# The meaning of 1 in *j(j+1)*

E. Gomez
*Instituto de Física, Universidad Autónoma de San Luis Potosi,*
*San Luis Potosi 78290,*
*e-mail: egomez@ifisica.uaslp.mx*

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The magnitude of the angular momentum ($J^2$) in quantum mechanics is greater than expected from a classical model. We explain this deviation in terms of quantum fluctuations. A standard quantum mechanical calculation gives the correct interpretation of the components of the angular momentum in the vector model in terms of projections and fluctuations. We show that the addition of angular momentum in quantum mechanics gives results consistent with the classical intuition in this vector model.

*Keywords:* Angular momentum; quantum mechanics; vector model.

La magnitud del momento angular ($J^2$) en mecánica cuántica es mayor que lo esperado en un modelo clásico. Explicamos esta diferencia en términos de las fluctuaciones cuánticas. Un cálculo estándar de mecánica cuántica da la interpretación correcta a las componentes del momento angular en el modelo vectorial en términos de proyecciones y fluctuaciones. Mostramos que la suma de momento angular en mecánica cuántica da resultados consistentes con la intuición clásica en este modelo vectorial.

*Descriptores:* Momento angular; mecánica cuántica; modelo vectorial.

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## 1. Introduction

The operator of angular momentum in quantum mechanics is always a confusing topic for new students. The quantum description of angular momentum involves differential operators or new algebra rules that seem to be disconnected from the classical intuition. For small values of angular momentum one needs a quantum description because the quantum fluctuations are as large as the angular momentum itself. In this regime, the simple classical models generally do not give the right result. In this paper I describe the use of fluctuations in the angular momentum components to produce a vector model compatible with the quantum mechanical result. I show that the addition of angular momenta from a standard quantum mechanical calculation is consistent with the classical intuition using the vector model. The paper is organized as follows: Section 2 shows the problems encountered with the vector model, Sec. 3 works out the details for a spin 1/2 particle, Sec. 4 explains the addition of angular momenta for two spin 1/2 particles, Sec. 5 describes the general case of addition of angular momenta and I give some conclusions at the end.

## 2. Vector model of angular momentum

The presentation of angular momentum in quantum mechanics textbooks demonstrates the following relations: [1]

$$\langle J^2 \rangle = j(j+1)\hbar^2 \tag{1a}$$

$$\langle J_z \rangle = m\hbar, \tag{1b}$$

with $-j \leq m \leq j$. It is usually said that the angular momentum comes in units of $\hbar$. This is consistent with Eq. (1b), since $m$ is an integer, but not with Eq. (1a). For example, if we have one unit of angular momentum ($j = 1$), then $\langle J^2 \rangle = 2\hbar^2$, that is, the magnitude of the angular momentum is $\sqrt{2}$ rather than 1 (from now on we shall express angular momentum in units of $\hbar$). Only the $z$ projection of the angular momentum comes in units of $\hbar$ and not the magnitude. How can we reconcile these two expressions? There is a nice derivation that explains the expression for $J^2$ by averaging the value of $J_z^2$ [2–4]. There are also ways to give an heuristic derivation of the properties of angular momentum [5]. We would like to gain some intuition as to where the extra 1 in Eq. (1a) comes from.

The vector model is often introduced to give a classical analogy to quantum angular momentum [6]. To describe the angular momentum classically by a vector, we must specify its three components $\mathcal{J}_x$, $\mathcal{J}_y$ and $\mathcal{J}_z$. The magnitude of the vector is obtained from the components. The problem with that scheme in quantum mechanics is that it is impossible to measure the three components of the angular momentum with absolute precision. If one measures $J_z$ and $J_y$ exactly, then the uncertainty in $J_x$ grows, that is, there is an uncertainty relation for the components of the angular momentum analogous to the uncertainty relation between position and momentum.

The natural choice for the components of angular momentum in the vector model would be $\mathcal{J}=(\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$. We shall show that this choice (choice A) gives an incorrect value for $\mathcal{J}^2$. A better choice (choice B) for the angular momentum vector is $\mathcal{J}=(\langle J_x^2 \rangle^{\frac{1}{2}}, \langle J_y^2 \rangle^{\frac{1}{2}}, \langle J_z^2 \rangle^{\frac{1}{2}})$. With this choice, the magnitude square of the angular momentum vector gives the correct value: $\mathcal{J}^2 = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle$. In the following section, we shall give a classical interpretation of the components of the angular momentum vector in terms of flucatiations, and we shall use this interpretation to explain the origin of the extra 1 in Eq. (1a).

## 3. Spin 1/2 case

The key point for explaining Eq. (1a) lies in the fluctuations. Take the case of a state with spin 1/2, ($j = s = 1/2$) and $m_s = 1/2$. The values of $\langle S_x\rangle$, $\langle S_y\rangle$ and $\langle S_z\rangle$ are 0, 0 and 1/2 respectively. Choice A for the vector model gives $\mathcal{S} = (0, 0, 1/2)$ and the magnitude square of this vector is $\mathcal{S}^2 = 1/4$, which differs from the result $\langle S^2\rangle = 3/4$ obtained from Eq. (1a). Choice B gives the correct value for $\mathcal{S}^2$ since it was constructed that way. What is the meaning of each component? $\mathcal{S}_z = \langle S_z^2\rangle^{\frac{1}{2}} = (\langle S_z\rangle^2)^{\frac{1}{2}} = \langle S_z\rangle$, and this component reduces to the $z$ projection of the operator S. For $\mathcal{S}_x$ we cannot use the same trick since we are not using an eigenstate of $S_x$. Still, we can relate that component to the fluctuations. The fluctuations of an operator $A$ in quantum mechanics are given by [1]

$$\Delta A^2 = \langle A^2\rangle - \langle A\rangle^2. \tag{2}$$

For the present state and the operator $S_x$ the result is

$$\Delta S_x^2 = \langle S_x^2\rangle. \tag{3}$$

Then $\mathcal{S}_x = \langle S_x^2\rangle^{\frac{1}{2}} = (\Delta S_x^2)^{\frac{1}{2}} = \Delta S_x$ and this component is equal to the fluctuations in the $x$ axis of the operator $S$. The $y$ component gives the same result. The meaning of the vector components in choice B is that $\mathcal{S}_x$ and $\mathcal{S}_y$ are fluctuations and $\mathcal{S}_z$ is the projection in the corresponding axis. The quantum mechanical calculation of the fluctuations gives

$$\Delta S_x^2 = \left\langle \frac{1}{2}\right| S_x^2 \left|\frac{1}{2}\right\rangle = \frac{1}{4}, \tag{4}$$

so that $\Delta S_x = 1/2$, and similarly $\Delta S_y = 1/2$. The vector is $\mathcal{S} = (1/2, 1/2, 1/2)$ and the magnitude square of the vector is $\mathcal{S}^2 = 3/4$, which is the correct value. The value of $\mathcal{S}^2$ in choice A is $s^2 = 1/4$. Instead in choice B, $\mathcal{S}_x$ and $\mathcal{S}_y$ contribute to $\mathcal{S}^2$ through the fluctuations, giving the value of $s(s+1) = 3/4$.

## 4. Addition of angular momenta

We construct any value of angular momentum by adding several spin 1/2 particles. We show how the vector model works for two spin 1/2 particles. The sum of two spin 1/2 particles gives a total angular momentum of $j = 1$ or $j = 0$. Take first the case of the state with $j = 1$ and $m = 1$. The state is represented in quantum mechanics by $|1/2, 1/2\rangle$ where the numbers represent the $z$ projection of the spin of particles 1 and 2 respectively. The objective is to calculate the value of $\langle J^2\rangle$, with $J = S_1 + S_2$, the sum of the spin contributions. The quantum mechanical result from Eq. (1a) is $\langle J^2\rangle = 2$, and we want to explain this in terms of the vector model.

The expression for $J^2$ is

$$J^2 = J_x^2 + J_y^2 + J_z^2 = (S_{x1} + S_{x2})^2 + (S_{y1} + S_{y2})^2 + (S_{z1} + S_{z2})^2, \tag{5}$$

where the subscripts 1 and 2 refer to particles 1 and 2 respectively. There is no question as to how to calculate the expectation values in quantum mechanics, but if we think in terms of the vector model we are in trouble since we have to add two vectors that are a mixture of projections and fluctuations. We show the correct recipe for adding these vectors from a quantum mechanical calculation and show that it is consistent with the classical intuition.

Take $\mathcal{J}_z$ first. The sum is again simplified since we use an eigenstate of the operator. We have

$$\mathcal{J}_z = (\langle(S_{z1} + S_{z2})^2\rangle)^{\frac{1}{2}} = \langle S_{z1}\rangle + \langle S_{z2}\rangle,$$

that is, $\mathcal{J}_z$ is just the direct sum of the individual projections. The $x$ component gives

$$\mathcal{J}_x = (\langle(S_{x1} + S_{x2})^2\rangle)^{\frac{1}{2}} = (\langle S_{x1}^2\rangle + 2\langle S_{x1}S_{x2}\rangle + \langle S_{x2}^2\rangle)^{\frac{1}{2}} = (\Delta S_{x1}^2 + \Delta S_{x2}^2)^{\frac{1}{2}}. \tag{6}$$

The two contributions add up in quadrature. This is to be expected since the $x$ component for each spin in the vector model corresponds to fluctuations (or noise), and the proper way to add uncorrelated noise is in quadrature. For a classical variable $w = u + v$, where $u$ and $v$ are fluctuating variables, the noise in $w$ is given by [7]

$$\sigma_w^2 = \sigma_u^2 + \sigma_v^2 + 2\sigma_{uv}^2. \tag{7}$$

The quantum mechanical expression for the fluctuations of $J_x = S_{x1} + S_{x2}$ for the present state is

$$\Delta J_x^2 = \langle S_{x1}^2\rangle + \langle S_{x2}^2\rangle + 2\langle S_{x1}S_{x2}\rangle, \tag{8}$$

where the similarity between the last two expressions is evident. The state we are considering has the two spins aligned. Since the two spins are independent, we expect their noise to be uncorrelated. The calculation of the correlation term [last term in Eq. (8)] gives

$$\left\langle \frac{1}{2}, \frac{1}{2}\right| S_{x1}S_{x2} \left|\frac{1}{2}, \frac{1}{2}\right\rangle = 0, \tag{9}$$

and the sum for $\mathcal{J}_x$ reduces to Eq. (6).

We can understand the addition of angular momentum in the vector model: the components that are projections add up directly, whereas the components that are fluctuations add up as noise. The noise can have different degrees of correlation as calculated by the last term in Eq. [8]. The noise for the present state happens to be uncorrelated [Eq. (9]. The vectors for the individual spins are $\mathcal{S}_1 = \mathcal{S}_2 = (1/2, 1/2, 1/2)$, and their sum gives $\mathcal{J} = (1/\sqrt{2}, 1/\sqrt{2}, 1)$, where we have added the $x$ and $y$ components in quadrature and the $z$ components directly. The magnitude square of the vector gives $\mathcal{J}^2 = 2$

in accordance with Eq. (1a). The result should be contrasted with a naive addition of the vectors $\mathcal{S}_1+\mathcal{S}_2=(1,1,1)$, which gives a magnitude square of 3. The case for the state with $j=1$ and $m=-1$ works the same way.

The state with $j=1$ and $m=0$ is more interesting. The state is the symmetric combination of the spins,

$$\left(\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle+\Big|-\frac{1}{2},\frac{1}{2}\Big\rangle\right)\Big/\sqrt{2}.$$

The vectors for the individual spins are $\mathcal{S}_1=(1/2,1/2,1/2)$ and $\mathcal{S}_2=(1/2,1/2,-1/2)$. We take the negative value of the square root in $\mathcal{S}_{2z}$ since the $z$ component of the two spins point in opposite directions. We choose $\mathcal{S}_{1z}$ ($\mathcal{S}_{2z}$) positive (negative), but the opposite is equally correct. In the direct sum of the $z$ components, $\mathcal{S}_{1z}$ and $\mathcal{S}_{2z}$ cancel each other giving 0. The correlation term in the $x$ component for this state gives

$$\frac{1}{\sqrt{2}}\left(\Big\langle\frac{1}{2},-\frac{1}{2}\Big|+\Big\langle-\frac{1}{2},\frac{1}{2}\Big|\right)\ S_{x1}S_{x2}$$
$$\times\left(\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle+\Big|-\frac{1}{2},\frac{1}{2}\Big\rangle\right)\frac{1}{\sqrt{2}}=\frac{1}{4}, \qquad (10)$$

and the calculation for the fluctuations from Eq. (8) gives $\Delta J_x^2=\Delta J_y^2=1$. The sum vector is $\mathcal{J}=(1,1,0)$, which gives the correct result for the magnitude square $\mathcal{J}^2=2$. The noise calculation for $\mathcal{J}_x$ and $\mathcal{J}_y$ tells us that we have perfectly correlated noise, so instead of adding the two contributions in quadrature, we add them directly (each one equal to 1/2 giving a total of 1). It is not surprising that the noise behaves in a correlated manner, since we use the symmetric combination of the spins.

Finally we have the case with $j=0$ and $m=0$. The state is the anti-symmetric combination of the spins, and we expect the noise to be anti-correlated. The vectors for the indivicual spins are still $\mathcal{S}_1=(1/2,1/2,1/2)$ and $\mathcal{S}_2=(1/2,1/2,-1/2)$. The correlation term in the $x$ component for this state is now

$$\frac{1}{\sqrt{2}}\left(\Big\langle\frac{1}{2},-\frac{1}{2}|-\Big\langle-\frac{1}{2},\frac{1}{2}|\right)\ S_{x1}S_{x2}$$
$$\times\left(\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle-\Big|-\frac{1}{2},\frac{1}{2}\Big\rangle\right)\frac{1}{\sqrt{2}}=-\frac{1}{4}, \qquad (11)$$

and the calculation for the fluctuations from Eq. (8) gives $\Delta J_x^2=\Delta J_y^2=0$. The sum vector is $\mathcal{J}=(0,0,0)$, with a magnitude square of $\mathcal{J}^2=0$ as expected. The anti-symmetric combination of the spins results in noise that is perfectly anti-correlated (due to the minus sign in the wave function). The noise subtracts directly ($1/2-1/2=0$) and not in quadrature for the $x$ and $y$ components. It seems that the noise in $J_x$, $J_y$ and $J_z$ is zero for the state. From the point of view of the sum, the individual perpendicular fluctuations are actually not zero, it is because of the correlations that the fluctuations of $J$ become zero.

## 5. General case

Any other value of angular momentum can be constructed using the same scheme. For example, to obtain $j=3/2$ we add three spin 1/2 particles. Each particle contributes some amount to the value of $\mathcal{J}_z$ and also to the fluctuations in the perpendicular components. There is some degree of correlation between the spins, depending on the $m$ value chosen. The correlation between spins can be calculated from the crossed term in Eq. (8). The correlation term between spins $i$ and $k$ in the $x$ component for the state with angular momentum $j$ and projection $m$ is

$$\langle j,m|\ S_{xi}S_{xk}\ |j,m\rangle. \qquad (12)$$

It is not trivial to predict the result of this calculation except for the maximum and minimum projections. All the spins are uncorrelated if $m=j$ or $m=-j$. For any other projection, there will be some intermediate degree of correlation between spins that can be calculated from Eq. (12). For the maximum projection, the $x$ (and $y$) components of all the individual spins add up in quadrature to give

$$\mathcal{J}_x=\sqrt{\Delta S_{x1}^2+\Delta S_{x2}^2+...+\Delta S_{x(2j)}^2}$$
$$=\sqrt{2j(1/4)}=\sqrt{j/2}. \qquad (13)$$

The vector sum is $\mathcal{J}=(\sqrt{j/2},\sqrt{j/2},j)$ with a magnitude square $\mathcal{J}^2=j/2+j/2+j^2=j(j+1)$, where the 1 comes from the perpendicular components.

## 6. Conclusions

We explain the 1 in the expression $j(j+1)$ in terms of the quantum fluctuations of the $x$ and $y$ components of the angular momentum. We include the fluctuations to describe the addition of angular momenta in the vector model. The vector components can be projections or fluctuations, and they have different formulas for addition. The correlations in the fluctuations cannot be ignored. Formula (1a) tells us that angular momentum does not come in units of $\hbar$, but instead it comes in units of $\sqrt{1+(1/j)}\hbar$. This is not even a uniform unit, but depends on the value of $j$ in a complicated way. This happens because some of the components of $J$ add up directly and others in quadrature. Only in the limit of a large $j$ can we recover the well-known $\hbar$ unit of angular momentum. For a small value of $j$, the quantum noise cannot be ignored.

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