

On the need to enhance physical insight via mathematical reasoning

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Recibido el 13 de agosto de 2007; aceptado el 4 de septiembre de 2007

It is becoming common to hear teaching advice about spending more time on the “physics of the problem” so that students will get more physical insight and develop a stronger intuition that can be very helpful when thinking about physics problems. Based on this type of justification, mathematical skills such as the ability to compute moments of inertia, center of mass, or gravitational fields from mass distributions, and electrical fields from charge distributions are considered “distracting mathematics” and therefore receive less attention. Based on published cited research on the subject, we’ll argue a) that this approach can have a negative influence on student reasoning when dealing with questions of rotational dynamics, a highly non-intuitive subject where even instructors may fail to provide correct answers, and b) that exposure of students to mathematical reasoning and to a wide range of computational techniques to obtain the moment of inertia of different mass distributions will make students more comfortable with the subject of rotational dynamics, thus improving their physical insight on the topic.

Keywords: Physical intuition; physics learning; teaching of physics; mathematical reasoning; student performance.

Se está haciendo común escuchar sobre estrategias de enseñanza en los cursos de física que hacen énfasis en la “física del problema”, justificándose en la esperanza que de esa forma los estudiantes obtendrían mejor intuición física y desarrollarían mejor sus capacidades al pensar en problemas de ciencias e ingenierías. De acuerdo con estas estrategias de enseñanza, habilidades matemáticas como por ejemplo la capacidad de calcular e interpretar momentos de inercia, centro de masa, campos gravitacionales de distribuciones de masa o campos eléctricos de distribuciones de carga se consideran como “matemáticas que distraen” y por lo tanto han de recibir menos atención. Con apoyo en estudios reportados en la literatura que referenciamos, en este trabajo discutimos a) que estas estrategias de enseñanza influyen negativamente en el razonamiento de los estudiantes cuando intentan describir aspectos relacionados con la dinámica rotacional, un tema altamente no-intuitivo donde incluso instructores con amplia experiencia enseñando la temática pueden no poder proporcionar respuestas correctas a problemas relacionados con el tema y b) que una exposición de los estudiantes al razonamiento matemático que les permita discernir con certeza cuantitativa y cualitativa sobre la dependencia de la inercia rotacional de distribución de masa referente al eje de rotación les permitiría sentir más confianza, no solo cuando tratan del tema de la dinámica rotacional, sino también de otros temas afines de estudios en los cursos introductorios de física.

Descriptores: Intuición en física; aprendizaje en física; enseñanza de la física; razonamiento matemático; rendimiento estudiantil.

PACS: 01.40.gb; 01.40.Ha; 01.40.Fk

1. Introduction

A common piece of advice that the Nobel prize winning physicist Lev Davidovich Landau often offered to students and colleagues asking him about what and how to study, particularly to those interested in physics, was about the importance of mastering first the techniques of working in the field of interest because “fine points will come by themselves.” In his own words, “You must start with mathematics which, you know, is the foundation of our science. [...] Bear in mind that by ‘knowledge of mathematics’ we mean not just all kinds of theorems, but a practical ability to integrate and to solve in quadratures ordinary differential equations, etc.” Or, in another response, “What is needed is not all kinds of existence theorems, on which mathematicians lavish so much praise, but mathematical techniques, that is, the ability to solve concrete mathematical problems” [1].

The importance of being able to express, interpret and manipulate physical results in mathematical terms was also stressed by the great physicist Lord Kelvin: “I often say that when you can measure something and express it in numbers, you know something about it. When you can not measure it, when you cannot express it in numbers, your knowledge is of

a meager and unsatisfactory kind. It may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the state of science, whatever it may be” [2]. Freeman Dyson was more eloquent: “...mathematics is not just a tool by means of which phenomena can be calculated; it is the main source of concepts and principles by means of which new theories can be created” [3].

The remarks of these great physicists bring to mind the challenges of undergraduate students learning rotational dynamics in the science and engineering fields while in a typical calculus-based physics course. Students face the problem of not finding sufficient detailed solved textbook examples of rotational inertia (I) computations. Despite the fact that in general “intuition” cannot be used as a good guide for computing I (although it can be of help when computing, for instance, the center of mass of a continuous body), no more than two or three worked examples can be found in most calculus-based physics textbooks [4–7]. Accordingly, based on published recent results [9] we would argue that students’ limited exposure to or poor training in computing I for several rigid body geometries with rotational motion about different axes, can limit their intuition [10] when asked

to answer questions involving the comparison, for instance, of the rotational kinetic energy of different objects in similar situations.

2. About the Problem

The difficulties that students have in realizing the dependence of I on the mass distribution about the rotating axis is evident when they are asked to compute the rotational inertia of a thin rod about one of the rod's ends and compare it with the corresponding I of a thin rectangular plate rotating about one of its edges. Students become uneasy about the unexpected result that, in the latter case, I does not depend on the length of the edge about which the rotation takes place, and that I has the same dependence on the length of the edges perpendicular to the rotational axis as the thin rod. Moreover, in spite of learning that the moment of inertia of a system of particles is the sum of the rotational inertia of individual particles about the same rotational axis, students are amused by the idea that to compute I for an axis through the center of mass and perpendicular to the plane of a rectangular thin plate of uniform mass M , for instance, we can divide the object into four rectangular smaller plates, each one of mass $M/4$ rotating around an axis through a corner and perpendicular to the surface of each lamina. Thus the desired I is four times the rotational inertia of one of the smaller plates.

Their anxiety diminishes somewhat when they can compute I using different methods such as direct integration, using the parallel axis theorem, and dividing a complicated body into smaller pieces of known I about the axis of interest. They gain even more confidence about the correctness of the result after finding the right answer in a textbook or after measuring a prediction of a physical quantity that depends on I (*i.e.* the period of oscillation of the corresponding physical pendulum for both objects). Nevertheless, the experimental approach also requires students to have first found I mathematically. In other words, it is required that students have some conceptual knowledge before they carry out experiments [11, 12].

Analogous “surprising results” can be shown to confuse students when doing similar computations of, for instance, the moment of inertia of a hollow thin cylinder and a solid cylinder, each one rotating around their respective axis. How can it be that I is smaller for the latter case than for the former case?

To be able to tackle the above mentioned difficulties with some success, students need to believe that these results are neither obvious nor intuitive. Moreover, even for instructors many rotational dynamics outcomes are not intuitive [10]. Accordingly, the old maxim “*A repetitio studiorum mater est*” applies, but it has to be “guided by critical feedback and *deliberate* attempts to improve” [13]. That is, the only way for students not to be surprised by surprising results is by doing more rotational inertia computations, which many colleagues dismiss as “*distracting mathematics*”, which hardly help students gain physical insight.

We might agree that in standard calculus courses students will learn, among other mathematical skills, the formal techniques in how to carry out some integrals. But it is in physics courses where students start to apply what they have learned in their math classes and to find new non-formal approaches to performing computations [14]. Some warnings on the traditional instruction of math in physics are also available [15, 16]. We believe that the mathematical understanding of a problem is a process that involves meaningful learning which goes beyond the merely application of rote procedures and involves “higher order thinking skills” [17]. Moreover, using properly designed quantitative problems requiring students to illustrate their conceptual learning and understanding will reveal much to the teachers and provide invaluable feedback [18–21], and they can also be a powerful means for helping students to understand the concepts of physics [17, 21].

3. About the evidence

In a study on students' understanding of rotational dynamic concepts, Rimoldini and Singh [9] point out some difficulties that students encounter when dealing with the subject. In particular, when students they interviewed were asked about how the angular velocity or the period of rotating objects would depend on the mass distribution of the object, “Many did not use the concept of rotational inertia correctly. Some said that they vaguely remember that the distribution of mass matters but did not remember the exact relation.” This observation is a reflection of the limited exposure students have to the computation of the rotational inertia of different mass distributions and how it actually depends on the mass distribution about the rotational axis.

This difficulty is further made explicit in a written test of thirty multiple-choice questions that the authors [9] also administered to 652 students from calculus- and algebra-based introductory physics courses, which includes an honor class of 97 students and an upper-level class of 17 physics majors enrolled in an intermediate mechanics course. The test questions, a total of thirty, are available as Appendix B of the report [9], and students were required to provide justification for their answers.

In addition to two classes of student difficulties identified by Rimoldini and Singh, a) those sharing a common ancestry with linear motion and b) those uniquely related to the more intricate nature of rotational motion, we could add a third category c) those associated with insufficient training of students on the mathematical computation of rotational inertia.

Rimoldini and Singh found that some of the students they interviewed were uncertain about the meaning of I , which likely indicates that the students had little practice in computing I and had not mastered the techniques of computing I . Recalling that mathematically solving problem involves a “higher-order thinking skills,” [17] we also share the idea expressed by Rigden in the sense that “a student's ability to discuss the problem—to do so in words of their own choosing,

to do so clearly and accurately—indicates an understanding in which we can have confidence.” [17] And to get to that level some practice is required [22]. In fact, in studies comparing conceptual learning and problem solving skills, students enrolled in courses based on traditional instruction scored on average higher in quantitative problems than students enrolled in courses emphasizing conceptual learning [23–25]. As pointed out by L.D. Landau “I can only emphasize that you must perform all the calculations by yourself, and must not leave it to the authors of the books you have read” [1]. Though the ability to do so is a matter of training in “distracting mathematical computations” such as computing rotational inertia, the training should also involve a proper physical interpretation of the quantitative results obtained [19,26].

To make our third category more evident, we first concentrate on the responses to the thirty questions given by students of the honor class. The answers can be divided into two sets: those that were answered correctly by eighty or more percent of the students (high level of correct response) and those that were answered correctly by less than eighty percent of the students. We find 14 questions in the first group. In the second group fall the remaining 16 questions. The answers to the questions are given on page 7 of the report [9].

It is expected that students of an honor class would be able to perform standard computations involving integration of not too complicated expressions. Correspondingly, according to this expectation we would expect that all 7 of the questions involving knowledge of the rotational inertia of the rotating solid (*i.e.* questions 1, 3, 4, 20, 24, 25, and 29) would have received a higher level of correct responses (to make the article self-contained we are including these 7 questions as appendix A and the respective responses to these questions are shown in Table I). Such an expectation is reasonable because the required computations are fairly easy. The results show that only question number 20 was answered in conformity with this expectation, and it was answered correctly by 85 percent of the students of the honor class (because the computation requires a simple integration and, more importantly, the rotational inertia of a homogeneous cylinder is one

of the few textbook-worked examples, this level of response should have been much better). By similar reasoning, questions 1, 3, and 4 should have received a higher level of correct responses by the honor class.

We believe that if students of the honor class had received enough training in computing the rotational inertia in composite and simple objects (like the ones involved in these questions), they would have given a higher level of correct responses on 7 of the questions requiring these simple computations [17, 19, 20, 24, 26]. This would have reduced the gap between the questions receiving a high level of correct responses and those that did not.

These observations also apply to the other group of students that took the test. For example, while 76 percent of honors students provided the correct response to question 3 (see Table I), physics majors in an upper-level class did not perform better on this than non-honors introductory students (41 and 45 percent correct responses respectively). The upper level class did better in answering question 4 than the other two. Regarding rotational inertia, these were the only two questions common to each group. On average, 71 percent of the students from the honor class answered all the questions regarding rotational inertia correctly while 56 percent of the non-honor class answered the same questions correctly.

We feel that, since the mathematical computations and algebraic manipulations involved in these questions are not really demanding, the failure of honor students to respond correctly is a consequence of neglecting mathematical skills over physical insight [13, 14, 16, 27]. In addition, in their study Hoellwarth *et al.* [23] conclude that “students must be taught both concepts and problem solving skills explicitly if we want students to be proficient at both.” In this sense, proactive teaching strategies should lead to the identification of quantitative problems helpful to recognize both conceptual and quantitative understanding of students. In fact, some fruitful ideas have been advanced on how to properly address the design of instruction so that the involved learning cognitive mechanism of the students are triggered, leading to more effective teaching outcomes [13, 18, 19, 21, 28, 29].

TABLE I. Multiple-choice questions (included in Appendix A) were administered to a total of 669 students. The performance of 559 general (calculus- and algebra-based) introductory non-honor students (GI) is distinguished from an honor class (HC) of 93 introductory students, and an upper-level (UL) class of 17 physics majors enrolled in an intermediate mechanics course (who were administered a subset of 11 questions). The table presents the average percentage (rounded to the nearest integer) of students selecting the answer choices (a)–(e) for each question of the test (bold numbers refer to the correct responses).

Answers	(a)			(b)			(c)			(d)			(e)		
	GI	HC	UL	GI	HC	UL	GI	HC	UL	GI	HC	UL	GI	HC	UL
1	16	10		23	16		57	73		4	1		1	0	
3	18	27	29	45	71	41	35	2	18	1	0	6	1	0	6
4	13	7	6	22	15	6	61	76	82	3	0	6	0	2	0
20	2	2		3	2		71	85		18	9		6	2	
24	33	61		32	24		3	0		4	0		28	15	
25	28	58		32	21		4	2		34	18		2	1	
29	19	9		60	75		3	0		1	1		17	15	

4. Concluding remarks

We suggest that the common difficulty students have in answering correctly questions of physical quantities involving rotational inertia is likely rooted in the limited exposure students have to computing and analyzing these quantities because these techniques, while essential, are considered to be “distracting mathematics,” and their importance is not emphasized by instructors [13, 18, 19]. Another reason for students’ difficulty is that textbooks used by students provide just one or two simple examples as models for students to learn these computational techniques [7]. Considering the non-intuitiveness of rotational dynamics, even for instructors having a wide experience teaching the subject [10], both reasons conspire against students reasoning on this subject.

In the analysis of the collected data, on page 6 of the report [9], Rimoldini and Singh pointed out that “many students were unsure about this concept. For example, many did not know that I is a function of the mass distribution about an axis and that the rotational kinetic energy depends on I and not just on the total mass.[...] Interviews showed that this type of difficulty was partly due to the students’ unfamiliarity with I ”.

In some sense, the research of Rimoldini and Singh somehow supports the idea that because of an overemphasis [23,27,30] on the qualitative (conceptual) physical aspects of the problems, standard mathematical abilities, which are essential for understanding the whole physical process, are not taught because, to rephrase a passage from a recent editorial [31], they interfere with the students’ emerging sense of physical insight.

Thus, if instructors do not have enough time to train students relevant computational techniques, textbook publishers should not leave mathematical computations only to the students. In addition to rotational inertia, textbooks should also include more solved illustrative examples on computing center of mass, gravitational and electric fields [7] and constantly point out that the involved techniques are essentially the same [14]. This is an important requirement for a textbook because innovative active-learning teaching strategies require students to acquire basic and fundamental knowledge through reading a textbook. Certainly, innovative teaching strategies will help in handling thicker and heavier textbooks, with lots of physical and mathematical insights within them [20, 23, 32–34].

To paraphrase Heron and Meltzer, learning to approach problems in a systematic way starts from teaching and learning the interrelationships among conceptual knowledge, mathematical skills and logical reasoning [35]. In physics, this necessarily requires the teaching of a good deal of “distracting mathematical computations.”

Acknowledgments

We are grateful to Dr. Cheryl Pahaham, who kindly provided useful comments on improving this article.

Appendix

A. Test problems

To make the article self-contained, in this section we are including the 7 multiple choice test problems we are analyzing [9].

1. Two copper disks (labeled “A” and “B”) have the same radius, but disk B is thicker with four times the mass of disk A. They spin on frictionless axles. If disk A is rotating twice as fast as disk B, which disk has more rotational kinetic energy?

- (a) The faster rotating disk A.
- (b) The thicker disk B.
- (c) Both disks have the same rotational kinetic energy.
- (d) It depends on the actual numerical values of the angular speeds of the disks.
- (e) None of the above.

3. An aluminum disk and an iron wheel (with spokes of negligible mass) have the same mass M and radius R . They are spinning around their frictionless axles with the same angular speed as shown. Which of them has more rotational kinetic energy?

- (a) The aluminum disk.
- (b) The iron wheel.
- (c) Both have the same rotational kinetic energy.
- (d) It depends on the actual numerical value of the mass M .
- (e) None of the above.

4. Consider the moment of inertia, I , of the rigid homogeneous disk of mass M shown below, about an axis through its center (different shadings only differentiate the two parts of the disk, each with equal mass $M/2$). Which one of the following statements concerning I is correct?

- (a) The inner and outer parts of the disk, each with mass $M/2$ (see figure), contribute equal amounts to I .
- (b) The inner part of the disk contributes more to I than the outer part.
- (c) The inner part of the disk contributes less to I than the outer part.
- (d) The inner part of the disk may contribute more or less to I than the outer part depending on the actual numerical value of the mass M of the disk.
- (e) None of the above.

20. The moment of inertia of a rigid cylinder

- (a) does not depend on the radius of the cylinder.
- (b) does not depend on the mass of the cylinder.
- (c) depends on the choice of rotation axis.
- (d) depends on the angular acceleration of the cylinder.
- (e) can be expressed in units of kg.

Setup for the next three questions An aluminum disk and an iron wheel (with spokes of negligible mass) have the same radius R and mass M as shown below. Each is free to rotate about its own fixed horizontal frictionless axle. Both objects are initially at rest. Identical small lumps of clay are attached to their rims as shown in the figure (the figure shows each rim

on vertical position and the small mass attached to the right of the rim on the horizontal diameter).

24. Which one of the following statements about their angular accelerations is true?

- (a) The angular acceleration is greater for the disk+clay system.
- (b) The angular acceleration is greater for the wheel+clay system.
- (c) Which system has a greater angular acceleration depends on the actual numerical values of R and M.
- (d) There is no angular acceleration for either system.
- (e) The angular accelerations of both systems are equal and non-zero.

25. Which one of the following statements about their maximum angular velocities is true?

- (a) The maximum angular velocity is greater for the disk+clay system.
- (b) The maximum angular velocity is greater for the wheel+clay system.
- (c) Which object has a greater maximum angular velocity is determined by the actual numerical values of R and M.
- (d) The maximum angular velocities of both systems are equal and non-zero.

(e) There is no angular velocity for either system so the question of a maximum value does not arise.

Setup for the next two questions Two copper disks of different thicknesses have the same radius but different masses as shown below. Each disk is free to rotate about its own fixed horizontal frictionless axle. Both disks are initially at rest. Identical small lumps of clay are attached to their rims as shown in the figure. (the figure shows each rim on vertical position and the small mass attached to the right of the rim on the horizontal diameter).

29. Which one of the following statements about their angular accelerations is true?

- (a) The angular acceleration is greater for the system in which the disk has larger mass.
- (b) The angular acceleration is greater for the system in which the disk has smaller mass.
- (c) Which system has a greater angular acceleration depends on the actual numerical values of their masses.
- (d) There is no angular acceleration for either system.
- (e) The angular accelerations of both systems are equal and non-zero.

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1. E.M. Lifshitz, *Am. J. Phys.* **45** (1977) 415.
 2. P.G. Hewitt, *Conceptual Physics* (Addison-Wesley, 7th edition, 1993).
 3. F.J. Dyson, *Scientific American* 211 (1964) 128.
 4. D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics* (John Wiley & Sons, 6th edition, 2000).
 5. R.A. Serway and J.W. Jewett, *Physics for Scientists and Engineers*, 6th. edition (Thomson Learning, 2003).
 6. P.A. Tipler and G. Mosca, *Physics for Scientists and Engineers*, 5th edition (WH Freeman and Co, 2003).
 7. For instance, in a recent article [8] about the proper way to compute the electric field of a finite line of uniform charge, the author mentions that the problem is 'either ignored or handled incompletely in most textbooks' [8]. Similar findings will be obtained in searching for rotational inertia problems. Now, in terms of teaching, recall that the geometry involved in handling the line charge problem is essentially the same as computing the rotational inertia of a finite line of uniform mass around an axis through the field point and perpendicular to the plane containing the finite line and the field point. Since the latter is first covered in a typical introductory calculus-based physics course, if properly presented, students could transfer their knowledge from one problem to another [18, 19].
 8. R.J. Rowley, *Am. J. Phys.* **74** (2006) 1120.
 9. L.G. Rimoldini and C. Singh, *Phys. Rev. ST. Phys. Educ. Res.* **1** (2005) 010102.
 10. C. Singh, *Am. J. Phys.* **70** (2002) 1103.
 11. C.v. Aufschnaiter and S.v. Aufschnaiter, *Eur. J. Phys.* 28 (2007) S51.
 12. One could also implement non-passive classroom demonstrations as discussed by C.H. Crouch, A.P. Fagen, J.P. Callan, and E. Mazur, *Am. J. Phys.* **72** (2004) 835.
 13. D. Hestenes, *Am. J. Phys.* **71** (2003) 104.
 14. F.R. Yeatts and J.R. Hundhausen, *Am. J. Phys.* **60** (1992) 716.
 15. E.F. Redish, Problem Solving and the Use of Math in Physics Courses, *ArXiv Physics e-prints*, 2006.
 16. B.L. Sherin, *Cognition and Instruction* 19 (2001) 479.
 17. J.S. Rigden, *Am. J. Phys.* **55** (1987) 877.
 18. F. Reif, *Phys. Teach.* **19** (1981) 310; see also references there in.
 19. F. Reif and L.A. Scott, *Am. J. Phys.* **67** (1999) 819; see also references there in.
 20. K.C. Yap and C.L. Wong, *Phys. Educ.* **42** (2007) 50.
 21. J.W. Dunn and J. Barbanel, *Am. J. Phys.* **68** (2000) 749.
 22. For instance, one could carefully design three problems dealing with rotational inertia computations involving linear, superficial and volumetric mass distributions about some axis, each one to be carefully solved using different techniques, emphasizing the teaching of identified thought processes [18, 19] involved to interpret the concepts presented in the solution of the problem (one of them could be the analogy of the quantitative analysis on solving the problems with the computation of center of mass). To speed the time on presenting the solution and to spend more time on the physical interpretation of the results, instructors could hand out the solution in advance so a kind of dialog in the classroom could be established between the instructor and the students as a way of determining, bookkeeping,

- and properly addressing students' misconceptions or confusing ideas (a kind of teaching strategy by thinking aloud) [18,28].
23. M.J. Moelter, C. Hoellwarth, and R.D. Knight, *Am. J. Phys.* 73 (2005) 459.
 24. S.M. Lea, *Am. J. Phys.* 62 (1994) 627.
 25. Let us mention that in the study of Lea [24] we are looking at the data corresponding to the traditional honor class (emphasizing quantitative understanding) and the inquire-based class (emphasizing conceptual understanding). Table IV of the report [24] shows that the honors students did far better in the quantitative problem, while Table III shows that in a conceptual problem the inquire-based class did better than the honor class. Accordingly, these results are in accordance with the views expressed in this article in the sense that 'students must be taught both concepts and problem-solving skills explicitly if we want students to be proficient at both.' [23].
 26. Rephrasing a remark of Professor Hestenes [13], conscious learning requires deliberate practice with critical feedback.
 27. R. Muallem and B.S. Eylon, *Phys. Teach.* 45 (2007) 158; see also references there in.
 28. E.F. Redish and R.N. Steinberg, *Phys. Today* 52 (1999) 24; see also references there in.
 29. R.E. Scherr, *Am. J. Phys.* 75 (2007) 272.
 30. The results of Rimoldini and Singh study [9] might be a response to the question raised by Hoellwarth *et al.* [23] about 'what happens to problem-solving skills as conceptual understanding is increasingly emphasized? Are we sacrificing problem-solving development to make gains in conceptual understanding?'
 31. D. Klein, *Am. J. Phys.* 75 (2007) 101.
 32. C.H. Crouch and E. Mazur, *Am. J. Phys.* 69 (2001) 970.
 33. M.C. James, *Am. J. Phys.* 74 (2006) 689.
 34. W. Cerbin and B. Kopp, *IJTLHE* 18 (2006) 250; available at <http://www.isetl.org/ijtlhe/>.
 35. P.R.L. Heron and D.E. Meltzer, *Am. J. Phys.* 73 (2005) 390.