An alternative solution to the general tautochrone problem

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In 1658, Blaise Pascal put forward a challenge for solving the area under a segment of a cycloid and also its center of gravity. In 1659, motivated by Pascal challenge, Huygens showed experimentally that the cycloid is the solution to the tautochrone problem, namely that of finding a curve such that the time taken by a particle sliding down to its lowest point, under uniform gravity, is independent of its starting point. Ever since, this problem has appeared in many books and papers that show different solutions. In particular, the fractional derivative formalism has been used to solve the problem for an arbitrary potential and also to put forward the inverse problem: what potential is needed in order for a particular trajectory to be a tautochrone? Unfortunately, the fractional derivative formalism is not a regular subject in the mathematics curricula for physics at most of the Universities we know. In this work we develop an approach that uses the well-known Laplace transform formalism together with the convolution theorem to arrive at similar results.

Keywords: Tautochrone; Laplace transform; convolution theorem.

En 1658, Blaise Pascal lanzó el reto de determinar el área debajo de la curva de un segmento de cicloide, así como su centro de gravedad. En 1659, motivado por el reto de Pascal, Huygens demuestra experimentalmente que la cicloide es la solución al problema de la tautócrona, es decir, al problema de encontrar una curva tal que, si una partícula engarzada en ella se mueve por la acción del campo gravitacional uniforme, su tiempo de descenso es independiente de la posición inicial. Desde entonces, este problema ha sido tratado en muchos libros y artículos con diferentes soluciones. En particular, el formalismo de derivadas fraccionales ha sido utilizado para resolver el problema en el caso de un potencial arbitrario así como el problema inverso: ¿qué potencial se requiere para que una trayectoria, en particular, sea una tautócrona? Desafortunadamente, el formalismo de derivadas fraccionales no forma parte de la currícula de la carrera de Física de muchas de las Universidades que conocemos. En este trabajo desarrollamos un cálculo que utiliza el bien conocido formalismo de la transformada de Laplace, que junto con el teorema de convolución, nos lleva a resultados similares.

Descriptores: Tautócrona; transformada de Laplace; teorema de convolución.

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1. Introduction

As stated in the abstract, Huygens showed experimentally that a cycloid is the tautochrone curve for a particle sliding without friction in the uniform gravitational field. He published this result in his book *Horologium oscillatorium* [1] and some years later, when J. Bernoulli found that the cycloid is also a brachistochrone, he wrote [2]:

> *Before I end I must voice once more the admiration I feel for the unexpected identity of Huygens' tautochrone and my brachistochrone. I consider it especially remarkable that this coincidence can take place only under the hypothesis of Galileo, so that we even obtain from this a proof of its correctness. Nature always tends to act in the simplest way, and so it here lets one curve serve two different functions, while under any other hypothesis we should need two curves…*

The problem has been, and still is, of interest, especially because of its similarity with the brachistochrone problem [3-5]. Great mathematicians such as Joseph Louis Lagrange and Leonhard Euler looked for an analytical solution to the problem and Niels H. Abel used the Laplace formalism to solve it [6]. Ever since, many papers have been published dealing with tautochrone curves under special physical conditions, such as: the tautochrone with friction [7], the relativistic tautochrone [8], the tautochrone in rotating frames of reference [9] and the tautochrone under an arbitrary potential [10]. In this last paper, Flores and Osler introduced the fractional derivatives formalism to solve the problem and to generalize to an arbitrary potential energy function, and also to solve the inverse problem. However, the fractional derivative formalism is not a regular subject in the under graduate syllabus of physics or mathematics studies. In this paper we develop the same type of generalization, but using the more accessible and well-known formalism of Laplace transform.

2. The Laplace transform formalism

Before we move toward the tautochrone problem, we shall recall a pair of definitions and the convolution theorem [11]:

1.- The Laplace transform $f(s)$ of a function $F(t)$ is defined as:

$$f(s) = \mathrm{L}\left\{F(t)\right\} = \int_0^\infty e^{-st}F(t)dt \tag{1}$$

and the convolution between two functions as:

$$\int_0^t F_1(z)F_2\left(t-z\right)dz = F_1 * F_2 \tag{2}$$

2.- The convolution theorem:

If $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$, respectively, then

$$f_1\left(s\right)f_2\left(s\right) = L\left\{\int_0^t F_1\left(z\right)F_2\left(t-z\right)dz\right\} = L\left(F_1 * F_2\right) \tag{3}$$

3. The Laplace formalism in the Tautochrone problem

The path-time employed in a given trajectory defined by an arc element dσ

$$d\sigma=\sqrt{dx^2+dy^2}=\sqrt{1+\left(dx/dy\right)^2}dy=\sqrt{1+x'^2}dy \tag{4}$$

with a velocity v

$$v = \sqrt{(2/m)\left[U\left(y_0\right)-U\left(y\right)\right]}, \tag{5}$$

where m is the mass of the particle and U(y) is the potential energy function, can be written as:

$$\int_{t_0}^t dt = -\int_\sigma \frac{d\sigma}{v} = -\int_{y_0}^y \frac{d\sigma}{dy}\sqrt{(2/m)\left[U\left(y_0\right)-U\left(y\right)\right]}dy; \tag{6}$$

the minus sign is because $d\sigma/dt < 0$. As stated before, the standard way to solve the problem using the Laplace formalism was put forward by Abel [6], and it can be seen, for instance, in Arfken's book [11]. However, we include it here for the sake of completeness:

In the constant and uniform gravitational potential case, $U(y)$ =mg$y$, the last integral of Eq. (6) is from the top of the trajectory ($y = y_0$, t= t_0 = 0) to its bottom; (y = 0, t = T), where the ending point has been taken, without lost of generality as y = 0, so that equation can be written as:

$$\int_0^T \sqrt{2g}dt = \sqrt{2g}T = -\int_{y=y_0}^{y=0}\left(y_0-y\right)^{-1/2}\left(\frac{d\sigma}{dy}\right)dy = \int_0^{y_0}\left(y_0-y\right)^{-1/2}\left(\frac{d\sigma}{dy}\right)dy \tag{7}$$

For a tautochrone, the time of descent T is to be constant independent of y_0, so that the upper limit of Eq. (7) is arbitrary and this fact makes it possible to take the integrand as the convolution of $d\sigma/dy$. The convolution theorem then states that

$$\sqrt{2g}T = y^{-1/2} * \frac{d\sigma}{dy}, \tag{8}$$

and calculating the corresponding Laplace transforms one obtains

$$\sqrt{2g}T\frac{1}{s} = L\left\{\frac{d\sigma}{dy}\right\}L\left\{y^{-1/2}\right\} = L\left\{\frac{d\sigma}{dy}\right\}\sqrt{\frac{\pi}{s}} \tag{9}$$

because the Laplace transform of a constant A is A/s and that of $y^{-1/2}$ is $\sqrt{\pi/s}$. Applying the inverse transform yields:

$$\frac{d\sigma}{dy} = \frac{\sqrt{2g}T}{\pi}y^{-1/2} \tag{10}$$

Squaring, separating variables, and integrating we arrive at the parametric equations of a cycloid passing through the origin [11]. This is the usual method for solving the tautochrone in the homogeneous gravitational field. The same method can be used to solve the general case, in which the potential energy function is arbitrary. We give the solution in the next section.

4. General case

In the general case for an arbitrary potential, we cannot use the convolution theorem because the left side of Eq. (6) does not contain a term $y_0 - y$ [as in Eq. (7)]. However, making $z = U(y)$ we can write

$$\frac{d\sigma}{dy}dy=\frac{d\sigma}{dz}\frac{dz}{dy}dy=\frac{d\sigma}{dz}dz \quad \text{and} \quad dz=\frac{dU}{dy}dy = U'dy \tag{11}$$

so then Eq. (6) takes the form

$$\sqrt{\frac{2}{m}}\int_0^T dt=\sqrt{\frac{2}{m}}T=-\int_{z_0}^z\left(z_0-z\right)^{-1/2}\left(\frac{d\sigma}{dz}\right)dz \tag{12}$$

Under the isochronal condition, the left side of Eq. (12) is a constant, so the right side does not depend on the integration limits and we can apply the convolution theorem; then

$$L\left\{\sqrt{\frac{2}{m}}T\right\} = \sqrt{\frac{2}{m}}\frac{T}{s} = L\left\{z^{-1/2}\right\}L\left\{\frac{d\sigma}{dz}\right\} = \sqrt{\frac{\pi}{s}}L\left\{\frac{d\sigma}{dz}\right\} \tag{13}$$

because the Laplace transform of a constant A is A/s and that of $z^{-1/2}$ is $\sqrt{\pi/s}$. Solving for $L\left\{d\sigma/dz\right\}$ we obtain:

$$L\left\{\frac{d\sigma}{dz}\right\} = \left(\sqrt{\frac{2}{m}}\frac{T}{\pi}\right)\sqrt{\frac{\pi}{s}}. \tag{14}$$

Applying the inverse transform, and substituting $z = U(y)$, yields

$$\frac{d\sigma}{dU} = \sqrt{\frac{2}{m}}\frac{T}{\pi}U^{-1/2} \tag{15}$$

so that

$$\sqrt{\frac{2}{m}}\frac{T}{\pi}\int_0^U U^{1/2}dU = \int_\sigma d\sigma, \tag{16}$$

where $U = 0$ is taken at the end point of the trajectory. Then:

$$\sigma = 2\sqrt{\frac{2}{m}}\frac{T}{\pi}U^{1/2} = A\sqrt{U} \tag{17}$$

and because

$$\frac{d\sigma}{dU} = \frac{d\sigma}{dy}\frac{dy}{dU} = \frac{1}{U'}\frac{d\sigma}{dy} = \sqrt{\frac{2}{m}}\frac{T}{\pi}U^{-1/2} \tag{18}$$

we obtain, from Eq. (4):

$$x = \int_0^y \sqrt{\frac{2T^2}{m\pi^2}\frac{U'^2}{U} - 1}dy \tag{19}$$

Equation (17) makes it possible to find the potential energy function that “makes” a given curve a tautochrone. As was stated in the introductory section, the results established in Eqs. (17) and (19) were previously derived using the fractional derivative formalism [10]. However, this formalism is far from usual at the undergraduate level. On the other hand, the Laplace transform formalism is a main part of any mathematic methods course, not only in Physics and Mathematics, but also in Engineering.

5. Generalization to central potentials U(r)

In the case of central potential $U(r)$, the movement of a particle is in a plane (ϕ = constant), so the arc element $d\sigma$ in polar coordinates is:

$$d\sigma = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}dr \tag{20}$$

and the speed of the particle along the trajectory is:

$$v = \sqrt{\frac{2}{m}\left[U\left(r_0\right) - U\left(r\right)\right]} \tag{21}$$

so the time from the initial point to the bottom of the trajectory will be:

$$T=\int_0^T dt=-\int_\sigma \frac{d\sigma}{v}=-\int_{r_0}^r \frac{d\sigma/dr}{\sqrt{\frac{2}{m}\left[U\left(r_0\right)-U\left(r\right)\right]}}dr \tag{22}$$

Proceeding as in the Cartesian coordinates case, we call $z = U(r)$ so that

$$\sqrt{\frac{2}{m}}T = \int_z^{z_0}\left(z_0 - z\right)^{-1/2}\frac{d\sigma\left(z\right)}{dz}dz \tag{23}$$

Repeating the previous steps one obtains:

$$\frac{d\sigma}{dz} = \sqrt{\frac{2}{m}}\frac{T}{\pi}z^{-1/2}. \tag{24}$$

So then

$$\sigma = 2\sqrt{\frac{2}{m}}\frac{T}{\pi}U^{1/2} = B\sqrt{U} \tag{25}$$

and because

$$\frac{d\sigma}{dU} = \frac{d\sigma}{dr}\frac{dr}{dU} = \frac{1}{U'}\frac{d\sigma}{dr} = \sqrt{\frac{2}{m}}\frac{T}{\pi}U^{-1/2} \tag{26}$$

we arrive at

$$\theta = \int\sqrt{\frac{2T^2}{m\pi^2}\frac{U'^2}{U} - 1}\frac{dr}{r}. \tag{27}$$

Equations (25) and (27) are the polar equivalents of Eqs. (17) and (19) for central potential energy functions.

The purpose of this work is to develop a simpler alternative to the fractional derivative method [9]. As was mentioned in the introduction, several examples of tautochrone curves have been published using their results, so it is not worth extending this paper with those examples. We shall only point out that once Eqs. (19), and (27) are known, the direct problem is formally solved, as shown in the following examples.

6. Examples

6.1. Potential energy functions of the (Ay+B)^n form

In this case

$$x = \int\sqrt{\frac{2T^2}{m\pi^2}n^2A^2\left(Ay + B\right)^{n-2} - 1}dy \tag{28}$$

which has simple solutions for the following cases:

$i) n = 1$

$$\begin{aligned} x &= \int\sqrt{\frac{2T^2}{m\pi^2}\frac{A^3}{(Ay + B)} - 1}dy \\ &= \int\sqrt{\frac{C - \left(Ay + B\right)y}{(Ay + B)}}\,dy \\ &= \frac{1}{a}\int\sqrt{\frac{C - u}{u}}\,du \end{aligned} \tag{29}$$

with u = Ay + B and C = 2T^2/mπ^2 the solution is an inverted cycloid and corresponds to the constant uniform gravitational field with A = mg. This is Arfken's result, obtained in one line!

$ii) n = 2$

$$x = \int \sqrt{\frac{8T^2A^2}{m\pi^2} - 1}\, dy = Cy \tag{30}$$

which is a straight line and corresponds to the linear harmonic oscillator potential.

$iii) n = 3$

$$\begin{aligned} x &= \int \sqrt{\frac{18T^2A^2}{m\pi^2}(Ay+B) - 1}\, dy \\ &= \int \sqrt{Dy+E}\, dy \\ &= \frac{2}{3D}\sqrt{(Dy+E)^3} \end{aligned} \tag{31}$$

$iv) n = 4$

$$\begin{aligned} x &= \int \sqrt{\frac{32T^2A^2}{m\pi^2}(Ay+B)^2 - 1}\, dy \\ &= \int \sqrt{cy^2+by+a}\, dy \\ &= \frac{(2cy+b)\sqrt{cy^2+by+a}}{4c} \\ &+ \frac{1}{c}\sinh^{-1}\left(\frac{2cy+b}{\sqrt{4ac-b^2}}\right) \end{aligned} \tag{32}$$

6.2. Central Potential energy functions

In the case of central force fields, there are also some cases in which the integral of the trajectory, Eq. (27) has simple solutions:

If $U = \pm \mathrm{A} r^2$, the U'^2/U = 4A^2, so the square root of the integral is a constant C and the trajectory is

$$r = r_0 \exp(-k\theta) \tag{33}$$

where k = 1/C.

If the potential is such that

$$\frac{2T^2}{m\pi^2}\frac{U'^2}{U} - 1 = Br \tag{34}$$

then the trajectory will be

$$\theta = 2B\left(r_0^{1/2} r^{1/2}\right) \tag{35}$$

The potential energy function that satisfies Eq. (34) is readily obtained by solving that equation and it is

$$U = \frac{2B^2}{9}\frac{m\pi^2}{T^2}\left[(Br_0+1)^{3/2} - (Br+1)^{3/2}\right]^2. \tag{36}$$

7. Conclusions

An alternative method to solve the tautochrone problem for an arbitrary potential energy function based on the Laplace transform formalism, instead of the fractional derivative formalism, has been developed. The method has the advantage of being accessible to students of physics, mathematics and engineering at the undergraduate level. Moreover, the method is also developed for polar coordinates, useful in central potential problems.

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