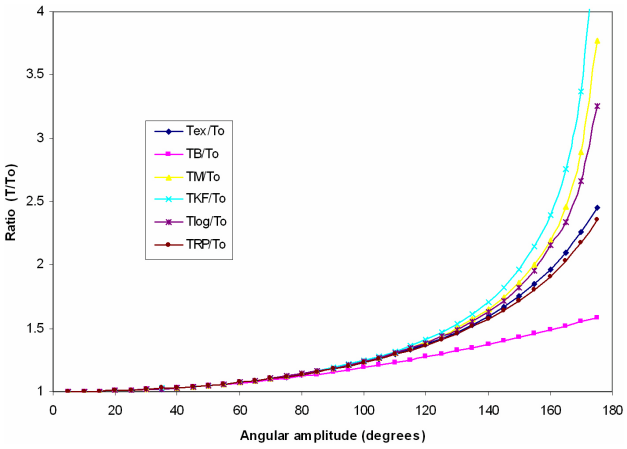

FIGURE 3. Comparison of ratios for exact period (T_{ex}/T_0) and approximation expressions (T/T_0) in the interval $0° \leq \theta \leq 180°$..

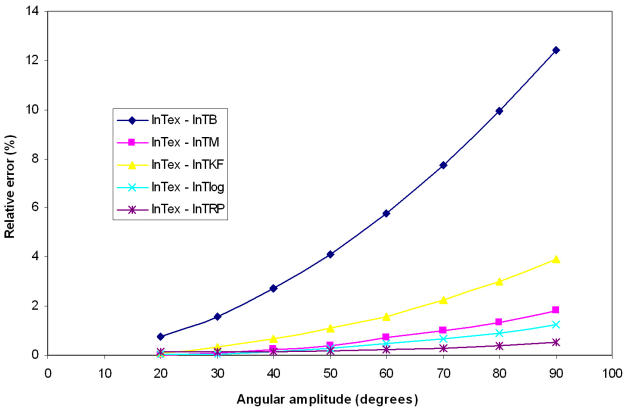

FIGURE 4. Comparison of the relative errors for approximation expressions in the interval $0° \leq \theta \leq 90°$.

We attempted to quantify the degree of increase of each approximation and give physics students and teachers the accuracy of these formulae with respect to the exact period. From the plots of the exact period linearization ($\ln T_{ex}$) as a function of the linearization of each approximation ($\ln T$), we have obtained slopes, intercepts and correlation coefficients for the straight lines in 10° steps from 20 to 180 degrees. The results of slopes with their corresponding relative errors, and intercepts with their corresponding relative errors, are presented in Table II and III**,** respectively, over an interval of angular amplitude $0° \leq \theta \leq 180°$.

The relative errors obtained for the approximation expressions investigated in this work are shown in Figs. 4 and 5 for angular amplitudes ($0° \leq \theta \leq 90°$) and ($90° \leq \theta \leq 180°$), respectively. As can be seen from these plots, for angular amplitudes less than 20°, their corresponding error curves tend to zero, and for larger amplitudes the curves increase at a different rate. In other words, the error increase is different for each curve.

To illustrate the accuracy of each approximation expression compared to the exact period, the error values of these formulae for different angular amplitudes (20° to 180°) are presented in Table IV. From the results obtained in Table IV and Figs. 4 and 5, it may be said that the best approximation expression discussed in the text is that given by Parwani, Eq. (10); it presents relative errors of 0.12, 0.16, 0.24, 0.39, 0.51, 1.17, 2.69, 4.11 and 4.45% at amplitudes of 20, 40, 60, 80, 90, 120, 150, 170 and 180°, respectively. These relative errors are smaller than for the other approximation formulae mentioned in this work, except in amplitude range 20° to 40°, where the logarithmic approximation given by Lima and Arun, Eq. (11) offers a better approximation. This formula has relative errors of 0.04, 0.15, 0.46, 0.68, 0.93, 1.23, 2.57,

TABLE IV. Comparison of the relative errors for the approximation studies in this work.

Angular amplitudes (degrees)	Relative errors (%) ; [ln T_{exact}– ln T $_{expression}$)/ ln T_{exact}]				
	T_B expression	$T_M$expression	T_{KF} expression	T_{log} expression	T_P expression
20	0.76	0.01	0.1	0.04	0.12
30	1.59	0.1	0.34	0.04	0.13
40	2.7	0.25	0.66	0.15	0.16
50	4.1	0.4	1.08	0.29	0.19
60	5.78	0.7	1.6	0.46	0.24
70	7.73	1	2.24	0.68	0.31
80	9.94	1.36	3.01	0.93	0.39
90	12.41	1.8	3.93	1.23	0.51
100	15.15	2.33	5.04	1.6	0.67
110	18.14	2.98	6.37	2.04	0.88
120	21.41	3.78	8	2.57	1.17
130	25.0	4.8	10.02	3.27	1.55
140	28.81	6.19	12.66	4.162.06	
150	32.98	8.24	16.33	5.51	2.69
160	37.46	11.7	22.03	7.83	3.42
170	42.22	18.88	32.85	12.67	4.11
180	47.17	67.95	99.68	38.71	4.45

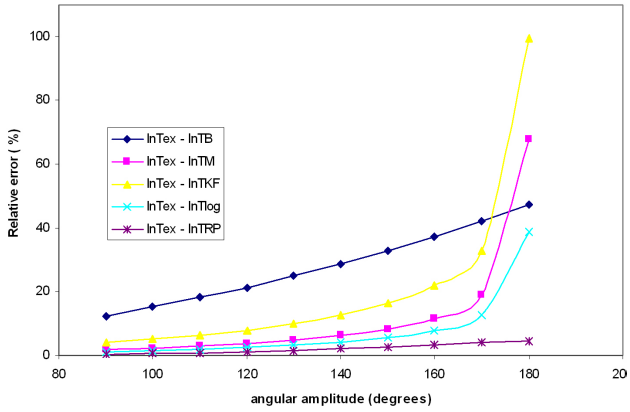

FIGURE 5. Comparison of the relative errors for approximation expressions in the interval $90° \leq \theta \leq 180°$.

5.51, 12.67 and 38.71% for amplitudes 20, 40, 60, 70, 80, 90, 120, 150, 170 and 180°, respectively. The approximation given by Molina, Eq. (8), is the most accurate of all the formulae studied with a relative error of 0.01% at a smaller angle of displacement of 20°.

5. Conclusion

It seems that the reported errors in approximating the exact period of the approximation formulae for large amplitudes of a simple pendulum in the intervals $0° \leq \theta \leq 90°$ and $0° \leq \theta \leq 180°$ are not very precise or accurate for physics students and teachers who perform the simple pendulum experiment. For this reason, we have attempted to investigate the accuracy of each approximation expression in this text by using a linear form, *i.e.* taking the log of both sides of the expressions, and plotting the relative error (ln T_{ex} – ln $T_{expression}$)/ T $_{ex}$) as a function of amplitudes in degrees.

As can be seen in Table III, the best approximation, of all the expressions studied, that approximates the exact period in the intervals $0° \leq \theta \leq 90°$and $0° \leq \theta \leq 180°$ is Parwani's, Eq. (10), where the error for the exact period is approximately 0.51% for amplitude 90°. Equation (8) better approximates the exact period for angular amplitude 20° with an error of 0.01 %. The second best formula in approximating the exact period is Eq. (11), called the log approximation expression. It presents a relative error of 1.23% at amplitude 90°.

The analysis considered in this paper reveals the accuracy and precision of each approximation expression for the exact period that has some pedagogical interest for undergraduate physics students.

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