Answer to the Comment on "Continuous groups of transformations and time-dependent invariants"

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In this paper we answer to the Comment on Continuous groups of transformations and time-dependent invariants. Additionally, in order to apply Hegel´s *Aufhebung* concept in physics, some remarks about what we consider the right way to write a "Comment" are given.

Keywords: Lie groups; Lorenz group; dynamical systems; Noether's theorem; infinitesimal transformations.

En el presente trabajo damos respuesta a los comentarios que se hicieron sobre el artículo Continuous groups of transformations and time-dependent invariants. Adicionalmente, para aplicar en física el concepto hegeliano de *Aufhebung*, hacemos algunas observaciones acerca de la que consideramos es la mejor manera de escribir un "Comment".

Descriptores: Grupos de Lie; grupo de Lorentz; sistemas dinámicos; teorema de Noether; transformaciones infinitesimales.

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1. Errata

1.- *For* Eq. (6) of [1]

$$\bar{x}^j = x^j + \xi_a^j\,(x)\,t,$$

read

$$\bar{x}^j = x^j + \xi^j\,(x)\,t, \tag{1}$$

2. *For* Eq. (8) of [1]

$$\bar{x}_1^j = \frac{\partial f}{\partial x^i} x_1^i$$

read

$$\bar{x}_1^j = \frac{\partial f^j}{\partial x^i} x_1^i \tag{2}$$

3. *For*

$$\frac{\partial}{\partial x_1^1} = -x^{2'}\frac{\partial}{\partial x_1^1}\;\partial\, x^{2'}$$

and

$$\frac{\partial}{\partial x_1^2} = \frac{\partial}{\partial x_1^1}\;\partial x^{2'}$$

in the text after Eq. (11) of Ref. 1 *read*

$$\frac{\partial}{\partial x_1^1} = -\frac{x^{2'}}{x_1^1}\frac{\partial}{\partial x^{2'}}$$

and

$$\frac{\partial}{\partial x_1^2} = \frac{1}{x_1}\frac{\partial}{\partial x^{2'}},$$

respectively.

4.- *For* $-t\bar{e}\cdot\bar{\nabla}$ in Eq (23) of Ref. 1 *read* $+t\bar{e}\cdot\bar{\nabla}$

5.- *For* $\omega = \sqrt{\frac{k}{m}}$ after Eq. (26) of Ref. 1 *read* $\omega = \sqrt{\frac{k}{m_0}}$

2. Answers

In the following, we will answer the questions and remarks of Dr. Torres del Castillo concerning the paper "Continuous groups of transformations and time-dependent invariants".

For the statement: "A very clear and detailed treatment of the subject can be found in Ref. 2." **We recommend also reading Ref. 2 of this answer.**

For the statement: "At the beginning of Sec. 2 a one-parameter group of transformations… but the objects appearing on the right hand side of Eqs. (3) are not defined at all."

In Refs. 3 to 5 of this answer it is explained exhaustively, why is it possible to do what has been done in Eq. (2) and (3) of [1]. Of special importance is the topic *fundamental differential equations of a group***. In the first chapter of [6] of this answer this question is also explained.**

For the statement: "Similarly, the functions ξ_a^j, appearing in the…"

If the functions ξ^j are assumed to be regular in the domain of x^j, the integrals of Eq. (5) of [1] can be written in the form

$$\bar{x}^j = x^j + \xi^j(x)t + \xi^i(x)\frac{\partial \xi^j}{\partial x^i}\frac{t^2}{2} + \dots \tag{3}$$

For an infinitesimal value of t, neglecting terms of higher order, one obtains (1) or

$$\bar{x}^j = x^j + \xi^j(x)\,\delta t, \tag{4}$$

where δt is another representation of the infinitesimal value of t (see also Ref. 6 of this answer).

For the statements: "Equation (11) and the following displayed equations ... is not explained at all.", and "From Eqs. (15) one might guess... as a function of x^1." **Our remark is: Let**

$$F(x^1, x^2, x_1^1, x_1^2) = 0 \tag{5}$$

be any differential equation homogeneous in x_1^1 and x_1^2 which admits a one parameter group, if F is an invariant, absolute or relative, of the extended group. In order to give another form to this problem we can write Eq. (5) as

$$f(x^1, x^2, x^{2'}) = 0 \tag{6}$$

where

$$x^{2'} = \frac{dx^2}{dx^1} = \frac{x_1^2}{x_1^1}. \tag{7}$$

On the other hand

$$\frac{\partial}{\partial x_1^1} = \frac{\partial x^{2'}}{\partial x_1^1}\frac{\partial}{\partial x^{2'}} = -\frac{x^{2'}}{x_1^1}\frac{\partial}{\partial x^{2'}} \tag{8}$$

and

$$\frac{\partial}{\partial x_1^2} = \frac{\partial x^{2'}}{\partial x_1^2}\frac{\partial}{\partial x^{2'}} = \frac{1}{x_1^1}\frac{\partial}{\partial x^{2'}} \tag{9}$$

are used to obtain Eq. (11) of Ref. 1. This simplifies the algebra without introducing new quantities.

For the statement: "Also the origin or meaning of Eq. (16)..."

Dr. Torres del Castillo is right in that the f used in (16) is not the same as that used in Eq. (8) of Ref. 1.

For the statement: "It is not explained in which way the *physical meaning...*"

We recommend reading Ref. 8 of this answer and Refs. 3, 4, 6, and 7 of Ref. 1. The problem with most of the mathematical invariants is that they do not provide physical information. However, if an invariant provides useful information so that someone can give it a physical interpretation, or it can be associated with laws of nature, or it can be measured in the laboratory, then we say that it is an invariant with a physical meaning. One can make an arbitrary selection of the parameters of the equation of motion, substitute them in Eq. (19) of Ref. 1 and test if the conditions given above are satisfied.

For the statement: "The meaning of the symbols appearing in Eq. (21) ..." **We recommend reading Ref. 7 of this answer. From Eq. (2) of Ref. 1 it is easy to infer that $x^3 = y$ and $x^4 = z$ and for the meaning of $\bar{\nabla}$ see Ref. 9. At this point, we would like to stress the importance of reading the literature given at the end of a paper. The references play a fundamental role in the definition of the background in which the paper was written and often the literature helps to understand the nomenclature of the equations of a paper. In order to clarify ideas, we have given additional literature in this answer, with the hope that it will be read.**

We understand Dr. Torres del Castillo's fervor for constraining the meaning of the invariant obtained in Eq. (26) to the framework of a mechanical system, namely that constants of motion are quantities which are constant during motion. However, the invariant obtained in Ref. 1 is original because it is the result of the action of the group on the trajectory but not on the time coordinate [10, 11].

For the statement: "... however, not necessarily each of these terms has to be a constant of motion separately."

Nowhere in Ref. 1 is it mentioned that each term of (26) is a constant of motion; we have only given an interpretation of each term of the invariant given by Eq. (26) of Ref. 1.

For the statement: "By contrast with the claim in Sec. 4..." **To obtain the angular momentum conservation law it is necessary to generalize the method to, at least, two spatial coordinates and Eq. (24) determines the selection of the corresponding ξ_a^j in Eq. (10). Clearly, Eq. (24) does not represent the angular momentum conservation law but it is the generator of infinitesimal rotations containing components corresponding to spatial rotations in all three coordinate planes and a finite rotation can be obtained by exponentiation of this equation [6]. In addition, Eq. (24) leaves the Hamiltonian unchanged and hence it is associated with a conserved quantity, in this case, the angular momentum.**

We shall now give some general comments.

On the one hand, we wish to thank Dr. Torres del Castillo for the observations concerning the typing errors contained in the first section of this answer. On the other hand, since this answer is intended to appear in the "Sección de Enseñanza de la Revista Mexicana de Física", we finish with a remark about what we consider the right way to write a "Comment". Under the premise that a researcher must try to interpret the reality and because of the specialization in the scientific work, in our opinion, a "Comment" should contain three main parts:

1.- a constructive analysis of the work,

2.- the lacks, unfinished proposals, errors, and unsatisfied objectives,

3.- an explanation of the achievement of the work in spite of the errors, lacks and other adverse points indicated in the second part.

In addition, the analysis of the work from a different point of view should serve to enrich it. In our opinion the "Comment" of Dr. Torres del Castillo centers mainly on the second point.

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