Complete pure dipole spheroidal electrostatic fields and sources

E. Ley-Koo
Instituto de Física, Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México, D.F., 01000, México,
e-mail: eleykoo@fisica.unam.mx

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Pure dipole distributions of electric charges on the surfaces of prolate and oblate spheroids are identified from the construction of the respective electrostatic potential and intensity fields, inside and outside the spheroids. The Euler connection between the respective prolate and oblate dipole spheroidal harmonics is emphasized; their transition via the spherical harmonics is also recognized; and their limits of a needle surface and a disk surface, respectively, are readily obtained.

Keywords: Pure dipole electrostatic fields and sources; prolate and oblate spheroidal harmonics; Euler connection.

Se identifican distribuciones puramente dipolares de carga eléctrica sobre la superficie de esferoides prolatos y oblatos a partir de la construcción de los respectivos campos elctrostáticos de potencial e intensidad, dentro y fuera de los esferoides. Se destaca la conexión a la Euler entre los respectivos armónicos esferoidales prolatos y oblatos; se reconoce la transición entre ellos vía los armónicos esféricos; y sus límites de superficies de aguja y de disco, respectivamente, se obtienen fácilmente.

Descriptores: Campos y fuentes electrostáticos dipolares puros; armónicos esferoidales prolatos y oblatos; conexión de Euler;

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1. Introduction

The writing of this paper has been motivated by the investigations on electron capture by polar molecules, in which the latter have been modeled as point and finite electric dipole moments [1]. The interested reader may profit by reading [2], in which the connection with Fermi and Teller's work on "The capture of Negative Mesotrons in Matter" [3] recognizes the discovery by these authors of the minimum dipole moment required to bind an electron, twenty years earlier than the molecular theorists. Two recent works, with the titles "Electron structure of a dipole-bound anion confined in a spherical box" and "Addendum to 'Electron structure of a dipole-bound anion confined in a box': the case of a finite dipole" [4,5], are the motivation for reviewing some simple dipole electrostatic field configurations and their sources, emphasizing their different behaviours in the internal and external regions of the sources [6,7].

Most textbooks introduce the finite electric dipole with two point charges of the same magnitude, one negative -q and one positive q, and the relative position vector $\vec{d}$ from the first to the second, by defining the electric dipole moment as [8-10]:

$$\vec{p} = q\vec{d}\cdot \tag{1}$$

Then the point dipole moment is obtained in the limit in which $q \to \infty$ and $d \to 0$.

For other charge distributions, the extensions of the definition of Eq. (1) are directly obtained:

$$\vec{p} = \sum_{i=1}^{N} q_i \vec{r}_i \tag{2}$$

for a collection of point charges;

$$\vec{p} = \int_V dv \rho\left(\vec{r}\right) \vec{r} \tag{3}$$

for charges with a volume density $\rho\left(\vec{r}\right)$ inside a volume V; and

$$\vec{p} = \int_A da \sigma\left(\vec{r}\right) \vec{r} \tag{4}$$

for charges with a surface density $\sigma(\vec{r})$ over an area A.

The electrostatic potential for any of the above dipole moments, in the regions far from the sources, has the familiar form associated with the point dipole,

$$\phi(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{r^2}\cdot \tag{5}$$

Its form follows from the application of the superposition principle and the multipole expansion of the Coulomb potential, keeping only the terms with $l = 1, m = 0$ [8-10].

Reference 11 recognizes that the multipole expansion is also valid for the regions inside the sources. For instance, in the case of a sphere of radius $r' = a$ with a surface charge density

$$\sigma = \sigma_0 \cos\theta, \tag{6}$$

the associated electrostatic potential inside the sphere is

$$\phi\left(\vec{r}\right) = \int da'\, \sigma_0 \cos\theta' \;\frac{\hat{r}'\cdot\vec{r}}{a^2}, \tag{7}$$

where the radial unit vector is given by

$$\hat{r'} = \hat{i}\sin\theta'\cos\varphi' + \hat{j}\sin\theta'\sin\varphi' + \hat{k}\cos\theta', \tag{8}$$

and the area element is $da' = a^2 \sin\theta'\, d\theta'\, d\varphi'$. The integrations over φ' vanish for the terms in cos φ' and sin φ', with the net result:

$$\phi\left(\vec{r}\right) = \frac{4\pi\sigma_0}{3} r \cos\theta = \frac{4\pi\sigma_0}{3}\, z \tag{9}$$

Correspondingly, the electric intensity field in the interior of the sphere is uniform:

$$\vec{E}\left(\vec{r}\right) = -\hat{k}\,\frac{4\pi\sigma_0}{3} \tag{10}$$

It is also instructive to evaluate the charges associated with each hemisphere and the dipole moment of the distribution of Eq. (6):

$$q = \int da\sigma\left(\vec{r}\right) = \int_0^{\pi/2}\int_0^{2\pi} a^2 \sin\theta d\theta d\varphi \sigma_0 \cos\theta = \pi a^2 \sigma_0, \tag{11}$$

$$\vec{p} = \hat{k}\int_0^{\pi}\int_0^{2\pi} a^2 \sin\theta d\theta d\varphi \sigma_0 \cos^2\theta = \hat{k}\frac{4\pi a^3 \sigma_0}{3}\cdot \tag{12}$$

Comparison of Eq. (1) with Eqs. (11) and (12) permits the identification of the separation of the centers of positive and negative charge of the respective hemispheres as $d = 4a/3$ in the equivalent finite dipole.

This contribution extends the exact results of Eqs. (5)-(12) for spheres to the cases of prolate and oblate spheroids, identifying their respective dipole surface charge distributions and alternative sources. The electrostatic potential is constructed in Sec. 2 as the solutions to the Laplace equation in the corresponding spheroidal coordinates with multipolarity restricted to $l = 1, m = 0$, subject to the conditions of being well-behaved inside and outside the charged spheroidal surfaces, and being continuous at the surface. The electric intensity field is evaluated in Sec. 3, as the negative gradient of the electrostatic potential both inside and outside the spheroid, checking that its tangential components at the surface are also continuous. In Sec. 4, it is recognized that its normal components at the surface are discontinuous, and according to Gauss's law their discontinuity measures the surface charge distribution; the charges associated with each hemisphere and the dipole moment of the charge distribution are also evaluated and interpreted. Section 5 illustrates the connection, the transition, and the limits anticipated in the abstract. For the sake of continuity in the reading of the main text, the details about the spheroidal coordinates and harmonic functions, emphasizing the Euler connection, are described in the Appendix. This work is also complementary, in methodology and content, to the lecture on "Electrostatics of Prolate and Oblate Spheroidal Conductors à la Euler" [12].

2. Pure dipole electrostatic potential inside and outside prolate and oblate spheroids

The geometry of the spheroids is incorporated into the respective coordinates defined by Eqs. (A.1) and (A.2). In the prolate case, fixed values of ξ correspond to confocal spheroids with foci at $(x=0, y=0, z=\pm f)$ and eccentricity $1/\xi$, fixed values of η correspond to two-sheath confocal hyperboloids with the same foci and eccentricity $1/\eta$, and fixed values of φ are the usual meridian half-planes. In the oblate case the foci are on a circle on the xy plane at $(x=f\cos\varphi, y=f\sin\varphi, z=0)$, fixed values of ζ correspond to confocal spheroids with eccentricity $1/\sqrt{\zeta^2+1}$, fixed values of η correspond to one-sheath confocal hyperboloids with eccentricity $1/\eta$, and the already mentioned common φ coordinate.

The general solutions to the Laplace equation in spheroidal coordinates of Eqs. (A.18) and (A.20), leading to pure electrostatic dipole fields involve non-vanishing coefficients only for $l = 1$ and $m = 0$. Additionally, the solutions for the interior of the spheroids can involve only Legendre polynomial in both the spheroidal and hyperboloidal coordinates:

$$\phi\left(\xi \leqslant \xi_0, \eta, \varphi\right) = N_p^{int} P_1\left(\xi\right) P_1\left(\eta\right) = N_p^{int}\xi\eta \tag{13}$$

$$\phi\left(\zeta \leqslant \zeta_0, \eta, \varphi\right) = -iN_0^{int} P_1\left(i\zeta\right) P_1\left(\eta\right) = N_0^{int}\zeta\eta \tag{14}$$

using their forms from Eqs. (A.21) and (A.22). The same equations for the Legendre fuctions of the second kind, which are well-behaved far away from the sources, determine the exterior solutions:

$$\begin{aligned}\phi\left(\xi \geqslant \xi_0, \eta, \varphi\right) &= N_p^{ext} Q_1\left(\xi\right) P_1\left(\eta\right)\\ &= N_p^{ext}\left[\frac{\xi}{2}\ln\frac{\xi+1}{\xi-1} - 1\right]\eta,\end{aligned} \tag{15}$$

$$\begin{aligned}\phi\left(\zeta \geqslant \zeta_0, \eta, \varphi\right) &= N_0^{ext} Q_1\left(i\zeta\right) P_1\left(\eta\right)\\ &= N_0^{ext}\left[\zeta\left(\frac{\pi}{2} - \tan^{-1}\zeta\right) - 1\right]\eta\cdot\end{aligned} \tag{16}$$

The imaginary unit i in Eq. (14) is included anticipating that all the physical quantities involved are real. Comparison of Eqs. (9), (13) and (14) show the common linear potential in the z coordinate for the spherical and both spheroidal dipole harmonics. Notice that Eq. (16) does not need the imaginary unit associated with Eq. (14) because $Q_1(i\xi)$ is a real function, Eq. (A.22).

The continuity of the electrostatic potential at the surface of the charged spheroids permits the rewriting of Eqs. (13) and (15) in the form

$$\phi(\xi, \eta, \varphi) = N_p P_1(\xi_<) Q_1(\xi_>) P_1(\eta) \tag{17}$$

where $\xi_<$ and $\xi_>$ are the the smaller and larger of ξ and ξ_0. Similarly Eqs. (14) and (16) become

$$\phi(\zeta, \eta, \varphi) = -iN_0 P_1(i\zeta_<) Q_1(i\zeta_>) P_1(\eta) \tag{18}$$

with the corresponding relationships among $\zeta_<, \zeta_>$, and ζ_0.

The determination of the proportionality constants N_p and N_0 in Eqs. (17) and (18) for the electrostatic potentials associated with the respective prolate and oblate spheroids is implemented in the following sections.

3. Pure dipole electric intensity field inside and outside prolate and oblate spheroids

The evaluation of the electric intensity field inside and outside the charged spheroidal surfaces is obtained as the negative gradient of the respective potential functions of Eqs. (17) and (18). The successive results are:

$$\vec{E}\left(\xi\leqslant\xi_0,\eta,\varphi\right)=-N_p\left[\frac{\hat{\xi}}{h_\xi}\frac{dP_1\left(\xi\right)}{d\xi}Q_1\left(\xi_0\right)P_1\left(\eta\right)+\frac{\hat{\eta}}{h_\eta}P_1\left(\xi\right)Q_1\left(\xi_0\right)\frac{dP_1\left(\xi\right)}{d\xi}\right]$$
$$=-N_pQ_1\left(\xi_0\right)\left[\frac{\hat{\xi}\eta\sqrt{\xi^2-1}+\hat{\eta}\xi\sqrt{1-\eta^2}}{\sqrt{\xi^2-\eta^2}}\right]=-N_pQ_1\left(\xi_0\right)\hat{k} \tag{19}$$

$$\vec{E}\left(\xi\geqslant\xi_0,\eta,\varphi\right)=-N_p\left[\frac{\hat{\xi}}{h_\xi}P_1\left(\xi_0\right)\frac{dQ_1\left(\xi\right)}{d\xi}P_1\left(\eta\right)+\frac{\hat{\eta}}{h_\eta}P_1\left(\xi_0\right)Q_1\left(\xi\right)\frac{dP_1\left(\eta\right)}{d\eta}\right]$$
$$=-N_p\xi_0\left[\frac{\hat{\xi}}{h_\xi}\left(\frac{1}{2}\,ln\frac{\xi+1}{\xi-1}-\frac{\xi^2}{\xi^2-1}\right)+\frac{\hat{\eta}}{h_\eta}\left(\frac{\xi}{2}\;\ln\frac{\xi+1}{\xi-1}-1\right)\right] \tag{20}$$

$$\vec{E}\left(\zeta\leqslant\zeta_0,\eta,\varphi\right)=+iN_0\left[\frac{\hat{\zeta}}{h_\zeta}\frac{dP_1\left(i\zeta_0\right)}{d\zeta}Q_1\left(i\zeta_0\right)P_1\left(\eta\right)+\frac{\hat{\eta}}{h_\eta}P_1\left(i\zeta\right)Q_1\left(i\zeta_0\right)\frac{dP_1\left(\eta\right)}{d\eta}\right]$$
$$=-N_0Q_1\left(i\zeta_0\right)\left[\frac{\hat{\zeta}\eta\sqrt{\zeta^2+1}+\hat{\eta}\zeta\sqrt{1-\eta^2}}{\sqrt{\zeta^2+\eta^2}}\right]=-N_0Q_1\left(i\zeta_0\right)\hat{k} \tag{21}$$

$$\vec{E}\left(\zeta\geqslant\zeta_0,\eta,\varphi\right)=+iN_0\left[\frac{\hat{\zeta}}{h_\zeta}P_1\left(i\zeta_0\right)\frac{dQ_1\left(i\zeta\right)}{d\zeta}P_1\left(\eta\right)+\frac{\hat{\eta}}{h_\eta}P_1\left(i\zeta_0\right)Q_1\left(i\zeta\right)\frac{dP_1\left(\eta\right)}{d\eta}\right]$$
$$=-N_0\zeta_0\left[\frac{\hat{\zeta}}{h_\zeta}\left(\frac{\pi}{2}-\tan^{-1}\zeta-\frac{\zeta^2}{1+\zeta^2}\right)+\frac{\hat{\eta}}{h_\eta}\left(\zeta\left(\frac{\pi}{2}-\tan^{-1}\zeta\right)-1\right)\right] \tag{22}$$

where the last form of each equation uses the explicit forms of the Legendre functions of Eqs. (A.21)-(A.22). Notice also the uniform fields in the interior of the spheroids, Eqs. (19) and (21), where the unit vector $\hat{k}$ along their rotational axis is obtained from Eqs. (A.6) and (A.7); of course, they also follow directly from the linear electrostatic potentials of Eqs. (13)-(14), respectively.

The determination of the proportionality constants N_p and N_0 requires the analysis of the contuinity of the tangential components and the discontinuity of the normal components of the electric intensity field at the $\xi=\xi_0$ and $\zeta=\zeta_0$ spheroidal boundaries, respectively. Equations (19) and (20), and (21) and (22) immediately show the continuity of the respective internal and external $\hat{\eta}$ components. The same pairs of equations exhibit the discontinuities of the normal components, $\hat{\xi}$ and $\hat{\zeta}$, which determine the respective surface charge densities according to Gauss's law, as shown next.

4. Surface electric charge densities on prolate and oblate spheroids

Gauss's law, in its boundary condition form at the prolate and oblate spheroidal surfaces, becomes:

$$\left[\vec{E}\left(\xi=\xi_0^+,\eta,\varphi\right)-\vec{E}\left(\xi=\xi_0^-,\eta,\varphi\right)\right]\cdot\hat{\xi}=4\pi\sigma\left(\xi=\xi_0,\eta,\varphi\right) \tag{23}$$

$$\left[\vec{E}\left(\zeta=\zeta_0^+,\eta,\varphi\right)-\vec{E}\left(\zeta=\zeta_0^-,\eta,\varphi\right)\right]\cdot\hat{\zeta}=4\pi\sigma\left(\zeta=\zeta_0,\eta,\varphi\right)\cdot \tag{24}$$

Then the respective substitutions of Eqs. (19) and (20) in (23), and of Eqs. (21) and (22) in (24) lead to

$$N_p\frac{1}{h_{\xi_0}}\left[-P_1\left(\xi_0\right)\frac{dQ_1\left(\xi_0\right)}{d\xi_0}+\frac{dP_1\left(\xi_0\right)}{d\xi_0}Q_1\left(\xi_0\right)\right]P_1\left(\eta\right)=4\pi\sigma\left(\xi=\xi_0,\eta,\varphi\right) \tag{25}$$

$$iN_0\frac{1}{h_{\zeta_0}}\left[-P_1\left(i\zeta_0\right)\frac{dQ_1\left(i\zeta_0\right)}{d\left(i\zeta_0\right)}+\frac{dP_1\left(i\zeta_0\right)}{d\left(i\zeta_0\right)}Q_1\left(i\zeta_0\right)\right]P_1\left(\eta\right)=4\pi\sigma\left(\zeta=\zeta_0,\eta,\varphi\right) \tag{26}$$

The expressions inside the brackets are identified as the known Wronskians of the Legendre functions [13], which can also be evaluated from the explicit forms of Eqs. (19)-(22), with the values $1/(\xi^2-1)$ and $-i/(\zeta^2+1)$. Therefore, the surface electric charge densities take on the respective forms:

$$\sigma(\xi=\xi_0,\eta,\varphi)=\frac{N_p}{4\pi}\frac{1}{f\sqrt{(\xi_0^2-\eta^2)(\xi_0^2-1)}}P_1(\eta)$$
$$=\frac{fN_p}{4\pi h_\eta(\xi_0,\eta)\,h_\varphi(\xi_0,\eta)}P_1(\eta), \quad (27)$$

$$\sigma(\zeta=\zeta_0,\eta,\varphi)=\frac{N_0}{4\pi}\frac{1}{f\sqrt{(\zeta_0^2+\eta^2)(\xi_0^2-1)}}P_1(\eta)$$
$$=\frac{fN_0}{4\pi h_\eta(\zeta_0,\eta)\,h_\varphi(\zeta_0,\eta)}P_1(\eta), \quad (28)$$

where Eqs. (A.4) and (A.5) for the scale factors have been used. The expressions in Eqs. (27) and (28) are the counterparts of Eq. (6), sharing the same Legendre polynomial of order one "angular" dependence, and differing in the scale factors associated with the "radial" area element.

It is also straightforward to evaluate the charges of the hemispheres and the dipole moments of both types of spheroids, as their counterparts of Eqs. (11) and (12) for the spheres, with the succesive results:

$$q=\int_0^1\int_0^{2\pi}\sigma(\xi_0,\eta,\varphi)\,h_\eta h_\varphi d\eta d\varphi=\frac{fN_p}{4} \quad (29)$$

$$q=\int_0^1\int_0^{2\pi}\sigma(\zeta_0,\eta,\varphi)\,h_\eta h_\varphi d\eta d\varphi=\frac{fN_0}{4} \quad (30)$$

$$\vec{p}=\hat{k}\int_{-1}^1\int_0^{2\pi}\sigma(\xi_0,\eta,\varphi)\,f\xi_0\eta h_\eta h_\varphi d\eta d\varphi$$
$$=\hat{k}\frac{f^2N_p\xi_0}{3}=\hat{k}q\frac{4f\xi_0}{3} \quad (31)$$

$$\vec{p}=\hat{k}\int_{-1}^1\int_0^{2\pi}\sigma(\zeta_0,\eta,\varphi)\,f\zeta_0\eta h_\eta h_\varphi d\eta d\varphi$$
$$=\hat{k}\frac{f^2N_0\zeta_0}{3}=\hat{k}q\frac{4f\zeta_0}{3} \quad (32)$$

According to Eqs. (29)-(30) the proportionality constants are determined by the electric charges of the hemispheres and the focal distances of the respective spheroids. On the other hand, the centers of positive and negative charge are located at

$$z_\pm=\pm\frac{2}{3}f\xi_0, z_\pm=\pm\frac{2}{3}f\xi_0 \quad (33)$$

according to Eqs. (31) and (32), respectively. Notice that $f\xi_0$ and $f\zeta_0$ are the major and minor radius along the axis of rotation for the respective spheroids. Both become the radius of a sphere in the limits of $f\to O$ and $\xi_0\to\infty$, $\zeta_0\to\infty$, coinciding with a in Eq. (12).

5. Discussion on asymptotic and near fields

The initial discussion in this section is centered on the asymptotic fields, focusing on the potential functions of Eqs. (15)-(16) in the limits of ξ and ζ becoming infinite. The discussion of the near fields outside the spheroids is better illustrated using the electric intensity fields of Eqs. (20) and (22), for $\xi=\xi_0$ and $\zeta=\zeta_0$, respectively. Discussions of extensions or connections with other electrostatic configurations are also included.

The potential function of Eq. (15), using the normalization constant in terms of the magnitude of the dipole moment in Eq. (31), and the asymptotic form of $Q_1(\xi)$ in Eq. (A.21), takes the form for the prolate spheroid

$$\phi(\xi,\eta,\varphi)=\frac{p\eta}{f^2\xi^2} \quad (34)$$

which is the same as Eq. (9) for the sphere.

Similarly, for the oblate spheroid the use of the corresponding Eqs. (16) ,(32), and (A.22) for $Q_1(i\zeta)$, leads to

$$\phi(\zeta,\eta,\varphi)=\frac{p\eta}{f^2\zeta^2} \quad (35)$$

which is also equivalent to Eqs. (9) and (34) far away from the dipole. Notice the differences in signs in the normalization constants of Eqs. (17) and (18), and in the asymptotic forms of the Q_1 function of Eqs. (A.21) and (A.22), behind the equivalence in Eqs. (34) and (35).

Comparison of the surface charge densities of Eq. (6) for the sphere and of Eqs. (27) and (28) for the prolate and oblate spheroids, shows their common cosθ or η dependence, and their differences associated with their respective scale factors. They are positive and negative in the northern and southern hemispheres, with maximum and minimun value at the north and south poles, and vanishing at the equator. In the limit of $\xi_0\to 1$, for which the spheroid becomes like a needle with the foci at its ends, the charge densities become infinite at both poles $\eta\to\pm 1$, on account of the vanishing of the scale factors in the denominator of Eq. (27), describing the point effects in the limiting geometry. The limits of $f\to 0$, $\xi_0\to\infty$ and $f\to 0$, $\zeta_0\to\infty$ with $f\xi_0\to a$ and $f\zeta_0\to a$, describe the transition of the respective prolate and oblate spheroids to spheres, with $\eta\to\cos\theta$. Notice also that the limit in which the oblate spheroid becomes a disk, $\zeta_0=0$, its edge with $\eta\to 0$ also has a vanishing factor in the denominator of Eq. (28) for the surface charge density; nevertheless, the latter is zero at the equator as discussed in general in the second sentence of this paragraph.

The electric dipole intensity field from the electrostatic potential of Eq. (5) has the familiar form

$$\vec{E}(\vec{r})=\frac{\left(2\hat{r}\cos\theta-\hat{\theta}\sin\theta\right)}{r^3} \quad (36)$$

By taking its value at the surface of the sphere with $r = a$, the radial direction $\hat{r}$ as that of the vertical and the tangential direction $\hat{\theta}$ as that of the horizontal, the inclination of the electric field is given by its tangent function:

$$\tan i = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta \tag{37}$$

The counterparts for the prolate and oblate spheroids follow from Eqs. (20) and (22), respectively:

$$\tan i = \sqrt{\frac{\xi_0^2 - 1}{1-\eta^2}} \frac{\frac{1}{2}\ln\frac{\xi_0+1}{\xi_0-1} - \frac{\xi_0^2}{\xi_0^2-1}}{\frac{\xi_0}{2}\ln\frac{\xi_0+1}{\xi_0-1} - 1}\eta \tag{38}$$

$$\tan i = \sqrt{\frac{\zeta_0^2 + 1}{1-\eta^2}} \frac{\frac{\pi}{2} - \tan^{-1}\zeta_0 + \frac{\zeta_0^2}{\zeta_0^2+1}}{\zeta_0\left(\frac{\pi}{2} - \tan^{-1}\zeta_0\right) - 1}\eta \tag{39}$$

The common factor in Eqs. (37)-(39) is the one in the polar angle or hyperbolidal coordinates, and the differences are associated with the respective logarithmic derivatives of the Q_1 functions in the respective spheroidal coordinates. The inclination is 90° at the poles and 0°at the equator; it is only latitude dependent for the sphere, but for the spheroids it also depends on their eccentricities.

The grounded prolate and oblate spheroidal conductors in an external uniform electric field studied in Ref. 12 also involve electric dipole fields and surface charges complementary to the ones involved in the present study. For the conductors the fields vanish in the interior, and are the superposition of the uniform field and the respective fields of Eqs. (20) and (22) in the exterior. The superposed fields are perpendicular to the surface of the spheroids.

Appendix

The prolate $(1 \leqslant \xi < \infty, -1 \leqslant \eta \leqslant 1, 0 \leqslant \varphi \leqslant 2\pi)$ and oblate $(0 \leqslant \zeta < \infty, -1 \leqslant \eta \leqslant 1, 0 \leqslant \varphi \leqslant 2\pi)$ spheroidal coordinates are defined via their transformation equations to cartesian coordinates [13]:

$$x = f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \qquad y = f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi, z = f\xi\eta \tag{A.1}$$

$$x = f\sqrt{(\zeta^2+1)(1-\eta^2)}\cos\varphi, \qquad y = f\sqrt{(\zeta^2+1)(1-\eta^2)}\sin\varphi, z = f\zeta\eta. \tag{A.2}$$

Their scale factors and unit vectors follow from

$$d\vec{r} = \hat{i}dx + \hat{j}d\eta + \hat{k}dz = \xi\hat{h}_\xi d\xi + \hat{\eta}h_\eta d\eta + \hat{\varphi}h_\varphi d\varphi = \zeta\hat{h}_\zeta d\zeta + \hat{\eta}h_\eta d\eta + \hat{\varphi}h_\varphi d\varphi \tag{A.3}$$

taking the respective forms

$$h_\xi = f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \quad h_\eta = f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \quad h_\varphi = f\sqrt{(\xi^2-1)(1-\eta^2)} \tag{A.4}$$

$$h_\zeta = f\sqrt{\frac{\zeta^2+\eta^2}{\zeta^2+1}}, \quad h_\eta = f\sqrt{\frac{\zeta^2+\eta^2}{1-\eta^2}}, \quad h_\varphi = f\sqrt{(\zeta^2+1)(1-\eta^2)} \tag{A.5}$$

$$\hat{\xi} = \frac{(\hat{i}\cos\varphi + \hat{j}\sin\varphi)\xi\sqrt{1-\eta^2} + \hat{k}\eta\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}},$$

$$\hat{\eta} = \frac{-(\hat{i}\cos\varphi + \hat{j}\sin\varphi)\eta\sqrt{\xi^2-1} + \hat{k}\xi\sqrt{1-\eta^2}}{\sqrt{\xi^2-\eta^2}}, \quad \hat{\varphi} = -\hat{i}\sin\varphi + \hat{j}\cos\varphi \tag{A.6}$$

$$\hat{\zeta} = \frac{(\hat{i}\cos\varphi + \hat{j}\sin\varphi)\zeta\sqrt{1-\eta^2} + \hat{k}\eta\sqrt{\zeta^2+1}}{\sqrt{\zeta^2+\eta^2}},$$

$$\hat{\eta} = \frac{-(\hat{i}\cos\varphi + \hat{j}\sin\varphi)\eta\sqrt{\zeta^2+1} + \hat{k}\zeta\sqrt{1-\eta^2}}{\sqrt{\zeta^2+\eta^2}}, \quad \hat{\varphi} = -\hat{i}\sin\varphi + \hat{j}\cos\varphi \tag{A.7}$$

The respective Laplace operator and Laplace equation in the prolate and oblate spheroidal coordinates are constructed from Eqs. (4),(5), and take the respective forms:

$$\left\{\frac{1}{f^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(\xi^2-1\right)\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\right] + \frac{1}{f^2(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\varphi^2}\right\}\phi(\xi,\eta,\varphi) = 0 \tag{A.8}$$

$$\left\{\frac{1}{f^2(\zeta^2+\eta^2)}\left[\frac{\partial}{\partial\zeta}\left(\zeta^2+1\right)\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}\right] + \frac{1}{f^2(\zeta^2+1)(1-\eta^2)}\frac{\partial^2}{\partial\varphi^2}\right\}\phi(\xi,\eta,\varphi) = 0 \tag{A.9}$$

Both equations admit factorizable solutions

$$\phi(\xi,\eta,\varphi)=\Xi(\xi)\,H(\eta)\,\Phi(\varphi) \tag{A.10}$$

$$\phi(\zeta,\eta,\varphi)=Z(\zeta)\,H(\eta)\,\Phi(\varphi) \tag{A.11}$$

each factor satisfying the respective ordinary differential equations

$$\left[\frac{d}{d\xi}\left(\xi^2-1\right)\frac{d}{d\xi}-\frac{m^2}{\xi^2-1}\right]\Xi(\xi)=l\,(l+1)\,\Xi(\xi) \tag{A.12}$$

$$\left[\frac{d}{d\eta}\left(1-\eta^2\right)\frac{d}{d\eta}-\frac{m^2}{1-\eta^2}\right]H(\eta)=-l\,(l+1)\,H(\eta) \tag{A.13}$$

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi \tag{A.14}$$

$$\left[\frac{d}{d\zeta}\left(\zeta^2+1\right)\frac{d}{d\zeta}-\frac{m^2}{\zeta^2+1}\right]Z(\zeta)=l\,(l+1)\,Z(\zeta) \tag{A.15}$$

Here m^2 and $l(l+1)$ are the separation constants with the restrictions $m=0,1,2,3,$...due to the periodicity of the φ coordinate, and $l=0,1,2,3,$...in order to ensure good behavior of the regular solutions at the hyperboloidal coordinate regular singular points $\eta=\pm1$.

The solutions to the eigenvalue Eq. (A.14) are the familiar Fourier basis with the Euler connection:

$$e^{im\varphi}=\cos m\varphi+i\sin m\varphi \tag{A.16}$$

The solutions to Eq. (A.13) are the associated Legendre polynomals $P_l^m(\eta)$ and associated Legendre functions of the second kind $Q_l^m(\eta)$, divergent at $\eta=\pm1$, which may be familiar to the reader in spherical coordinates via the identification $\eta=\cos\theta$. Notice that Eqs. (A.13) and (A.12) have the same form, except for the domains of their respective variables, Eq. (A.1). Therefore, the most general solution to the Laplace Eq. (A.8) is the linear combination of prolate spheroidal harmonics:

$$\Phi(\xi,\eta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[A_l^mP_l^m(\xi)+B_l^mQ_l^m(\xi)\right]\left[C_l^mP_l^m(\eta)+D_l^mQ_l^m(\eta)\right]\left[E_m\cos m\varphi+D_m\sin m\varphi\right] \tag{A.18}$$

Notice also that if in Eq. (A.15) we make the change of variable $z=i\zeta$, we obtain

$$-\left[\frac{d}{dz}\left(-z^2+1\right)\frac{d}{dz}-\frac{m^2}{-z^2+1}\right]Z=l\,(l+1)\,Z \tag{A19}$$

which is of the same form as Eqs. (A.12) and (A.13). Correspondingly the potential function of Eq. (A.9) is the linear combination of oblate spheroidal harmonics

$$\Phi(\zeta,\eta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[A_l^mP_l^m(i\zeta)+B_l^mQ^m(i\zeta)\right]\left[C_l^mP_l^m(\eta)+D_l^mQ_l^m(\eta)\right]\left[E^m\cos m\varphi+F^m\sin m\varphi\right] \tag{A.20}$$

For the analysis of the problems in Secs. 2 and 3 the $l=1$, $m=0$ spheroidal harmonics are needed, so we list their prolate and oblate dependencies:

$$P_1(\xi)=\xi,\qquad Q_1(\xi)=\frac{\xi}{2}\ln\frac{\xi+1}{\xi-1}-1 \tag{A.21}$$

$$P_1(i\zeta)=i\zeta,\qquad Q_1(i\zeta)=\frac{i\zeta}{2}\ln\frac{i\zeta+1}{i\zeta-1}-1=\zeta\left(\frac{\pi}{2}-\tan^{-1}\zeta\right)-1 \text{ for } \zeta>1$$

$$Q_1(i\zeta)=\frac{i\zeta}{2}\ln\frac{1+i\zeta}{1-i\zeta}-1=-\zeta\,\tan^{-1}\zeta-1 \text{ for } \zeta<1 \tag{A.22}$$

The last equations distinguish the intervals $\zeta > 1$ and $0 < \zeta < 1$, in order to obtain the correct phase of the respective fractions. The interested reader can work out the details of going from the logarithmic forms to the form in terms of the arc-tangent function of $Q_1(i\zeta)$ for any value of $0 \leqslant \zeta < \infty$. The difference between the values of $Q_1(i\zeta)$ for $\zeta > 1$ and $\zeta < 1$ is the linear term in ζ. Its presence guarantees that $Q_1(i\zeta)$ vanishes in the limit of $\zeta \to \infty$. The continuity of the potential function at $\zeta = 1$, requires that the solutions in the interval $0 < \zeta < 1$ be the surperposition $(\pi/2)\zeta + Q_1(i\zeta)$, which extends the validity of $Q_1(i\zeta)$ for $\zeta > 1$ to the interval under discussion.

In order to illustrate and appreciate the Euler connection in the context of harmonic functions, we may consider the solutions to the Laplace equation in two dimensions and cartesian coordinates:

$$\begin{aligned}\Phi(x,y) &= \left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{iky} + De^{-iky}\right) \\ &= [(A+B)\ \cos kx + (A-B)\sinh kx] \\ &\times [(C+D)\cos ky + i\,(C-D)\sin ky] \end{aligned} \tag{A.23}$$

The important mathematical elements behind the connection are:

i) the second order of the Laplace equation and of the ordinary differential equations in which it separates, and

ii) the opposite signs of the separation constants in the latter equations. The respective consequences follow:

iii) the equations in each independent coordinate admit two independent solutions, and

iv) the solutions in the respective coordinates are connected via the analytical continuation $x \to iy$.

The reader can identify the corresponding elements and consequences in the spheroidal harmonic functions. In fact, while Eqs. (A.13) and (A.14) are common for both types of spheroidal coordinates; in the prolate case Eq. (A.12) is the same as Eq. (A.13) but in different real domains, and in the oblate case analytical continuation of either of those equations, from the real to the imaginary domain $i\zeta$, leads to Eq. (A.15). The Euler connection within Eq. (A.20) in the oblate spheroidal coordinates is similar to that within the cartesian coordinates in Eq. (A.23). The Euler connection between the prolate and oblate spheroidal coordinates is in operation when going between Eqs. (A.12) and (A.19). The explicit forms of the connection for $l = 1$, $m = 0$ are exhibited by Eqs. (A.21)-(A.22), in which the logarithmic versus the $\zeta\tan^{-1}\zeta$ behaviors of the respective Q_1 functions is the analog of the real versus imaginary exponential behaviours in Eq. (A.23).

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