n-order perturbative solution of the inhomogeneous wave equation

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The exact solution of the inhomogeneous wave equation in one dimension, when the square of the velocity is a linear function of the position, can be written in terms of Bessel functions of the first kind. We use this solution as the zero order approximation for a perturbation expansion and apply it to the case when the square of the velocity can be written as a polynomial in the position. The first and second order perturbation terms, corresponding to quadratic and cubic terms for the square of the velocity, are obtained. A closed formula for the n-order correction in terms of integrals of the Bessel functions of the first kind was also explicitly obtained, this expression can be solved analytically for the first and second order corrections and numerically for higher terms.

Keywords: Inhomogeneous media; perturbation theory; wave propagation.

La solución exacta de la ecuación de onda inhomogénea en una dimensión, cuando el cuadrado de la velocidad es una función lineal de la posición, puede escribirse en términos de las funciones Bessel de primera especie. Usamos esta solución como la aproximación de orden cero de un desarrollo perturbativo y lo aplicamos al caso cuando el cuadrado de la velocidad puede escribirse como un polinomio de grado n. Obtuvimos explícitamente las perturbaciones de primer y segundo orden correspondientes a los términos cuadráticos y cúbicos para el cuadrado de la velocidad. También se encontró una expresión cerrada para la corrección a orden n en términos de integrales de funciones Bessel de primera especie; ésta puede resolverse analíticamente para el primer y segundo orden y numéricamente para ordenes superiores.

Descriptores: Medios inhomogéneos; teoría de perturbaciones; propagación de ondas.

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1. Introduction

The inhomogeneous wave equation describes a great variety of physical systems, such as: mechanical systems, elastic systems, electromagnetic propagation and transmission, electronic devices, quantum systems, and others [1]. In several cases the propagation or transmission of a physical quantity can be modelled by a wave equation (WE) in which the velocity is a function of the propagation coordinate: for instance, the cases of electromagnetic waves in normal incidence on a region whose electric permeability depends on the position in the medium, thin film coating of optical surfaces where antireflection is of practical interest, radio wave reflection, propagation or transmission of electromagnetic field in the ionosphere, optical systems with variable index of refraction, etc. For this kind of system, several dielectric constant profiles have been solved analytically: the inverse squared profile [2], exponential, linear and quadratic polynomials (*e.g.*, V. Ginzburg [3] and references therein) and the inverse of an even polinomial of the fourth degree [4]. In elastic media, under certain physical conditions, wave motion can be modelled by a WE with a velocity depending on the coordinate of propagation; for example, a stratified fluid, a solid with a density or elastic coefficients depending on the position, a solid subject to a strong temperature or pressure gradient, etc.

When the WE cannot be solved analytically several approximative methods are available. The first one, widely used, is the geometrical optics approximation or WKB (Wentzel-Kramers-Brillouin). In this method the homogeneous wave equation is the starting point, and the correction is made by changing in the plane wave solution the dielectric constant of the homogeneous media by a function to be determined; the first approximation is obtained when the undetermined function is the square root of the dielectric function. This approximation is equivalent to the first order correction in perturbation theory [3]. A second approach is the numerical solution of the WE either by direct solution [5] or by means of the Green function [6, 7]. These methods have become extensively used due to the power of modern computer machines and techniques, providing very realistic descriptions for several systems; however, without an analytical solution the physical interpretation and analysis becomes cumbersome. Other useful methods are the transfer matrix theory [8] and iterative methods [9], these methods share the same problem with numerical solutions. An important method is the perturbative approximation, which depends on the goodness of the zero order solution and the possibility of solving the remaining differential equations, this method provides analytical solutions that can be interpreted and analyzed [10, 11].

In this work, we use the solution of the WE where the square of the velocity is a linear function of the position, which is given in terms of the Bessel functions of the first kind; then, we use this solution as the zero order approximation and apply perturbation theory to solve the case of a polynomial of any degree for the square of the velocity. The first perturbative approximation includes the quadratic term while the second one corresponds to the cubic term, etc.; a closed expression is obtained for higher order approximations, where numerical integration is required.

2. Model equation

A wide range of physical phenomena taking place in inhomogeneous media can be modelled by the inhomogeneous WE with a velocity function; for many of these systems, it is enough to study this phenomenon in one dimension. For this kind of system, we assume the following WE:

$$v^2(x)\frac{\partial^2\Psi(x,t)}{\partial x^2}=\frac{\partial^2\Psi(x,t)}{\partial t^2}; \tag{1}$$

the function $\Psi(x,t)$ describes the quantity of interest: electric or magnetic field, dilatation, pressure, etc. The solution to Eq. (1) can be expanded in Fourier components:

$$\Psi(x,t)=\exp(-i\omega t)\psi(x). \tag{2}$$

If the velocity function $v^2(x)$ (VF) is an analytical function, a Taylor series expansion can be performed so that the WE can be written as:

$$(k_0^2+k_1^2x+k_2^2x^2+k_3^2x^3+...)\frac{d^2\psi(x)}{dx^2}=-\omega^2\psi(x), \tag{3}$$

where the parameter k_0 is the velocity in the corresponding homogeneous medium, ω is the frequency in the Fourier expansion, and k_i^2 are just coefficients in the VF.

As the first step, we take into account the linear velocity profile assuming that the linear term adequately represents the phenomenon or can be taken as the first order approximation; in this case the WE becomes:

$$(k_0^2+k_1^2x)\frac{d^2\psi(x)}{dx^2}=-\omega^2\psi(x). \tag{4}$$

A change of variable transforms this to the Bessel equation:

$$k_0^2k^2\left[u\frac{d}{du}\left(\frac{1}{u}\frac{d\psi(u)}{du}\right)\right]+k_0^2k^2\psi(u)=0 \tag{5}$$

with

$$u=2\sqrt{\frac{k_0^2k^2}{k_1^4}}\sqrt{k_0^2+k_1^2x},\quad \omega^2=k_0^2k^2, \tag{6}$$

where k is the wave number. The solution to Eq. (5) is a linear combination of the Bessel functions of first kind $J_1(u)$ and $Y_1(u)$:

$$\psi(u)=C_1uJ_1(u)+C_2uY_1(u); \tag{7}$$

the border condition will provide the constants C_1 and C_2.

3. Perturbative approach

Analytical solutions for a velocity dependence beyond the quadratic term in Eq. (1) are not possible; however, when the VF is a slowly varying function of the position, a perturbative approach could be useful. Although there is an analytical solution to the second order polynomial VF in terms of hypergeometric functions (Ginzburg [3] and references therein), we choose the solution to the linear velocity profile of Eq. (4) as the zero order solution; the reason for this is that Bessel functions are easer to handle and compute than hypergeometric ones. We write the VF in terms of a smallness parameter ϵ that would depend of the physical properties of the specific system:

$$v^2(x)=\left(k_0^2+k_1^2x\right)+\epsilon k_2^2x^2+\epsilon^2k_3^2x^3+... \tag{8}$$

then, Eq. (3) becomes:

$$\left\{k_0^2+k_1^2x+\epsilon k_2^2x^2+\epsilon^2k_3^2x^3+\dots\right\}\nabla^2\psi=-\omega^2\psi(x). \tag{9}$$

The solution to this equation can also be expanded in terms of the smallness parameter [10] in the following way:

$$\psi(x)=\psi_0(x)+\epsilon\psi_1(x)+\epsilon^2\psi_2(x)+..., \tag{10}$$

where $\psi_0(x)$ is the solution to Eq. (4). After the substitution of (8) and (10) the general WE can be written as:

$$\begin{aligned}&(k_0^2+k_1^2x+\epsilon k_2^2x^2++\epsilon^2k_3^2x^3+...)\\&\times\frac{d^2\left(\psi_0(x)+\epsilon\psi_1(x)+\epsilon^2\psi_2(x)+...\right)}{dx^2}\\&=k_0^2k^2\left(\psi_0(x)+\epsilon\psi_1(x)+\epsilon^2\psi_2(x)+...\right)=0.\end{aligned} \tag{11}$$

In order to find the first order approximation we equate all terms in the first power of ϵ and neglecting all terms in higher powers of ϵ, this gives the differential equation for the first order approximation denoted by ψ_1:

$$\begin{aligned}\epsilon\left(k_2^2x^2\right)\frac{d^2\psi_0(x)}{dx^2}+\epsilon\left(k_0^2+k_1^2x\right)\frac{d^2\psi_1(x)}{dx^2}\\+\epsilon k_0^2k^2\psi_1(x)=0.\end{aligned} \tag{12}$$

If in the previous equation: we change to the variable u given by (6), include the solution (7) for ψ_0, and arrange terms, we obtain the differential equation for ψ_1;

$$u\frac{d}{du}\left(\frac{1}{u}\frac{d}{du}\psi_1(u)\right)+\psi_1(u)=f\left(u\right), \tag{13}$$

where the function on the right hand side is defined by

$$f(u)=\left(au^2-b-\frac{c}{u^2}\right)\left[C_1uJ_1(u)+C_2uY_1(u)\right], \tag{14}$$

and the parameters involved depend only on the wave number and the parameters of the VF:

$$a=-\frac{k_2^2}{4k_0^2k^2}\quad b=\frac{2k_2^2k_0^2}{k_1^4}\quad c=-\frac{4k^2k_0^6k_2^2}{k_1^8}. \tag{15}$$

The method of the undetermined parameters [12] provides the solution ψ_1; which can be expanded in terms of the Bessel functions J_0, J_1, Y_0 and Y_1 as follows:

$$\psi_1(u) = \sum_{m=0} \left(A_{m,0} u^m J_0\left(u\right) + B_{m,0} u^m Y_0\left(u\right)\right) + \sum_{n=0} \left(C_{n,1} u^m J_1\left(u\right) + D_{n,1} u^m Y_1\left(u\right)\right). \quad (16)$$

If we define the second order differential operator:

$$\delta_{uu} = \left[u\frac{d}{du}\left(\frac{1}{u}\frac{d}{du}\right)\right], \quad (17)$$

apply it to the Bessel functions of Eq. (16), and use the properties for the derivatives of the Bessel functions [13] we arrive at the following equations:

$$\begin{aligned}\delta_{uu}\left(u^m\left(J_0\left(u\right)\right)\right) &= -2(m-1)u^{m-1}\left(J_1\left(u\right)\right)\\ &+ -u^m\left(J_0\left(u\right)\right)\\ &+ m(m-2)x^{m-2}\left(J_0\left(u\right)\right) \quad (18)\end{aligned}$$

$$\begin{aligned}\delta_{uu}\left(u^m\left(Y_0\left(u\right)\right)\right) &= -2(m-1)u^{m-1}\left(Y_1\left(u\right)\right)\\ &+ -u^m\left(Y_0\left(u\right)\right)\\ &+ m(m-2)u^{m-2}\left(Y_0\left(u\right)\right) \quad (19)\end{aligned}$$

$$\begin{aligned}\delta_{uu}\left(u^n\left(J_1\left(u\right)\right)\right) &= (n-1)\left(n-3\right)u^{n-2}\left(J_1\left(u\right)\right)\\ &+ 2\left(n-1\right)u^{n-1}\left(J_0\left(u\right)+\right)\\ &- u^n\left(J_1\left(u\right)\right) \quad (20)\end{aligned}$$

$$\begin{aligned}\delta_{uu}\left(u^n\left(Y_1\left(u\right)\right)\right) &= (n-1)\left(n-3\right)u^{n-2}\left(Y_1\left(u\right)\right)\\ &+ 2\left(n-1\right)u^{n-1}\left(Y_0\left(u\right)\right)\\ &- u^n\left(Y_1\left(u\right)\right). \quad (21)\end{aligned}$$

Using these results, the first term in the left hand side of Eq. (13) becomes:

$$\begin{aligned}\delta_{uu}\left(\psi_1(u)\right) = &\sum_m A_{m,0}\left[-2(m-1)u^{m-1}\left(J_1\left(u\right)\right) - u^m\left(J_0\left(u\right)\right) + m(m-2)u^{m-2}\left(J_0\left(u\right)\right)\right]\\ &+ \sum_m B_{m,0}\left[-2(m-1)u^{m-1}\left(Y_1\left(u\right)\right) - u^m\left(Y_0\left(u\right)\right) + m(m-2)u^{m-2}\left(Y_0\left(u\right)\right)\right]\\ &\times \sum_m C_{n,1}\left[(n-1)\left(n-3\right)u^{n-2}\left(J_1\left(u\right)\right) + 2\left(n-1\right)u^{n-1}\left(J_0\left(u\right)\right) - u^n\left(J_1\left(u\right)\right)\right]\\ &+ \sum_n D_{n,1}\left[(n-1)\left(n-3\right)u^{n-2}\left(Y_1\left(u\right)\right) + 2\left(n-1\right)u^{n-1}\left(Y_0\left(u\right)\right) - u^n\left(Y_1\left(u\right)\right)\right]. \quad (22)\end{aligned}$$

Substitution of Eq. (22) into Eq. (13) imposes the condition for the coefficients in (16). Taking into account that the set of products $u^l J_i(u)$ and $u^l Y_i(u)$ are linearly independent, we arrive at the conditions for the zero order Bessel functions

$$\begin{aligned}&\sum_m A_{m,0}\left[m(m-2)u^{m-2}\left(J_0\left(u\right)\right)\right] + B_{m,0}\left[m(m-2)u^{m-2}\left(Y_0\left(u\right)\right)\right]\\ &+ \sum_n C_{n,1}\left[2\left(n-1\right)u^{n-1}\left(J_0\left(u\right)\right)\right] + D_{n,1}\left[2\left(n-1\right)u^{n-1}\left(Y_0\left(u\right)\right)\right] = 0, \quad (23)\end{aligned}$$

and for the first order Bessel functions

$$\begin{aligned}&\sum_n A_{m,0}\left[-2(m-1)u^{m-1}\left(J_1\left(u\right)\right)\right] + B_{m,0}\left[-2(m-1)u^{m-1}\left(Y_1\left(u\right)\right)\right]\\ &\times \sum_n \left[C_{n,1}\left[(n-1)\left(n-3\right)u^{n-2}\left(J_1\left(u\right)\right)\right] + D_{n,1}\left[(n-1)\left(n-3\right)u^{n-2}Y_1\left(u\right)\right]\right]\\ &+ \left(au^3 + bu + \frac{c}{u}\right)\left(C_1 J_1\left(u\right) + C_2 Y_1\left(u\right)\right) = 0. \quad (24)\end{aligned}$$

Collecting terms in $u^l J_i(u)$ and $u^l Y_i(u)$ and equating them to zero we arrive at the following conditions:

$$\begin{aligned}&A_{m,0} = 0 \quad \text{if} \quad m \neq 0,\, 2\,, 4 \qquad B_{m,0} = 0 \quad \text{if} \quad m \neq 0,\, 2\,, 4\\ &C_{n,1} = 0 \quad \text{if} \quad n \neq 1,\, 3\,, 5 \qquad D_{n,1} = 0 \quad \text{if} \quad n \neq 1,\, 3\,, 5, \quad (25)\end{aligned}$$

and in addition we find that

$$C_{5,1} = 0 \qquad D_{5,1} = 0. \tag{26}$$

The non-null coefficients are:

$$A_{0,0} = \frac{-c}{2}C_1 \quad A_{2,0} = \frac{b}{2}C_1 \quad A_{4,0} = \frac{a}{6}C_1$$
$$B_{0,0} = \frac{-c}{2}C_2 \quad B_{2,0} = \frac{b}{2}C_2 \quad B_{4,0} = \frac{a}{6}C_2$$
$$C_{1,1} = \frac{-b}{2}C_1 \quad C_{3,1} = \frac{-a}{3}C_1$$
$$D_{1,1} = \frac{-b}{2}C_2 \quad D_{3,1} = \frac{-a}{3}C_2. \tag{27}$$

With this result the first order approximation becomes

$$\psi_1(u) = -\left(u\left(\frac{1}{3}au^2 + \frac{b}{2}\right)\right)\left[C_1 J_1(u) + C_2 Y_1(u)\right] + \frac{1}{2}\left(-c + bu^2 + \frac{1}{3}au^4\right)\left[C_1 J_0(u) + C_2 Y_0(u)\right], \tag{28}$$

where the coefficients a, b, and c are given by Eq. (15). The first approximation is a combination of Bessel functions J_1, Y_1, J_0 and Y_0, multiplied by polynomials in the new variable u. For further purposes, it is useful to write this solution as follows:

$$\psi_1 = \left[C_1 u J_1(u) + C_2 u Y_1(u)\right] - \frac{\pi}{2}u\left[J_1(u)\int_{u_0}^{u} Y_1(x) f(x)\,dx\right] + \frac{\pi}{2}u\left[Y_1(u)\int_{u_0}^{u} J_1(x) f(x)\,dx\right], \tag{29}$$

where $f(x)$ is defined by (14) and the lower limit of integration corresponds to $x = 0$,

$$u_0 = 2\sqrt{\frac{k_0^4 k^2}{k_1^4}}. \tag{30}$$

The proof of this statement is as follows: first, we substitute Eq. (29) into Eq. (13) and, taking into account that the Bessel functions satisfy the following identity:

$$J_1(u)Y_0(u) - J_0(u)Y_1(u) = \frac{2}{\pi u}, \tag{31}$$

we obtain the required result. Expression (29) is a formal solution obtained by the method of variation of parameters and will become very useful in the subsequent development of the perturbative calculation.

4. Second order approximation

The second order approximation is obtained by equating all terms in ϵ^2 from Eq. (11) and neglecting all terms in higher powers of ϵ, thus leading to the following condition for ψ_2:

$$\left(k_3^2 x^3\right)\frac{d^2\psi_0(x)}{dx^2} + \left(k_2^2 x^2\right)\frac{d^2\psi_1(x)}{dx^2} + \left(k_0^2 + k_1^2 x\right)\frac{d^2\psi_2(x)}{dx^2} + k_0^2 k^2 \psi_2(x) = 0. \tag{32}$$

Changing to the variable u, including the solutions for the zero and first order approximations, after a few algebraic steps we arrive at the following differential equation for the second order approximation:

$$\delta_{uu}\psi_2(u) + \psi_2(u) + \sum_{i=0}^{4}\left(\zeta_{i+1}^2 u^{2i-3}\left[C_1 J_1(u) + C_2 Y_1(u)\right]\right) + \sum_{i=0}^{4}\left(\phi_{i+1}^2 u^{2i-2}\left[C_1 J_0(u) + C_2 Y_0(u)\right]\right) = 0, \tag{33}$$

where the coefficients ζ_{i+1}^2 and ϕ_{i+1}^2 depend on the constants k_0, k_1, k_2, k_3 and k. In a similar procedure to that used in the last section, we propose the solution for the second order approximation as a series of Bessel functions, in this case involving J_ν and Y_ν with $\nu = 0, 1$ and 2. Using the fact that the products $u^l J_i(u)$ and $u^l Y_i(u)$ are linearly independent we obtain the following expression for ψ_2:

$$\psi_2(u) = \left(-\frac{\zeta_1^2}{4u} - \frac{\phi_3^2 u^3}{4} + \left(\frac{2\phi_5^2}{5} - \frac{\phi_4^2}{8} - \frac{\zeta_5^2}{10}u^5 - \frac{\phi_5^2}{12}\right)u^7\right)\left[C_1 J_1(u) + C_2 Y_1(u)\right]$$
$$- \frac{\zeta_2^2}{2}\left[C_1 J_0(u) + C_2 Y_0(u)\right] + \left(\frac{\zeta_3^2}{2} - \phi_3^2\right)u^2\left[C_1 J_0(u) + C_2 Y_0(u)\right]$$
$$+ \frac{1}{8}\left(\phi_1^2 + \zeta_1^2\right)\left[C_1 J_0(u) + C_2 Y_0(u) + C_1 J_2(u) + C_2 Y_2(u)\right]$$
$$+ \left(\frac{\phi_4^2}{2} - \frac{\zeta_4^2}{6} + \frac{2\zeta_5^2}{5} - \frac{8\phi_5^2}{5}\right)u^4\left[C_1 J_2(u) + C_2 Y_2(u)\right] + \left(\frac{2\phi_5^2}{5} - \frac{\zeta_5^2}{10}\right)u^6\left[C_1 J_2(u) + C_2 Y_2(u)\right]$$
$$- \frac{\pi\phi_2^2}{2}u\left[J_1(u)\int_{u_0}^{u} Y_1(u')\left[C_1 J_2(u') + C_2 Y_2(u')\right]du'\right] + \frac{\pi\phi_2^2}{2}u\left[Y_1(u)\int_{u_0}^{u} J_1(u')\left[C_1 J_2(u') + C_2 Y_2(u')\right]du'\right], \tag{34}$$

where u_0 is given by the formula (30), and we have used the well known recurrence formulas for Bessel functions:

$$J_\nu(u) = \frac{2\,(\nu \pm 1)}{u} J_{\nu\pm1}(u) - J_{\nu\pm2}(u). \tag{35}$$

The second approximation turns out to be a combination of Bessel functions for $\nu = 0, 1, 2$ multiplied by a polinomial in u. In analogy to the first order solution, the second one can also be written as follows:

$$\begin{aligned}\psi_2 &= [C_1 u J_1(u) + C_2 u Y_1(u)] \\ &- \frac{\pi}{2} u \left[J_1(u) \int_{u_0}^{u} Y_1(x) [f_2(x)]\, dx \right] \\ &+ \frac{\pi}{2} u \left[Y_1(u) \int_{u_0}^{u} J_1(x) [f_2(x)]\, dx \right], \end{aligned} \tag{36}$$

where

$$\begin{aligned} f_2(u) &= -k_2^2 \frac{\left(\frac{A^2u^2}{C^2} - B\right)^2}{A^2} \frac{1}{u}\frac{d}{du}\frac{1}{u}\frac{d}{du}(\psi_1) \\ &- k_3^2 \frac{\left(\frac{A^2u^2}{C^2} - B\right)^3}{A^2} \frac{1}{u}\frac{d}{du}\frac{1}{u}\frac{d}{du}(\psi_0), \end{aligned} \tag{37}$$

where the constants A, B, C are defined by

$$A = \frac{k_1^2}{2k_0 k}, \;\; B = \frac{k_0^2}{k_1^2}, \;\; C = k_1^2. \tag{38}$$

It is instructive to compare the similarities between the first order approximation given by (29) and the second order expressed in (36).

5. n-order approximation

Third and higher order approximations can be obtained with the procedure outlined above, although the algebraic steps involved and the expressions for each solution become more and more cumbersome as the order of the approximation increases. However, by an iterative procedure it is easy to show that the n-order approximation ψ_n satisfies the following differential equation:

$$u\frac{d}{du}\left(\frac{1}{u}\frac{d}{du}\psi_n(u)\right) + \psi_n(u) = f_n(u) \tag{39}$$

with the function $f_n(u)$ defined by:

$$\begin{aligned} f_n(u) = -\{ & k_2^2 \frac{\left(\frac{A^2u^2}{C^2} - B\right)^2}{A^2} \frac{1}{u}\frac{d}{du}\frac{1}{u}\frac{d}{du}(\psi_{n-1}) \\ & + k_3^2 \frac{\left(\frac{A^2u^2}{C^2} - B\right)^3}{A^2} \frac{1}{u}\frac{d}{du}\frac{1}{u}\frac{d}{du}(\psi_{n-2}) + \cdots \\ & + k_{n+1}^2 \frac{\left(\frac{A^2u^2}{C^2} - B\right)^{n+1}}{A^2} \frac{1}{u}\frac{d}{du}\frac{1}{u}\frac{d}{du}(\psi_0) \}. \end{aligned} \tag{40}$$

The solution to the differential equation (39) is

$$\begin{aligned}\psi_n(u) &= [C_1 u J_1(u) + C_2 u Y_1(u)] \\ &- \frac{\pi}{2} u \left[J_1(u) \int_{u_0}^{u} Y_1(y) [f_n(y)]\, dy \right] \\ &+ \frac{\pi}{2} u \left[Y_1(u) \int_{u_0}^{u} J_1(y) [f_n(y)]\, dy \right], \end{aligned} \tag{41}$$

where y is just a variable of integration and u is defined by Eq. (6). The proof of this assertion is easy: by substitution of (40) in (41) and of the latter in (39), and taking into account the property expressed by Eq. (31), we obtain an identity. Of course, the special cases when $n = 1$ and 2 are appropriately reproduced by this last result.

The n-order approximation given by the formula (41) presents a few difficulties. For the first values of n the integrals involved can be done analytically; however, as n increases the number of terms of the function $f_n(u)$ grows very rapidly and analytical integration is not longer possible. Besides, for $n \geq 3$ the n-order approximation involves multiple integration, quickly diminishing the accuracy of the calculation and making any further analysis difficult.

It is instructive to arrange the n-order correction to the wave motion in the following way:

$$\begin{aligned}\epsilon^n \psi_n(u) &= C_1 u J_1(u) \left[\epsilon^n \left(1 - \frac{\pi}{2C_1} g(u) \right) \right] \\ &+ C_2 u Y_1(u) \left[\epsilon^n \left(1 + \frac{\pi}{2C_2} h(u) \right) \right]. \end{aligned} \tag{42}$$

where $g(u)$ and $h(u)$ are the integrals on the second and third terms in the right hand side of Eq. (41) respectively. By comparison of this result with the zero order solution (corresponding to the linear VF) given by Eq. (7), we observe that the effect of the n order correction is to change the amplitude at each point of the unperturbed *wave*. These corrections depend, in the last instance, on the parameters k_i of the VF and the wave vector k in the corresponding homogeneous media.

6. Examples

In order to illustrate the general behavior of the perturbative solution we shall consider the case of a plane monochromatic wave travelling in an homogeneous medium $x < 0$ which is incident on an inhomogeneous region $x \geq 0$, with a linear VF; the wave amplitude in the homogeneous medium is written as

$$\Psi(x,t) = \exp(i(kx - \omega t)). \tag{43}$$

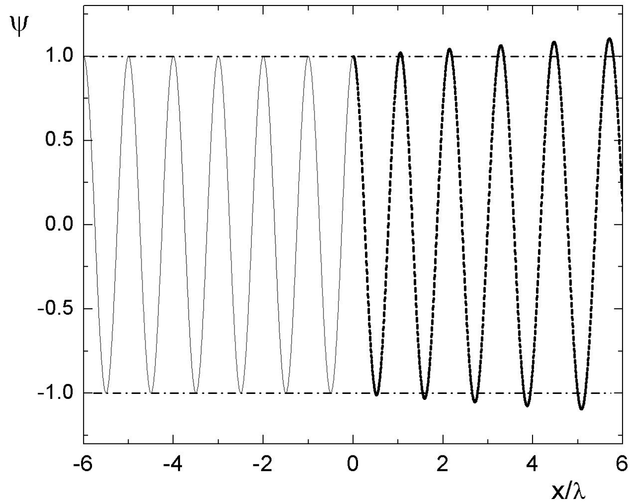

FIGURE 1. The amplitude of the wave at the passage from a homogeneous ($x < 0$) to an inhomogeneous medium ($x \geq 0$); the horizontal line is drawn to guide the eye.

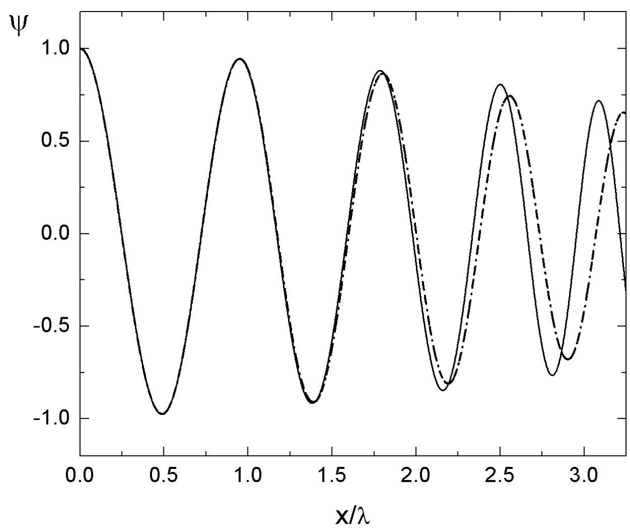

FIGURE 2. The doted curve is the exact solution to the quadratic VF, the continuos curve is the first perturbative correction.

The border conditions that define the constants in Eq. (7) are: continuity of the wave function and of its first derivative. The resulting amplitude for $x > 0$ becomes:

$$\psi(x) = C_1\sqrt{k_0^2 + k_1^2 x}(Y_0(u_0)J_1(u) - J_0(u_0)Y_1(u)) + C_2\left(\sqrt{k_0^2 + k_1^2 x}(Y_1(u_0)J_1(u) - J_1(u_0)Y_1(u))\right), \quad (44)$$

where u_0 is given as before and

$$C_1 = \frac{1}{k_0(J_1(u_0)Y_0(u_0) - J_0(u_0)Y_1(u_0))}$$

$$C_2 = i\frac{1}{k_0(J_1(u_0)Y_0(u_0) - J_0(u_0)Y_1(u_0))}\ . \quad (45)$$

In Fig. 1 we illustrate the wave behavior for the following parameters of the VF and wave number,

$$k = \sqrt{10},\ k_0^2 = 10, \qquad \frac{k_1^2}{k_0^2} = \frac{1}{10}. \quad (46)$$

As can be seen from this figure, the amplitude increases very slowly as the wave penetrates the medium, because the medium is non-dissipative and becomes *harder* for this set of parameters.

As a second example we consider the quadratic VF. The exact solution for it can be written in terms of confluent hypergeometric functions and has the following form [3]:

$$\psi(x) = C_1(2k_2x + k_1 + a_1)^{-a2}(2k_2x + k_1 - a_1) \times F(-b_1, b_2, b_3, 2\frac{a_1}{2k_2x + k_1 + a_1}) + C_2(2k_2x + k_1 + a_1)^{-a2}(2k_2x + k_1 - a_1) \times F(b_1, b_4, b_5, 2\frac{a_1}{2k_2x + k_1 + a_1}), \quad (47)$$

$F(\alpha, \beta, \gamma, x)$ is the hypergeometric function, the coefficients are listed below:

$$a_1 = \sqrt{k_1{}^2 - 4\,k_2 k_0}$$

$$a_2 = \frac{\sqrt{k_2} - i\sqrt{4\,\omega^2 - k_2}}{2\sqrt{k_2}}$$

$$b_1 = \frac{-3\,\sqrt{k_2} + i\sqrt{4\,\omega^2 - k_2}}{2\sqrt{k_2}}$$

$$b_2 = \frac{\sqrt{k_2} - i\sqrt{4\,\omega^2 - k_2}}{2\sqrt{k_2}}$$

$$b_3 = \frac{\sqrt{k_2} - i\sqrt{4\,\omega^2 - k_2}}{\sqrt{k_2}}$$

$$b_4 = \frac{\sqrt{k_2} + i\sqrt{4\,\omega^2 - k_2}}{2\sqrt{k_2}}$$

$$b_5 = \frac{\sqrt{k_2} + i\sqrt{4\,\omega^2 - k_2}}{\sqrt{k_2}}, \quad (48)$$

In Fig. 2 we compare the first order solution (28) with the exact solution given by (47) for the quadratic VF; the parameters for this case are defined in (46) where we changed the sign in k_1^2 and selected the value $k_2^2/k_0^2 = -1$, and $\epsilon = 0.01$, meaning that the medium in the positive half space is *softer* than the medium for $x < 0$. As expected, the amplitude decreases with the position. Agreement between the exact and approximate solution is very good for small values of the position; however, it fails for x values of the order of a few wavelengths $\lambda = 2\pi/k$. This can be understood by looking at Eqs. (28): the correction ψ_1 depends on u^n with $n = 1, 2, 3,$ and 4, and besides, it is proportional to k_2. As the wave penetrates the inhomogeneous medium this correction loses its smallness. However, the applicability of this method would depend on the specific problem; for example, thin film coating is usually smaller than a wavelength.

7. Concluding remarks

The novelty in the present approach is the solution of the inhomogeneous WE by perturbative series with a better zero order solution; we choose the Bessel functions of the first kind, solutions of an inhomogeneous wave equation with a linear VF, instead of the harmonic solutions of the homogeneous wave equation. The WE for a polynomial VF has been solved by a perturbative expansion, explicitly for the first and second order approximation, that includes the quadratic and cubic terms of the VF, and in a closed formula for any order; however, starting with the third correction the computation involves numerical integration, making any further analysis difficult. This method could be useful for some specific system when second and third order approximations are enough to describe the system and when the medium has a small finite size. It is worth noting that there is an analytical solution for the square velocity profile in terms of hypergeometric functions; therefore other zero order perturbative solution to the inhomogeneous wave equation would be constructed from hypergeometric functions. However, the differential equations and their solutions arising from the perturbative series become increasingly difficult.

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