Mathematics motivated by physics: the electrostatic potential is the Coulomb integral transform of the electric charge density

L. Medina[a] and E. Ley Koo [a,b]
[a]*Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, México D.F., 04510, México, e-mail: lumg@fciencias.unam.mx,*
[b]*Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 Mexico D.F., Mexico, e-mail: eleykoo@fisica.unam.mx*

Recibido el 19 de octubre de 2007; aceptado el 11 de marzo de 2007

This article illustrates a practical way to connect and coordinate the teaching and learning of physics and mathematics. The starting point is the electrostatic potential, which is obtained in any introductory course of electromagnetism from the Coulomb potential and the superposition principle for any charge distribution. The necessity to develop solutions to the Laplace and Poisson differential equations is also recognized, identifying the Coulomb potential as the generating function of harmonic functions. Correspondingly, the convenience of expressing the electrostatic potential in terms of its multipole expansion in spherical coordinates, or as integral transforms based on harmonic functions in different coordinate systems, is also established. These connections provide a motivation for teachers and students to acquire the necessary mathematics as a basic tool in the study of electromagnetic theory, optics and quantum mechanics.

Keywords: Electrostatics; Laplace and Poisson equations; spherical and circular cylindrical Harmonic functions.

Este artículo ilustra una manera práctica de conectar y coordinar la enseñanza y aprendizaje de la física y las matemáticas. El punto de partida es el potencial electrostático que se obtiene en el curso introductorio de electromagnetismo a partir del potencial de Coulomb y del principio de superposición para cualquier distribución de carga. También se reconoce la necesidad de construir soluciones de las ecuaciones diferenciales de Laplace y de Poisson, identificando al potencial de Coulomb como una función generadora de funciones armónicas. Correspondientemente, también se reconoce la conveniencia de expresar al potencial electrostático en términos de su desarrollo multipolar en coordenadas esféricas, o de transformadas integrales basadas en funciones armónicas en diferentes sistemas de coordenadas. Estas conexiones proporcionan una motivación para maestros y alumnos para adquirir las matemáticas necesarias como una herramienta básica en el estudio de la teoría electromagnética, la óptica y mecánica cuántica.

Descriptores: Electrostática; ecuaciones diferenciales de Laplace y de Poisson; funciones armónicas esféricas y cilíndricas circulares.

PACS: 41.20.Cv1

1. Introduction

Mathematics and physics have always been closely interwoven in a two-way process. The former is not only the language of the latter; in addition, it often determines to a large extent the content and meaning of physical concepts and theories themselves. Consequently, progress in the study of fundamental physics increasingly depends on the availability of new mathematical tools. It is a well-known fact that there has been a close interrelationship between mathematics and physics throughout their historical development. Modern mathematics and physics were born in the 17th century through Newton's formulation of the laws of mechanics and the invention of the infinitesimal calculus [1-2]. Newton changed the face of scientific research by placing the full force of mathematics at the service of physical enquiry, becoming a unique example of coordination in invention and discovery by a single individual. In contrast, Einstein had to learn Riemannian geometry in order to formulate the theory of general relativity [3], while Born recognized the matrix mathematics behind Heisenberg's formulation of quantum mechanics [4].

This article addresses the general problem of connecting and coordinating the study of physics and mathematics in different areas and on different levels. Specific facets of the problem have been explored in [5] and in a series of dialogues under the title of ”Mathematics motivated by physics” [6]. Emphasis was placed on the construction of mathematical bridges to make the transition from introductory courses in mechanics, fluids, thermodynamics, electromagnetism and quantum mechanics to their junior/senior/graduate level counterparts. The second half of the title of this manuscript blends the physical and mathematical elements to guide colleagues and students in their respective tasks of teaching and learning electrostatics, identifying and constructing the appropriate mathematical tools.

The starting points are the physical laws of electrostatics expressed in their integral and differential equation forms, reviewed in Sec. 2. Section 3 is devoted to the solutions to the Laplace equation in some illustrative coordinate systems, corresponding to the so-called harmonic function bases. In Sec. 4, the harmonic function expansions of the Coulomb potential in spherical and circular cylindrical coordinates are comparatively analyzed, contrasting their discrete and continuous natures, respectively. The physical and mathematical elements identified in Secs. 3 and 4 are the basis for representing the electrostatic potential as harmonic function expansions in Sec. 5, characterizing them according to the specific coordinates involved. Section 6 contains discussions of the extensions to other coordinates, and to other areas of electromagnetism.

2. The laws of electrostatics

Coulomb's law describes the radial and inverse-square of the distance force between two electrically charged point particles [7-12]:

$$\overrightarrow{F}_{1\rightarrow 2} = k_e\, q_1\, q_2 \frac{(\overrightarrow{r_1} - \overrightarrow{r_2})}{|\overrightarrow{r_1} - \overrightarrow{r_2}|^3} \tag{1}$$

The superposition principle applies in electrostatics, and states that the force of a collection of charges on a test charge is the vector sum of Coulomb forces:

$$\overrightarrow{F}_{\{\overrightarrow{q}_i,\overrightarrow{r_i}\}\rightarrow(\overrightarrow{q},\overrightarrow{r})} = \sum_{i=1}^{N} \overrightarrow{F_i}_{\{\overrightarrow{q}_i,\overrightarrow{r_i}\}\rightarrow(\overrightarrow{q},\overrightarrow{r})}$$

$$= k_e \sum_{i=1}^{N} q_i q \frac{(\overrightarrow{r} - \overrightarrow{r_i})}{|\overrightarrow{r} - \overrightarrow{r_i}|^3} \tag{2}$$

In the case in which the collection of charges is continuously distributed in a volume *V*, so that the charge element associated with a differential volume is $dq' = \rho(\overrightarrow{r'})d^3V'$, where $\rho(\overrightarrow{r'})$ is the charge volume density, the sum in Eq. (2) becomes an integral:

$$\overrightarrow{F}_{\{\rho,V\}\rightarrow(q,\overrightarrow{r})} = k_e q \int_V \rho(\overrightarrow{r'}) \frac{\left(\overrightarrow{r} - \overrightarrow{r'}\right)}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|^3} dV' \tag{3}$$

Since the forces in Eqs. (2) and (3), are proportional to the magnitude of the charge *q*, it is possible to identify

$$\overrightarrow{E}(\overrightarrow{r}) = \frac{\overrightarrow{F}}{q} = k_e \int_V \rho(\overrightarrow{r'}) \frac{\left(\overrightarrow{r} - \overrightarrow{r'}\right)}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|^3} dV' \tag{4}$$

as the electric intensity field produced by the charge distribution $\{\rho, V\}$.

Gauss' law of electrostatics follows from the evaluation of the flux integral of Eq. (4), becoming

$$\oint_S \overrightarrow{E} \cdot d\overrightarrow{a} = 4\pi k_e \int_V \rho(\overrightarrow{r'}) dV' = 4\pi k_e Q \tag{5}$$

where *S* is the closed surface bounding the volume *V*, and *Q* is the electric charge contained inside that volume. The differential equation form of Gauss' law is obtained by applying Gauss' flux or divergence theorem to Eq. (5):

$$\bigtriangledown \cdot \overrightarrow{E}(\overrightarrow{r}) = 4\pi k_e \rho(\overrightarrow{r}) \tag{6}$$

The Coulomb force of Eq. (1) is conservative and can be written as the negative of the gradient of the so-called Coulomb potential energy,

$$\overrightarrow{F}_{1\rightarrow 2} = -\bigtriangledown \left(k_e\, q_1\, q_2 \frac{1}{|\overrightarrow{r_1} - \overrightarrow{r_2}|} \right) = -\bigtriangledown (U_{12}) \tag{7}$$

Correspondingly, the electric intensity field of Eq. (4) can be written as

$$\overrightarrow{E} = -k_e \bigtriangledown \left(\int_V \frac{\rho(\overrightarrow{r'})}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|} dV' \right) = -\bigtriangledown \phi(\overrightarrow{r}) \tag{8}$$

where

$$\phi(\overrightarrow{r}) = k_e \int_V \frac{\rho(\overrightarrow{r'})}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|} dV' \tag{9}$$

is the electrostatic potential produced by the charge distribution $\{\rho,V\}$. This is the quantity giving rise to the second half of the title in this manuscript. The reader can appreciate the physical and mathematical elements in it. In Sec. 5, several alternative mathematical representations are introduced explicitly.

If the line integral of electric intensity field is evaluated using Eq. (8),

$$\int_{r_i}^{r} \overrightarrow{E} \cdot d\overrightarrow{r} = -\int_{r_i}^{r} \bigtriangledown\phi \cdot d\overrightarrow{r} = -\phi(\overrightarrow{r}) + \phi(\overrightarrow{r}_i) \tag{10}$$

the result depends only on the initial and final points of the integration path, and is independent of the path chosen. When the path is closed, and therefore both points coincide, the closed-line integral vanishes. Also, the curl of Eq. (8) vanishes:

$$\bigtriangledown \times \overrightarrow{E}(\overrightarrow{r}) = 0 \tag{11}$$

Equations (7)-(11) are different forms of expressing the conservative character of the electrostatic field. While Eq. (4) indicates that $\overrightarrow{E}(\overrightarrow{r})$ is the force per unit charge at a given position, Eqs. (7) and (8) point out that $\phi(\overrightarrow{r})$ is the potential energy per unit charge.

The reader may also inquire about the equation that $\phi(\overrightarrow{r})$ must satisfy. The answer follows from the substitution of Eq. (8) into Eq. (6) with the result:

$$\bigtriangledown^2\phi(\overrightarrow{r}) = -4\pi k_e \rho(\overrightarrow{r}), \tag{12}$$

the so-called Poisson equation, involving the Lapacian or Laplace operator $\bigtriangledown^2$.

At the points where there is no charge, Eq. (12) reduces to the so-called Laplace equation,

$$\bigtriangledown^2\phi(\overrightarrow{r}) = 0 \tag{13}$$

3. Harmonic functions in spherical and circular cylindrical coordinates

The solutions to the Laplace equation are called harmonic functions. Here we review the explicit forms of the equation in spherical and circular cylindrical coordinates, illustrating their separation and integration leading to the respective spherical harmonics and circular cylindrical harmonics [13,14]. In fact, the Laplace equation in the familiar spherical (r,ϑ,φ) and circular cylindrical (R,ϑ,φ) coordinates has the respective forms:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]\right\}\phi(r,\vartheta,\varphi)=0 \quad (14)$$

$$\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]\phi(R,\vartheta,z)=0 \quad (15)$$

In both cases, the equation is satisfied by separable solutions in the following forms:

$$\phi(r,\vartheta,\varphi)=f(r)\theta(\vartheta)\Phi(\varphi) \quad (16)$$

$$\phi(R,\vartheta,z)=g(R)\Phi(\varphi)Z(z) \quad (17)$$

The successive factors satisfy the ordinary differential equations

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi \quad (18)$$

$$\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}-\frac{m^2}{\sin^2\vartheta}\right]\theta(\vartheta)=-l(l+1)\theta(\vartheta) \quad (19)$$

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]f(r)=0 \quad (20)$$

$$\frac{d^2Z}{dz^2}=k^2Z \quad (21)$$

$$\left[\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}-\frac{m^2}{\rho}\right]g(R)=-k^2g(R) \quad (22)$$

where $-m^2$, $-l(l+1)$, and k^2 are the respective separation constants. Notice that Eqs. (18)-(22) have the form of eigenvalue equations, involving second-order differential operators which, upon application to the eigenfunction, lead to the same function multiplied by a constant, called the eigenvalue. The angular coordinate φ and the associated Eq. (18) are common to both cases.

The solutions are chosen to those of periodic $\Phi(\varphi+2\pi)=\Phi(\varphi)$, restricting the values of the separation constant to be an integer $m=0,\pm1,\pm2,\ldots$

The set of such solutions,

$$\Phi_m(\varphi)=\frac{e^{(im\varphi)}}{\sqrt{2\pi}} \quad (23)$$

is the well-known Fourier basis. The real and imaginary parts $\cos(m\varphi)$ and $\sin(m\varphi)$ correspond to the alternative even and odd (under $\varphi\to-\varphi$) Fourier bases. The orthonormality and completeness properties of the basis are expressed, by

$$\int_0^{2\pi}\frac{e^{(in'\varphi)}e^{(in\varphi)}}{2\pi}d\varphi=\delta_{\mathrm{n',n}} \quad (24)$$

$$\sum_{n=-\infty}^{\infty}\frac{e^{(in\varphi)}e^{(in\varphi')}}{2\pi}=\delta(\varphi-\varphi') \quad (25)$$

respectively, in terms of the Kronecker-delta symbol and the Dirac-delta function, where the latter is zero for $\varphi\neq\varphi'$, infinity for $\varphi=\varphi'$, and its integral is one in the integration interval includes the value $\varphi=\varphi'$.

The polar coordinate $0\leq\vartheta\leq\pi$ and its associated Eq. (19) determine the singularities at $\vartheta=0$ and π, or $\cos(\vartheta)=1$ and -1. Eq. (19) is known as the associated Legendre equation; its regular solutions are the associated Legendre polynomials

$$\theta(\vartheta)=P_\ell^{|m|}(\cos\vartheta)=(\sin\vartheta)^{|m|}\frac{d^{|m|}P_\ell(\cos\vartheta)}{d(\cos\vartheta)^{|m|}} \quad (26)$$

of degree $\ell-|m|$, where $\ell=0,1,2,\ldots$ and $P_\ell(\cos\vartheta)$ are the ordinary Legendre polynomials of degree ℓ. They also have a well-defined parity $(-1)^{\ell-|m|}$ under $\vartheta\to\vartheta-\pi$ or $\cos\vartheta\to-\cos\vartheta$.

The products of the angular functions

$$Y_{lm}(\vartheta,\varphi)=N_{lm}P_\ell^{|m|}(\cos\vartheta)\exp(im\varphi) \quad (27)$$

are known as the angular spherical harmonics, where N_{lm} is a normalization factor such that

$$\int_0^{\pi}\int_0^{2\pi}Y^\star_{\ell m}(\vartheta,\varphi)Y_{\ell m}(\vartheta,\varphi)\sin\vartheta d\varphi d\vartheta=1 \quad (28)$$

It is known that the Fourier basis in Eq. (23) is orthonormal. In a similar way, the angular spherical harmonic basis of Eq. (27) is orthonormal

$$\int_0^{\pi}\int_0^{2\pi}Y^\star_{\ell'm'}(\vartheta,\varphi)Y_{\ell m}(\vartheta,\varphi)\sin\vartheta d\varphi d\vartheta=\delta_{\ell',\ell}\delta_{m',m} \quad (29)$$

and complete

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y^\star_{\ell'm'}(\vartheta',\varphi')Y_{\ell m}(\vartheta,\varphi)=\frac{\delta(\varphi-\varphi')\delta(\vartheta-\vartheta')}{\sin\vartheta'} \quad (30)$$

The completeness property implies that any quadratically integrable function of ϑ and φ can be expanded in terms of such a basis,

$$F(\vartheta,\varphi)=\sum_\ell\sum_m a_{\ell m}Y_{\ell m}(\vartheta,\varphi) \quad (31)$$

where the expansion coefficients follow from Eq. (29):

$$a_{\ell m} = \int_0^{\pi}\int_0^{2\pi} Y^{\star}_{\ell m}(\vartheta,\varphi)F(\vartheta,\varphi)\sin\vartheta d\varphi d\vartheta \tag{32}$$

The radial Eq. (20) is satisfied by two independent power solutions

$$f(r) = Ar^{\ell} + Br^{-(\ell+1)} \tag{33}$$

for each given value of ℓ. The positive powers are regular at the origin and diverge as $r \to \infty$, and the inverse powers diverge at the origin and tend to zero asymptotically.

The products $r^{\ell}Y_{\ell m}(\vartheta,\varphi)$ are known as solid spherical harmonics. The axial coordinate and its associated Eq. (21) lead to the exponential solutions

$$Z(z) = Ce^{(kz)} + De^{(-kz)} \tag{34}$$

which are regular and singular at $z \to -\infty$, respectively, and the other way around at $z \to \infty$.

The corresponding solutions of Eq. (22) in the circular radial coordinate are the ordinary Bessel functions

$$g(R) = EJ_m(kR) + FN_m(kR) \tag{35}$$

of the first and second kind, respectively. They are regular and singular at the origin $R \to 0$,

$$J_0(kR) \to 1, \qquad N_0(kR) \to \frac{2}{\pi}\ln(kR) \tag{36}$$

$$J_m(kR) \to \frac{(kR)^{|m|}}{2^{|m|}\,|m|!},$$

$$N_m(kR) \to -2^{|m|}\frac{(m-1)!}{\pi}(kR)^{-|m|} \tag{37}$$

oscillate for intermediate and larger values of their argument, and their asymptotic behavior is of the form

$$J_m(kR) = \sqrt{\frac{2}{\pi}}\frac{\cos\left(kR - \left(m+\frac{1}{2}\right)\frac{\pi}{2}\right)}{\sqrt{kR}},$$

$$N_m(kR) \to \sqrt{\frac{2}{\pi}}\frac{\sin\left(kR - \left(m+\frac{1}{2}\right)\frac{\pi}{2}\right)}{\sqrt{kR}} \tag{38}$$

The ordinary Bessel functions are quadratically integrable and form an orthonormal and complete set of functions:

$$\int_0^{\infty} J_m(k'R)J_m(kR)RdR = \frac{\delta(k-k')}{k'} \tag{39}$$

$$\int_0^{\infty} J_m(kR)J_m(kR')kdk = \frac{\delta(R-R')}{R'} \tag{40}$$

In Eqs. (21) and (22) the separation constant could be chosen with the opposite sign, which we express by the analytical continuation $k \to i\kappa$. Then, instead of Eqs. (34) and (35), the solutions become

$$Z(z) = Ce^{(i\kappa z)} + De^{(-i\kappa z)} \tag{41}$$

$$g(R) = EI_m(\kappa R) + FK_m(\kappa R) \tag{42}$$

respectively, involving the Fourier basis in longitudinal coordinates, and the modified Bessel functions in radial coordinates. The latter are monotonically increasing and decreasing as their arguments change from $\kappa R = 0$ to $\kappa R \to \infty$.

In conclusion, the general solutions to the Laplace Eqs. (14) and (15) can be written as a superposition of spherical harmonics, from Eqs. (27) and (33),

$$\phi(R,\vartheta,z) = \sum_{\ell}\sum_{m}\left(a_{\ell}r^{\ell} + b_{\ell}r^{-(\ell+1)}\right)Y_{\ell m}(\vartheta,\varphi) \tag{43}$$

or circular cylindrical harmonics,

$$\phi(R,\varphi, = \sum_m e^{(im\varphi)}\left\{\int_0^{\infty}\left[a_{mk}J_m(kR) + b_{mk}N_m(kR)\right]\right.$$
$$\left.\times\left[c_k e^{(kz)} + d_k e^{(-kz)}\right]kdk\right\} \tag{44}$$

from Eqs. (23), (30) and (33), or

$$\phi(R,\varphi,z) = \sum_m e^{(im\varphi)}\left\{\int_0^{\infty}\left[a_{mk}I_m(\kappa R) + b_{mk}N_m(\kappa R)\right]\right.$$
$$\left.\times\left[c_{\kappa}e^{(i\kappa z)} + d_k e^{(-i\kappa z)}\right]\kappa d\kappa\right\} \tag{45}$$

using Eqs. (41) and (42).

The potential in Eq. (43) is expressed as the spherical harmonic multipole expansion in which the terms with each value of ℓ are designated as 2^{ℓ} order poles [12]. The potential in Eqs. (44) and (45) are expressed as Fourier series in the angular coordinate φ. Besides, Eq. (44) involves a Laplace-ordinary Bessel integral transform, and Eq. (45) involves a Fourier-modified Bessel integral transform in the longitudinal and radial coordinates, respectively. The expansion coefficients in Eqs. (43) - (45) are to be determined by the boundary conditions according to each specific application.

4. Harmonic expansions of the Coulomb potential

In order to solve the Poisson differential equation, Eq. (12), for any source distribution with electric charge density $\rho(\overrightarrow{r})$, it is sufficient to use the Green function technique. The Green function $G(\overrightarrow{r},\overrightarrow{r}')$ satisfies the Poisson equation for a unit electric point charge located at the position $\overrightarrow{r}'$:

$$\nabla^2 G(\overrightarrow{r},\overrightarrow{r}') = -4\pi\delta(\overrightarrow{r}-\overrightarrow{r}') \tag{46}$$

where the Dirac-delta function represents appropriately the electric charge density of the chosen source. Its volume integral that contains the point where the charge is located is

$$\int \delta(\vec{r} - \vec{r}\,')dV' = 1 \tag{47}$$

The solution to the Poisson Eq. (12) is given by the integral of the product of the electric charge density and the Green function:

$$\phi(\vec{r}) = \int \rho(\vec{r}\,')G(\vec{r}, \vec{r}\,')dV' \tag{48}$$

If the Laplace operator is applied to both sides of the last equation, the result is

$$\nabla^2\phi(\vec{r}) = \int \rho(\vec{r}\,')\nabla^2 G(\vec{r}, \vec{r}\,')dV'$$
$$= -4\pi \int \rho(\vec{r}\,')\delta(\vec{r} - \vec{r}\,')dV' = -4\pi\rho(\vec{r}) \tag{49}$$

where use is made of Eq. (46) and of the integration with the Dirac-delta function, showing that $\phi(\vec{r})$ is indeed a solution to Eq. (12).

On the other hand, the potential of the unit point charge is simply the Coulomb potential, with which the Green function can be identified:

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} \tag{50}$$

Then the reader can identify Eq. (50) with the Coulomb integral transform of the charge density already discussed in connection with Eq. (9), in Sec. 2.

The Coulomb potential, or inverse of the distance between the source point and the field point, is known to be the generating function of the spherical harmonics [13, 14]

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'(\hat{r}\cdot\hat{r'})}} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P(\hat{r}\cdot\hat{r'})$$
$$= \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}^\star(\vartheta', \varphi')Y_{\ell m}(\vartheta, \varphi) \tag{51}$$

where $r_<$ and $r_>$ represent the smaller and larger of r and r'. Expansion of the -1/2 power of powers $(r_</r_>)$ of the trinomial generates the Legendre polynomials with argument

$$\hat{r}\cdot\hat{r'} = \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi') + \cos\vartheta \cos\vartheta' \tag{52}$$

and the latter in turn generate the spherical harmonics, via the so-called addition theorem involving the sum over *m*.

The inverse of the distance in Eq. (9) is also the generating function of the circular cylindrical harmonics [15]:

$$\frac{1}{\sqrt{R^2 + R'^2 - 2RR'\cos(\varphi - \varphi') + (z - z')^2}} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \int_0^\infty J_m(kR)J_m(kR')e^{-k(z_> - z_<)}dk \tag{53}$$

$$\frac{1}{\sqrt{R^2 + R'^2 - 2RR'\cos(\varphi - \varphi') + (z - z')^2}} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \int_0^\infty I_m(\kappa R_<)K_m(\kappa R_>)e^{-i\kappa(z-z')}d\kappa \tag{54}$$

Notice that the spherical and circular cylindrical harmonic expansions of Eqs. (51), (53) and (54), are particular cases of the general solutions to the Laplace Equation described by Eqs. (43), (44) and (45) in Sec. 3. Notice also, that the Poisson Eq. (12) reduces to the Laplace equation for all points $r \neq r'$, making possible the harmonic expansions of the Coulomb potential.

5. Harmonic expansions of the electrostatic potential

Substitution of the harmonic expansions of the Coulomb potential Eqs. (51), (53) and (54) into Eq. (9) leads to the respective harmonic expansions of the electrostatic potential:

$$\phi(r,\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \int_0^\infty dr' \int_0^\pi \sin\vartheta' d\vartheta' \int_0^{2\pi} d\varphi' \rho(r',\vartheta',\varphi') \frac{r_<^\ell}{r_>^{\ell+1}} r'^2 Y_{\ell m}^\star(\vartheta',\varphi')Y_{\ell m}(\vartheta,\varphi) \tag{55}$$

$$\phi(R,\vartheta,z) = \frac{1}{\pi} \sum_{m=0}^{\infty} (2-\delta_{m,0}) \int_0^\infty dk J_m(kR) \int_0^\infty dR' J_m(kR')R' \int_0^{2\pi} d\varphi' \cos m(\varphi-\varphi') \int_{-\infty}^{\infty} dz' \rho(R',\varphi',z')e^{-k(z_> - z_<)} \tag{56}$$

$$\phi(R,\vartheta,z) = \frac{1}{\pi} \sum_{m=0}^{\infty} (2-\delta_{m,0}) \int_0^\infty d\kappa \int_0^\infty dR' I_m(\kappa R_<)K_m(\kappa R_>)R' \int_0^{2\pi} d\varphi' \cos m(\varphi-\varphi') \int_{-\infty}^{\infty} dz' \rho(R',\varphi',z') \cos\kappa(z-z') \tag{57}$$

Equation (55) gives the familiar spherical multipole expansion of the electrostatic potential, described in detail in Ref. 14 for the regions inside and outside the region where the sources are located. It must be pointed out that in most books, the study of this type of expansion is limited only to the region far from the sources. The expansions in circular cylindrical harmonics of Eqs. (56) and (57) involve Fourier series in angular coordinates, and integral transforms of the Laplace-ordinary Bessel and of the Fourier-modified Bessel types in longitudinal and radial coordinates, respectively.

Integrations of the charge density with harmonic functions over the primed coordinates provide the coefficients in the complementary harmonic expansion of $\rho(\overrightarrow{r})$. The steps followed from Eq. (9) of Sec. 2 passing through Secs. 3 and 4 and ending in Eqs. (55) - (57) of the present section provide the bridge for arriving at the mathematical integral transforms motivated by the physical integral transform of the electrostatic potential.

6. Discussion

The contents of this article, as anticipated in the Introduction, take the review of the laws of electrostatics, in particular the connection between the electric source density and the electrostatic potential, Eqs. (9) and (12), as the motivation for learning about the solutions to the Laplace equation in Sec. 3, the harmonic expansions of the Coulomb potential in Sec. 4, and the harmonic expansions of the electrostatic potential in Sec. 5. The outline presented here emphasizes the special place that harmonic functions play in the study of electrostatics and it is the aim of the authors to motivate the reader, whether teacher or student, to learn about them in detail. Although the center of the attention has been the electrostatic potential, once it is available, Eq. (9) serves to construct the electric intensity field and its harmonic expansions. For the sake of space and illustration, the analysis in this paper has been limited to spherical and circular cylindrical coordinates.

The interested reader may consult [15] for the harmonic expansions of the Coulomb potential harmonics in prolate spheroidal, oblate spheroidal, and paraboloidal harmonics. Also the relationship between the electrostatic potential and the electric charge density has its counterpart in the relationship between the magnetostatic potential and the electric current density, being connected by a Coulomb integral transform and by the Poisson equation [14]. The same ideas are extended to the case of electromagnetic fields. Instead of the Laplace and Poisson equations, the sourceless and source Helmholtz equations must be used. The Coulomb potential as a Green function is replaced by the outgoing spherical wave $e^{ik|\overrightarrow{r}-\overrightarrow{r}'|}/\overrightarrow{r}-\overrightarrow{r}'$. The complete electromagnetic multipole expansion is also available in Ref. 16.

It must also be pointed out that the mathematics of electromagnetism is also useful in quantum mechanics, where the superposition principle is also valid. For instance, the spherical harmonics are the eigenfunctions of angular momentum $\hat{\ell}^2$ and $\hat{\ell}_z$, with eigenvalues $\hbar^2\ell(\ell+1)$ and $m\hbar$ for the square of its magnitude and its components along the *z*-axis. Plane waves and spherical waves are the mathematical tools for describing the scattering of electromagnetic and quantum waves. In conclusion, the interested reader is invited to identify and develop the appropriate mathematics for the field of physics of his/her choice.

Acknowledgement

One of the authors (E.L.K.) on sabbatical leave at the University of Wisconsin-Milwaukee with support from DGAPA-UNAM wishes to thank Professor Hans Volkmer of the School of Mathematical Sciences for his hospitality.

1. I. Newton, *Philosophiae Naturalis Principia Matematica* (Pepys, Reg Sc. Præses, 1686).
2. I. Newton, *Method of Fluxions and Infinte Series* (Buddenbrooks, Inc., Boston, USA, 1779).
3. A. Einstein, *Relativity: The Special and General Theory* (New York: Crown, 1961).
4. M. Born, W. Heisenberg, and P. Jordan, *Z. Phys.* **35** (1926) 557.
5. E. Ley Koo, *Rev. Mex. Fís.* **27** (1981) 449.
6. Ciclo de Diálogos: Matemáticas motivadas por Física, Facultad de Ciencias UNAM, 2006: Mecánica. Espacios y vectores de posición, velocidad, aceleración... Darío Moreno y José Luis del Río. Electromagnetismo y Óptica: El potencial electrostático es la transformada integral de Coulomb de la densidad de carga. Lucía Medina y Eugenio Ley Koo. Termodinámica y Mecánica Estadística. Distribuciones de probabilidad, promedios y fluctuaciones. Marcos Ley Koo e Iván Santamaría. Mecánica Cuántica. Espacios de operadores y de funciones de onda. María de los Ángeles Ortíz y Rocío Jáuregui.
7. E. Purcell, *Electricity and Magnetism: Berkeley Physics Course Vol II* (McGraw-Hill, 1965).
8. P. Lorrain and D.R. Corson, *Electromagnetism Principle and Applications* (W.H. Freeman and Company, 1990).
9. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics Electromagnetism and Matter, Vol II* (Addison-Wesley, California 1964).
10. J.R.Reitz and F.J. Mildford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Mass, 1967).
11. D.J. Griffiths, *Introduction to Electrodynamics*, 3ed. (Prentice Hall, New Jersey, 1999).
12. J.D. Jackson, *Classical Electrodynamics*, 3rd Ed. (John Wiley & Sons, Inc., MA, 1999).
13. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, 6th Ed. (Elsevier Academic Press, Boston, USA, 2005).

14. E. Ley-Koo and A. Góngora, *Rev. Mex. Fís.* **34** (1988) 645.
15. E. Ley-Koo and A. Góngora, *Rev. Mex. Fís.* **39** (1993) 785.
16. A. Góngora and E. Ley-Koo, *Rev. Mex. Fís. E* **52** (2006) 188.