Brownian motion in a magnetic field and in the presence of additional external forces

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Our purpose in this paper is to solve exactly the Fokker-Planck-Kramers equation of a charged particle (heavy-ion) embedded in a fluid and under the influence of mechanical and electromagnetic forces. In this work the magnetic field is assumed to be constant and pointing along any direction of a Cartesian reference frame; the mechanical and electrical forces are both space-independent, but in general time-dependent. Our proposal relies upon two transformations of the Langevin equation associated with the charged particle's phase-space $(\mathbf{r}, \mathbf{u})$. The first one is a fixed rotation which transforms the $(\mathbf{r}, \mathbf{u})$-coordinates into other $(\mathbf{r}', \mathbf{u}')$-coordinates, and makes it possible to re-orientate the magnetic field along an appropriate direction (say along the z'-axis). The second one is a time-dependent rotation which transforms the $(\mathbf{r}', \mathbf{u}')$-coordinates into other $(\mathbf{r}'', \mathbf{u}'')$-coordinates, in which the resulting Langevin equation strongly resembles that of ordinary Brownian motion in the presence of external forces. Under these circumstances, the Fokker-Planck-Kramers equation can immediately be solved in the $(\mathbf{r}'', \mathbf{u}'')$ phase-space, following our methodology developed in Ref. [*Phys. Rev. E* **76** (2007) 021106].

Keywords: (FP) Fokker-Planck; (FPK) Fokker-Planck-Kramers.

Nuestro propósito en este artículo consiste en resolver de manera exacta la ecuación Fokker-Planck-Kramers de una partícula con carga eléctrica (ión pesado) inmersa en un fluido y bajo la influencia de fuerzas mecánica y electromagnética. En este trabajo se supone que el campo magnético constante apunta en cualquier dirección de un sistema de referencia Cartesiano; las fuerzas mecánica y eléctrica son ambas independientes de la posición pero en general dependientes del tiempo. Nuestra propuesta se basa en dos transformaciones de la ecuación de Langevin asociada al espacio fase $(\mathbf{r}, \mathbf{u})$ de la partícula cargada. La primera, es una rotación fija que transforma las coordenadas $(\mathbf{r}, \mathbf{u})$ en otro sistema de coordenadas $(\mathbf{r}', \mathbf{u}')$, la cual permite una re-orientación del campo magnético a lo largo de una dirección apropiada (digamos a lo largo del eje z'). La segunda, es una rotación que depende del tiempo, la cual transforma las coordenadas $(\mathbf{r}', \mathbf{u})'$ en otro sistema de coordenadas $(\mathbf{r}'', \mathbf{u}'')$ donde la ecuación de langevin resultante es muy semejante a la del movimiento Browniano ordinario en presencia de fuerzas externas. En éstas circunstancias, la ecuación de Fokker-Planck-Kramers se puede resolver de forma inmediata en el espacio fase $(\mathbf{r}'', \mathbf{u}'')$, siguiendo nuestra metodología desarrollada en la Ref. [*Phys. Rev. E* **76** (2007) 021106].

Descriptores: (FP) Fokker-Planck; (FPK) Fokker-Planck-Kramers.

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1. Introduction

Very recently, we solved both the Fokker-Planck (FP) and Fokker-Planck-Kramers (FPK) equations corresponding to a heavy ion embedded in a fluid in the presence of external fields [1]. In that work the magnetic field has been considered, for simplicity, as a constant vector pointing along the z-axis of a Cartesian reference frame, that is $\mathbf{B} = (0, 0, B)$; the mechanical and electrical forces are in general time-dependent only. The same considerations for the mechanical and electromagnetic forces were assumed by Simões and Lagos (SL) [2] to solve the same Fokker-Planck-Kramers equation, by combining both Czopnik and Garbaczewski's (CG) [3] "rotated" Stokes force and Ferrari's [4] gauge. As established in Ref. 2, the method of solution relies upon a transformation of the FPK equation into a similar field-free equation, in a similar manner to that advanced by Ferrari [4]. Accordingly, the solution to this type of field-free equation is obtained by applying CG's strategy, which simply consists in proposing a Gaussian distribution function for the correlation functions of the appropriate variables. Similar studies on anisotropic plasma diffusion has been proposed in Ref. 5. Our aim in this work is now to extend the proposal in Ref. 1 to the situation in which the constant magnetic field is allowed to point along any direction in a Cartesian reference frame, that is $\mathbf{B} = (B_x, B_y, B_z)$. We have given a preliminary study of this problem in Ref. 6, in which our main attention was focused on computing the x, y, and z mean-square displacements of the charged particle. In this work, to solve the FPK equation with a constant magnetic field pointing along an arbitrary direction, we proceed as follows. We apply two successive transformations to the Langevin equation associated with the heavy ion's phase-space $(\mathbf{r}, \mathbf{u})$. The first one consists in a fixed rotation which transforms the Langevin equation, given in the original space $(\mathbf{r}, \mathbf{u})$, into another Langevin equation for the $(\mathbf{r}', \mathbf{u}')$ phase-space. In this transformed space we will show that the original magnetic field is allowed to point along one of the axes of the transformed Cartesian reference frame (say along the z'-axis). In this sense, the Brownian motion of a charged particle in a magnetic field pointing along any direction is equivalent to that situation for which that magnetic field is re-orientated along the z'-axis. On the other hand, trying to solve the FPK equation in this transformed space is a difficult task, due to the coupling between the resulting equations. To avoid these difficulties we perform a second transformation the Langevin equation, which consists in a time-dependent rotation and makes it possible to pass from the $(\mathbf{r}', \mathbf{u}')$ phase-space to another $(\mathbf{r}'', \mathbf{u}'')$ phase-space, in which the behavior of the embedded particle is similar to that of the ordinary Brownian motion under the action of an external, non-magnetic field. It is in this second transformation that we can calculate the solutions to the FPK equation, following the same strategy as Ref. 1. An important point we wish to comment on here is that, in order to to attain the aforementioned objectives, it is necessary for the statistical properties of the transformed fluctuating forces to remain identical to that of the original one, which is the case if the latter satisfies the properties of Gaussian white noise (GWN). From the fundamental solution to the FPK equation, it is possible to calculate the fundamental solution to the Fokker-Planck (FP) equation in the $\mathbf{u}'$ velocity-space.

2. The Langevin equation of a heavy ion in the presence of external forces

The Langevin equation describing the diffusion process of a charged particle embedded in a fluid in the presence of electromagnetic (via Lorentz force) and mechanical $F_{\rm mec}$ forces can be written in the phase-space as

$$\dot{\mathbf{r}} = \mathbf{u}\,, \tag{1}$$

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \frac{q}{m}\mathbf{u}\times\mathbf{B} + \frac{q}{m}\mathbf{E} + \frac{\mathbf{F}_{\rm mec}}{m} + \mathbf{A}(t)\,, \tag{2}$$

where q denotes the charge of the particle of mass m and the fluctuating force $\mathbf{A}(t)$ satisfies the properties of a GWN with zero mean value and correlation function

$$\langle A_i(t)A_j(t')\rangle = 2\lambda\,\delta_{ij}\,\delta(t-t')\,, \tag{3}$$

where $\lambda = \beta\, k_B T/m$ is the noise intensity and k_B is the Boltzmann constant. $\mathbf{F}_{\rm mec}$ and $\mathbf{E}$ are, in general, both space-independent and time-varying external forces. The magnetic field is assumed to be a constant vector pointing along any direction of a Cartesian reference frame, that is $\mathbf{B} = (B_x, B_y, B_z)$. If we define the acceleration $\mathbf{a}(t) \equiv [\mathbf{F}_{\rm mec}(t) + q\mathbf{E}(t)]/m$, the Langevin equation for the velocity reads

$$\dot{\mathbf{u}} = -\beta\,\mathbf{u} + \mathbb{W}\mathbf{u} + \mathbf{a}(t) + \mathbf{A}(t)\,, \tag{4}$$

where $\mathbb{W}$ is a real antisymmetric matrix given by

$$\mathbb{W} = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}, \tag{5}$$

whose elements are defined as $\Omega_i = qB_i/mc$, this being known as the *Larmor frequency*. Subindex i may have values 1, 2, or 3, which represent the coordinates x, y, and z, respectively. By means of an appropriate change of variable, the Langevin equations (1), (3) can be transformed in such a way that the magnetic field is allowed to point along a privileged direction. This change in variable is $\mathbf{r}' = \mathbb{R}^T\mathbf{r}$, such that $\dot{\mathbf{r}}' = \mathbf{u}'$ and therefore

$$\mathbf{u}' = \mathbb{R}^T\mathbf{u}\,, \tag{6}$$

where $\mathbb{R}^T$ is the transpose of the rotation matrix $\mathbb{R}$ and $\mathbb{W}' = \mathbb{R}^T\mathbb{W}\mathbb{R}$ is another antisymmetric matrix (both are explicitly given in Appendix A of Ref. 6), such that

$$\mathbb{R} = \begin{pmatrix} \frac{-\Omega_1\Omega_3}{\Omega'\sqrt{\Omega_1^2+\Omega_2^2}} & \frac{\Omega_2}{\sqrt{\Omega_1^2+\Omega_2^2}} & \frac{\Omega_1}{\Omega'} \\ \frac{-\Omega_2\Omega_3}{\Omega'\sqrt{\Omega_1^2+\Omega_2^2}} & \frac{-\Omega_1}{\sqrt{\Omega_1^2+\Omega_2^2}} & \frac{\Omega_2}{\Omega'} \\ \frac{\Omega_1^2+\Omega_2^2}{\Omega'\sqrt{\Omega_1^2+\Omega_2^2}} & 0 & \frac{\Omega_3}{\Omega'} \end{pmatrix}, \tag{7}$$

and

$$\mathbb{W}' = \begin{pmatrix} 0 & \Omega' & 0 \\ -\Omega' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

with

$$\Omega'^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = q^2\,{B'}^2/m^2c^2$$

and $B'^2 = B_1^2 + B_2^2 + B_3^2$ is the square modulus of the external magnetic field $\mathbf{B}$. Thus, in the $(\mathbf{r}', \mathbf{u}')$ phase-space, the original Langevin equations are transformed into

$$\dot{\mathbf{r}}' = \mathbf{u}'\,, \tag{9}$$

$$\dot{\mathbf{u}}' = -\beta\,\mathbf{u}' + \mathbb{W}'\mathbf{u}' + \mathbf{a}'(t) + \mathbf{A}'(t)\,, \tag{10}$$

where the quantities

$$\mathbf{a}'(t) = \mathbb{R}^T\mathbf{a}(t), \qquad \mathbf{A}' = \mathbb{R}^T\mathbf{A}(t)\,, \tag{11}$$

evidently represent a rotation of both forces, the external $\mathbf{a}(t)$ and the fluctuating force $\mathbf{A}(t)$, respectively. We can observe that the second term of Eq. (10) can also be written as a cross product, so that $\mathbb{W}'\mathbf{u}' = (q/mc)\mathbf{u}'\times\mathbf{B}'$. In this case $\mathbf{B}'$ can be visualized as an external magnetic field pointing along the z' axis, that is $\mathbf{B}' = B'\,\hat{\mathbf{k}}'$, where B' has already been defined and $\hat{\mathbf{k}}'$ is the unitary vector along the z'-axis. Therefore, for any constant magnetic field pointing along any direction, it is possible to re-orientate, by means of rotation (6), that magnetic field along the z' direction, giving as a result the Langevin equation (10) which is then equivalent to Eq. (4).

So, Eqs. (9), (10) represent a coupled system of equations in the (x', y')-plane and independent of the z'-coordinate, for which the Langevin equation is the same as that of an ordinary Brownian motion in the presence of an external force $\mathrm{a}'_{z'}(t)$. In the following section, we shall solve the FPK equation associated with Eqs. (9), (10) and once this fundamental solution is determined, we can obtain the solution to the FP equation in the $\mathbf{u}'$ velocity-space, without needing to solve the FP equation explicitly.

3. The associated Fokker-Planck-Kramers equation

Equations (9), (10) can also be written as

$$\dot{\mathbf{r}}' = \mathbf{u}' , \tag{12}$$

$$\dot{\mathbf{u}}' = -\Lambda' \mathbf{u}' + \mathbf{a}'(t) + \mathbf{A}'(t) . \tag{13}$$

The associated FPK equation for the transition probability density $P'(\mathbf{r}', \mathbf{u}', t|\mathbf{u}'_0, \mathbf{r}'_0)$ of velocity $\mathbf{u}'$ and position $\mathbf{r}'$ at time t, given that $\mathbf{u}' = \mathbf{u}'_0$ and $\mathbf{r}' = \mathbf{r}'_0$ at time $t = 0$ is then [1, 9]

$$\frac{\partial P'}{\partial t} + \mathbf{u}' \cdot \mathrm{grad}_{\mathbf{r}'} P' + \mathbf{a}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = \mathrm{div}_{\mathbf{u}'}(\Lambda' \mathbf{u}' P') + \lambda \nabla^2_{\mathbf{u}'} P' \tag{14}$$

subject to the initial condition

$$P'(\mathbf{r}', \mathbf{u}', 0|\mathbf{u}'_0, \mathbf{r}'_0) = C_1\, \delta(\mathbf{r}' - \mathbf{r}'_0)\delta(\mathbf{u}' - \mathbf{u}'_0) , \tag{15}$$

with C_1 a constant. Just as in Ref. [1], the solution to Eq. (14) is not easy to calculate due to the coupling term given in the first term of the right-hand side. To proceed further we transform the Langevin equations (12) and (13) into another phase-space $(\mathbf{r}'', \mathbf{u}'')$ by means of the following change of variables:

$$\dot{\mathbf{r}}'' = \mathbf{u}'', \qquad \mathbf{u}'' = e^{-\mathbb{W}'t}\mathbf{u}' . \tag{16}$$

In this new velocity-space, the above Langevin equations reduce:

$$\dot{\mathbf{r}}'' = \mathbf{u}'' , \tag{17}$$

$$\dot{\mathbf{u}}'' = -\beta \mathbf{u}'' + \mathbf{a}''(t) + \mathbf{A}''(t) , \tag{18}$$

where

$$\mathbf{a}''(t) = \mathbb{R}'^{-1}(t)\mathbf{a}'(t), \qquad \mathbf{A}''(t) = \mathbb{R}'^{-1}(t)\mathbf{A}'(t) . \tag{19}$$

$\mathbb{R}'(t) = \mathrm{e}^{\mathbb{W}'t}$ is an orthogonal rotation matrix given by

$$\mathbb{R}'(t) = \begin{pmatrix} \cos\Omega' t & \sin\Omega' t & 0 \\ -\sin\Omega' t & \cos\Omega' t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

such that $\mathbb{R}'^{T}(t) = \mathbb{R}'^{-1}(t)$, *i.e.* the transpose is equal to its inverse; therefore $\mathbb{R}'^{-1}(t) = e^{-\mathbb{W}'t}$. The Langevin equations (47) and (18) display a strong resemblance to those associated with the ordinary Brownian motion in the presence of an external force $\mathbf{a}''(t)$, which in this case is nothing but a time-dependent rotation of the external force $\mathbf{a}'(t)$. Similarly, $\mathbf{A}''(t)$ accounts for a time-dependent rotation of the fluctuating force $\mathbf{A}'(t)$. It can be shown that the corresponding FPK equation for the transition probability density $P''(\mathbf{r}'', \mathbf{u}'', t|\mathbf{u}''_0, \mathbf{r}''_0)$ of the velocity $\mathbf{u}''_0$ and position $\mathbf{r}''_0$ at time t, given that $\mathbf{u}'' = \mathbf{u}''_0$ and $\mathbf{r}'' = \mathbf{r}''_0$ at time $t = 0$, is [1, 9]

$$\frac{\partial P''}{\partial t} + \mathbf{u}'' \cdot \mathrm{grad}_{\mathbf{r}''} P'' + \mathbf{a}'' \cdot \mathrm{grad}_{\mathbf{u}''} P'' = \beta\, \mathrm{div}_{\mathbf{u}''}(\mathbf{u}'' P'') + \lambda \nabla^2_{\mathbf{u}''} P'' , \tag{21}$$

together with the initial condition

$$P''(\mathbf{r}'', \mathbf{u}'', 0|\mathbf{u}''_0, \mathbf{r}''_0) \equiv \delta(\mathbf{u}'' - \mathbf{u}''_0)\delta(\mathbf{r}'' - \mathbf{r}''_0) . \tag{22}$$

The solution to Eq. (21) is very similar to that given in Appendix B of Ref. 1. Thus, if we define

$$P''(\mathbf{R}'', \mathbf{S}'') \equiv P''(\mathbf{r}'', \mathbf{u}'', t|\mathbf{u}''_0, \mathbf{r}''_0),$$

the fundamental solution to Eq. (21), with the initial condition (22), can be written as

$$P''(\mathbf{R}'', \mathbf{S}'') = \frac{1}{8\pi^3(FG - H^2)^{3/2}} \times \exp\left\{ - \frac{(F|\mathbf{S}'|^2 - 2H\, \mathbf{R}'' \cdot \mathbf{S}'' + G|\mathbf{R}''|^2)}{2(FG - H^2)} \right\} , \tag{23}$$

with the variables

$$\mathbf{S}'' = \mathbf{u}'' - e^{-\beta t}(\overline{\mathbf{a}''} + \mathbf{u}''_0) , \tag{24}$$

$$\mathbf{R}'' = \mathbf{r}'' - \mathbf{r}''_0 - \Gamma'' \mathbf{u}''_0 - \overline{\overline{\mathbf{a}''}} , \tag{25}$$

such that

$$\Gamma'' = \beta^{-1}\,(1 - e^{-\beta t}) , \tag{26}$$

$$\overline{\overline{\mathbf{a}''}}(t) = \int_0^t e^{-\Lambda' s}\, \overline{\mathbf{a}''}(s)\, ds , \tag{27}$$

$$\overline{\mathbf{a}''}(t) = \int_0^t e^{\beta s} \mathbf{a}''(s)\, ds . \tag{28}$$

The parameters F, G, and H are given by

$$F = \frac{\lambda}{\beta^3}(2\beta\, t - 3 + 4e^{-\beta t} - e^{-2\beta t}) , \tag{29}$$

$$G = \frac{\lambda}{\beta}(1 - e^{-2\beta t}) , \tag{30}$$

$$H = \frac{\lambda}{\beta^2}(1 - e^{-\beta t})^2 . \tag{31}$$

In a similar way to in Ref. 1, to return to the phase-space $(\mathbf{r}', \mathbf{u}')$, we introduce the variables $\mathbf{S}'$ and $\mathbf{R}'$ such that

$$\mathbf{S}' = \mathbf{u}' - e^{-\Lambda' t}(\overline{\mathbf{a}'} + \mathbf{u}'_0) \tag{32}$$

$$\mathbf{R}' = \mathbf{r}' - \mathbf{r}'_0 - \Gamma'\,\mathbf{u}'_0 - \overline{\overline{\mathbf{a}'}}\,, \tag{33}$$

where now

$$\Gamma' = \Lambda'^{-1}\,(1 - e^{-\Lambda' t})\,, \tag{34}$$

$$\overline{\overline{\mathbf{a}'}}(t) = \int_0^t e^{-\Lambda' s}\,\overline{\mathbf{a}'}(s)\,ds\,, \tag{35}$$

$$\overline{\mathbf{a}'}(t) = \int_0^t e^{\Lambda' s}\,\mathbf{a}'(s)\,ds\,, \tag{36}$$

and

$$\Lambda'^{-1} = \begin{pmatrix} \frac{\beta}{\Omega'^2+\beta^2} & \frac{\Omega'}{\Omega'^2+\beta^2} & 0 \\ -\frac{\Omega'}{\Omega'^2+\beta^2} & \frac{\beta}{\Omega'^2+\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{pmatrix}. \tag{37}$$

The transformation between $\mathbf{S}''$ and $\mathbf{S}'$, $\mathbf{R}''$ and $\mathbf{R}'$ can be established through the associated transformation between the variables $\mathbf{u}''$ and $\mathbf{u}'$, $\mathbf{r}''$ and $\mathbf{r}'$. According to Ref. 1, it can be shown that such transformations are given by

$$\mathbf{u}'' - e^{-\beta t}(\overline{\mathbf{a}''}(t) + \mathbf{u}''_0) = e^{-\mathbb{W}' t}[\mathbf{u}' - e^{-\Lambda' t}(\overline{\mathbf{a}'}(t) + \mathbf{u}'_0)]\,, \tag{38}$$

and

$$\mathbf{r}'' - \beta^{-1}\int_0^t \mathbf{a}''(s)\,ds - \mathbf{I}''_2 = \beta^{-1}\,e^{-\mathbb{W}' t}\,\Lambda'\left(\mathbf{r}' - \Lambda'^{-1}\int_0^t \mathbf{a}'(s)\,ds - \mathbf{I}'_2\right). \tag{39}$$

Therefore

$$\mathbf{S}'' = e^{-\mathbb{W}' t}\,\mathbf{S}', \qquad \mathbf{R}'' = e^{-\mathbb{W}' t}\,\Lambda'\,\mathbf{R}'\,. \tag{40}$$

So, the transformation between both P'' and P' is given by $P'\,d\mathbf{S}'d\mathbf{R}' = P''\,d\mathbf{S}''d\mathbf{R}''$, where the volume element transforms as $d\mathbf{S}'d\mathbf{R}' = J\,d\mathbf{S}''d\mathbf{R}''$. According to Eq. (40) it can be shown that

$$P' = \left(\frac{\beta^2 + \Omega'^2}{\beta^2}\right) P''\,. \tag{41}$$

The constant C_1, given in the initial condition (15), will then be equal to $C_1 = (\beta^2 + \Omega'^2)/\beta^2$.

To return to the phase-space $(\mathbf{r}', \mathbf{u}')$, let us denote the quantity $\widehat{\mathbf{x}'}$ as any vector in the $x'y'$-plane and $\widehat{P'}$ as the TPD describing the diffusion process in the same plane. So $\widehat{\mathbf{R}'} = (R'_1, R'_2)$ and $\widehat{\mathbf{S}'} = (S'_1, S'_2)$; R'_3 and S'_3 are the z'-components of vectors $\mathbf{R}'$ and $\mathbf{S}'$, respectively. Under these circumstances, it can shown that

$$|\mathbf{S}''|^2 = |\widehat{\mathbf{S}'}|^2 + S'^2_3\,, \tag{42}$$

$$|\mathbf{R}''|^2 = C_1\,|\widehat{\mathbf{R}'}|^2 + R'^2_3\,, \tag{43}$$

$$\mathbf{S}'' \cdot \mathbf{R}'' = \widehat{\mathbf{S}'} \cdot \widehat{\mathbf{R}'} + \frac{\Omega'}{\beta}(\widehat{\mathbf{S}'} \times \widehat{\mathbf{R}'})_{z'} + S'_3 R'_3\,, \tag{44}$$

where $(\widehat{\mathbf{S}'} \times \widehat{\mathbf{R}'})_{z'} = (S'_1 R'_2 - S'_2 R'_1)$ is the z'-component of the cross product and

$$\widehat{\mathbf{S}'} \equiv \widehat{\mathbf{u}'} - e^{-\widehat{\Lambda'} t}\,(\widehat{\overline{\mathbf{a}'}} + \widehat{\mathbf{u}'}_0)\,, \tag{45}$$

$$\widehat{\mathbf{R}'} \equiv \widehat{\mathbf{r}'} - \widehat{\mathbf{r}'}_0 - \widehat{\Gamma'}\,\widehat{\mathbf{u}'}_0 - \widehat{\overline{\overline{\mathbf{a}'}}}\,, \tag{46}$$

$$S'_3 \equiv u'_{z'} - e^{-\beta t}(\overline{\mathrm{a}'}_{z'} + u'_{0z'})\,, \tag{47}$$

$$R'_3 \equiv z' - z'_0 - \beta^{-1}(1 - e^{-\beta t})u'_{0z'} - \overline{\overline{\mathrm{a}'}}_{z'}\,. \tag{48}$$

$\widehat{\Lambda'}$ represents a 2×2 matrix and $\widehat{\Gamma'} = \widehat{\Lambda'}^{-1}\,(1 - e^{-\widehat{\Lambda'} t})$. Accordingly, if Eqs. (45)-(48) are substituted into Eq. (23), it can be shown that the solution to the FPK equation (14) can be written as the product of two independent TPD's, that is

$$P'(\mathbf{R}', \mathbf{S}') = \widehat{P'}(\widehat{\mathbf{R}'}, \widehat{\mathbf{S}'}) P'_{z'}(R'_3, S'_3)\,, \tag{49}$$

such that

$$\widehat{P'}(\widehat{\mathbf{R}'}, \widehat{\mathbf{S}'}) \equiv \widehat{\mathbf{P}'}(\widehat{\mathbf{r}'}, \widehat{\mathbf{u}'}, t|\widehat{\mathbf{r}'}_0, \widehat{\mathbf{u}'}_0)\,, \tag{50}$$

$$P'_{z'}(R'_3, S'_3) \equiv P'(z', u'_{z'}, t|z'_0, u'_{oz'}), \tag{51}$$

where

$$\widehat{P'}(\widehat{\mathbf{R}'}, \widehat{\mathbf{S}'}) = \frac{C_1}{4\pi^2(FG - H^2)} \exp\Big\{ - \big[F|\widehat{\mathbf{S}'}|^2 - 2H\,\widehat{\mathbf{R}'} \cdot \widehat{\mathbf{S}'} - 2\frac{\Omega'}{\beta} H(\widehat{\mathbf{S}'} \times \widehat{\mathbf{R}'})_{z'} + C_1 G|\widehat{\mathbf{R}'}|^2\big]/2(FG - H^2)\Big\} \tag{52}$$

is the planar TPD describing the diffusion process of a Brownian charged particle across the magnetic field under the action of a planar external force $\widehat{\mathbf{a}'}(t)$, with $\widehat{P'}(\widehat{\mathbf{r}'}, \widehat{\mathbf{u}'}, 0|\widehat{\mathbf{u}'}_0) = C_1\,\delta(\widehat{\mathbf{r}'} - \widehat{\mathbf{r}'}_0)\delta(\widehat{\mathbf{u}'} - \widehat{u'}_0)$ as the initial condition. The TPD $P'_{z'}$, being equal to

$$P'_{z'}(R'_3, S'_3) = \frac{1}{[4\pi^2(FG - H^2)]^{1/2}} \times \exp\Big\{-\frac{(FS'^2_3 - 2HR'_3S'_3 + GR'^2_3)}{2(FG - H^2)}\Big\}, \tag{53}$$

describes the diffusion process along the z'-axis, *i.e.* parallel to the magnetic field, in the presence of an external force $\mathrm{a}'_{z'}(t)$ with the initial condition

$$P'_{z'}(z', u'_{z'}, 0|z'_0, u'_{0z'}) = \delta(z' - z'_0)\delta(u'_{z'} - u'_{0z'}).$$

This TPD is the same as that of ordinary Brownian motion in the presence of an external force $a'_{z'}(t)$ without the influence of the magnetic field, as expected.

The immediate consequences of Eqs. (52) and (53) are the following:

(a) The planar velocity-space fundamental solution $\widehat{P'}(\widehat{\mathbf{S}'})$ and the fundamental solution $P'_{z'}$ can be calculated from the integrals

$$\widehat{P'}(\widehat{\mathbf{S}'}) = \int \widehat{P'}(\widehat{\mathbf{R}'}, \widehat{\mathbf{S}'})\, d\widehat{\mathbf{R}'}\,, \tag{54}$$

$$P'_{z'}(S'_3) = \int P'_{z'}(R'_3, S'_3)\, dR'_3\,. \tag{55}$$

The former integral yields

$$\widehat{P'}(\widehat{\mathbf{S}'}) = \frac{\beta}{2\pi\lambda(1-e^{-2\beta t})} \exp\left\{-\frac{\beta\,|\widehat{\mathbf{S}'}|^2}{2\,\lambda(1-e^{-2\beta t})}\right\} \tag{56}$$

and the latter

$$P'_{z'}(S'_3) = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{1/2} \exp\left\{-\frac{\beta S'^2_3}{2\lambda(1-e^{-2\beta t})}\right\} \tag{57}$$

so that the product of Eqs. (56) and (57) leads to the same fundamental solution to the Fokker-Planck equation in the $\mathbf{u}'$ velocity-space, that is

$$P'(\mathbf{S}') \equiv P'(\mathbf{r}', \mathbf{u}', t|\mathbf{r}'_0, \mathbf{u}'_0) = \widehat{P'}(\widehat{\mathbf{S}'}) P'_{z'}(S'_3),$$

such that

$$P'(\mathbf{S}') = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{3/2} \exp\left\{-\frac{\beta|\mathbf{S}'|^2}{2\,\lambda(1-e^{-2\beta t})}\right\}. \tag{58}$$

(b) Similarly, in the configuration space $\mathbf{r}'$ the planar-spatial TPD $\widehat{P'}(\widehat{\mathbf{R}'})$ and spatial TPD $P'_z(R'_3)$ defined by

$$\widehat{P'}(\widehat{\mathbf{R}'}) \equiv \widehat{P'}(\widehat{\mathbf{r}'}, t|\widehat{\mathbf{r}'}_0, \widehat{\mathbf{u}'}_0) \tag{59}$$

$$P'_{z'}(R'_3) \equiv P'(z', t|z'_0, u'_{0z'})\,, \tag{60}$$

can be calculated through the integrals

$$\widehat{P'}(\widehat{\mathbf{R}'}) = \int \widehat{P'}(\widehat{\mathbf{S}'}, \widehat{\mathbf{R}'})\, d\widehat{\mathbf{S}'} \tag{61}$$

$$P'_{z'}(R'_3) = \int P'_{z'}(R'_3, S'_3)\, dS'_3\,. \tag{62}$$

After a long but straightforward algebra they reduce to

$$\widehat{P'}(\widehat{\mathbf{r}'}, t|\widehat{\mathbf{r}'}_0, \widehat{\mathbf{u}'}_0) = \frac{\beta}{2\pi\, D'_e(2\beta\, t - 3 + 4e^{-\beta t} - e^{-2\beta t})} \times \exp\left\{-\frac{\beta|\widehat{\mathbf{r}'} - \widehat{\mathbf{r}'}_0 - \widehat{\Lambda}'^{-1}(1 - e^{-\widehat{\Lambda}' t})\widehat{\mathbf{u}'}_0 - \widehat{\overline{\overline{\mathbf{a}'}}}|^2}{2\, D'_e(2\beta\, t - 3 + 4e^{-\beta t} - e^{-2\beta t})}\right\}, \tag{63}$$

where $D'_e = D\,\beta^2/(\Omega'^2 + \beta^2)$ represents a rescaling of the Einstein diffusion constant ($D = \lambda/\beta^2 = k_B T/m\beta$) and

$$P'_{z'}(R'_3) = \frac{1}{[2\pi D(2\beta\, t - 3 + 4e^{-\beta t} - e^{-2\beta t})/\beta]^{1/2}} \times \exp\left\{-\frac{\beta\,(z' - z'_0 - \beta^{-1}(1-e^{-\beta t})u'_{0z} - \overline{\overline{a'}}_{z'})^2}{2D(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})}\right\}. \tag{64}$$

Clearly, the configurational fundamental solution is $P'(\mathbf{r}', t|\mathbf{r}'_0, \mathbf{u}'_0) = \widehat{P'}(\widehat{\mathbf{R}'}) P'_{z'}(R'_3)$.

(c) For any other initial velocity distribution $f'(\mathbf{u}', 0)$, a more general probability distribution, corresponding to the solution to the FP equation, can be calculated through the following integration

$$f'(\mathbf{u}', t) = \int_{\mathbf{u}'_0} f'(\mathbf{u}'_0, 0)\, P'(\mathbf{u}', t|\mathbf{u}'_0)\, d\mathbf{u}'_0\,. \tag{65}$$

If we choose, in the original $\mathbf{u}$ velocity-space, the particular case of an initial Maxwellian velocity distribution of the charged Brownian particle at a temperature T_0 different from the equilibrium temperature T, and with an assigned mean velocity $\langle \mathbf{u} \rangle_0$, then

$$f(\mathbf{u}, 0) = \left(\frac{m}{2\pi\, k_B T_0}\right)^{3/2} \exp\left\{-\frac{m|\mathbf{u} - \langle \mathbf{u} \rangle_0|^2}{2k_B T_0}\right\}. \tag{66}$$

The initial distribution for the transformed $\mathbf{u}'$ velocity-space can be constructed using the transformation (5). In this case it can be shown that $|\mathbf{u} - \langle \mathbf{u} \rangle_0|^2 = |\mathbf{u}' - \langle \mathbf{u}' \rangle_0|^2$ and therefore the initial distribution in the transformed space has exactly the same form as Eq. (66), but with the velocity $\mathbf{u}$ replaced by $\mathbf{u}'$. So, a more general solution to the FP equation after integration of Eq. (65) will be

$$f'(\mathbf{u}', t) = \left(\frac{m}{2\pi\, k_B T_t}\right)^{3/2} \times \exp\left\{\frac{m\,|\mathbf{u}' - e^{-\Lambda' t}(\overline{\mathbf{a}'}(t) + \langle \mathbf{u}' \rangle_0)\,|^2}{2k_B T_t}\right\}, \tag{67}$$

where

$$T_t = T\left[1 - \left(1 - \frac{T_0}{T}\right)e^{-2\beta t}\right]. \tag{68}$$

We note that, as $T_0 \to 0$, $f(\mathbf{u}', 0) = \delta(\mathbf{u}' - \mathbf{u}'_0)$ with $\mathbf{u}'_0 = \langle \mathbf{u}' \rangle_0$ and temperature $T_t = T(1 - e^{-2\beta t})$. In this case $f(\mathbf{u}', t)$ reduces to the fundamental solution (58). On the other hand, if $B_x = 0$ and $B_y = 0$, then the Larmor frequency is $\Omega' = qB/mc$, with B being the modulus of the magnetic field $\mathbf{B} = (0, 0, B)$. In this case, Eq. (67) reduces to the same probability density obtained by Ferrari [8] by another method. A more general solution for $f(\mathbf{r}, \mathbf{u}, t)$ is still under study.

4. Conclusions

For a constant magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ it is possible to show, by means of transformation (6), that the Langevin equations (1), (2) formulated in the $(\mathbf{r}, \mathbf{u})$ phase-space are equivalent to those given by Eqs. (9), (10) in the $(\mathbf{r}', \mathbf{u}')$ phase-space. It is also shown that in the transformed space $\mathbf{r}'$, the magnetic field is visualized as another vector pointing along the z'-axis of the Cartesian reference frame, *i.e.*, $\mathbf{B}' = (0, 0, B')$, where $B'^2 = B_1^2 + B_2^2 + B_3^2 = B^2$; that is, the modulus of the transformed magnetic field is equal to that of the original magnetic field. The equivalence between the two sets of equations is also due to the fact that the noise term $\mathbf{A}'(t)$ has the same statistical properties of GWN as the original noise $\mathbf{A}(t)$.

To solve the FPK equation in the transformed $(\mathbf{r}', \mathbf{u}')$ phase-space, a second transformation given by Eq. (16) was required. In this case, we observe in Eq. (18) that the effects of the magnetic field B' are transferred to both the external $\mathbf{a}'(t)$ and fluctuating $\mathbf{A}'(t)$ forces. Eqs. (17) and (18) have the same algebraic structure as that of the ordinary Brownian motion where we have shown that the fluctuating force $\mathbf{A}''(t)$ has the same statistical properties of GWN as $\mathbf{A}'(t)$. In the $(\mathbf{r}'', \mathbf{u}'')$ phase-space, the solution to the FPK equation is found immediately. To return to $(\mathbf{r}', \mathbf{u}')$ phase-space we use the transformation (40) to obtain the fundamental FPK solution given by Eq. (49). From the latter we can calculate the fundamental FP solution as shown in Eq. (58).

Eq. (67) is a more general solution to the FP equation. It has been calculated by assuming a Maxwellian initial distribution function, such as that given by Eq. (66). A more general solution to the FPK equation has not been found yet.

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