Motion of a falling drop with accretion using canonical methods

G. Hernandez and G. del Valle
Area de Física Atómica y Molecular Aplicada,
División de Ciencias Básicas e Ingeniería,
Universidad Autónoma Metropolitana-Azcapotzalco,
Apartado Postal 21-463, México, D.F., 04000 México,
e-mail: [gpe,ddg]@correo.azc.uam.mx

I. Campos
Departamento de Física, Facultad de Ciencias,
Universidad Nacional Autónoma de México,
Apartado Postal 21-463, México, D.F., 04000, México,
e-mail: iecampos@prodigy.net.mx

J.L. Jimenez
Departamento de Física, División de Ciencias Básicas e Ingeniería,
Universidad Autónoma Metropolitana-Iztapalapa,
Apartado Postal 21-463, México D.F., 04000, México,
e-mail: jlj@xanum.uam.mx

Recibido el 12 de junio de 2008; aceptado el 6 de octubre de 2008

The motion of a falling drop whose mass grows by accretion is studied with canonical methods. This approach requires the introduction of S-equivalent non natural Lagrangians. That is, we have to consider Lagrangians that give rise to equations of motion that are not exactly the same as the equations of interest, but anyway they share the same solutions. We study three examples of laws of accretion: mass growing linearly with time, mass growing linearly with the surface of the drop, and mass growing proportionally to the product of surface and velocity of the drop. In all cases we recover the results obtained by means of the Newtonian methods, which we expose in table I.

Keywords: Variable mass systems; accretion; S-equivalent Lagrangians; Hamilton-Jacobi formalism.

Se estudia, mediante métodos canónicos, la caida de una gota cuya masa crece por acreción. Este enfoque requiere la introducción de Lagrangianos S-equivalentes, no naturales. Esto es, tenemos que considerar Lagrangianos que conducen a ecuaciones de movimiento que no son exáctamente las mismas que las de interés, pero que comparten con ellas las mismas soluciones. Estudiamos tres ejemplos de leyes de acreción: incremento lineal de la masa con el tiempo, incremento de la masa proporcional a la superficie de la gota, e incremento proporcional al producto de la superficie de la gota por la velocidad de la misma. En todos los casos recobramos los resultados obtenidos mediante los métodos Newtonianos, los cuales presentamos en la tabla I.

Descriptores: Sistemas de masa variable; acreción; Lagrangianos S-equivalentes; Formalismo de Hamilton-Jacobi.

PACS: 45.20. Jj; 02.30.Jr

1. Introduction

Since the 1960's there has been an increasing interest in classical mechanics with the study of chaos in non linear systems and also with the study of non-autonomous systems, which include dissipative and open systems (see for example Refs. 1 to 11). In particular there has recently been a great interest in applying canonical methods to dissipative systems, which requires the introduction of S-equivalent Lagrangians. In this work we apply this technique to an open, dissipative and non-autonomous system.

As we have argued elsewhere [12,13] (see also Refs. 13 to 16), the equation of motion appropriate for treating variable mass systems is Cauchy's equation of motion for continuous media:

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \rho\mathbf{b} + \nabla\cdot\overleftrightarrow{T}, \tag{1}$$

where ρ is the mass density, $\mathbf{v}$ is the velocity of a "particle" of the medium, $\mathbf{b}$ is the body force per unit mass and $\overleftrightarrow{T}$ is the stress tensor, associated with the force that comes from the stress at the surface that surrounds the system. In our case this term would result in the buoyancy force which we discard, assuming a small drop. With the aid of the continuity equation this equation can be transformed into the Eulerian or spatial form,

$$\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t} = \rho\mathbf{b} - \nabla\cdot\rho\mathbf{v}\mathbf{v}. \tag{2}$$

A volume integration gives, as a result,

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F}_b - \oint_S \rho\mathbf{v}\mathbf{v}\cdot\hat{n}dS, \tag{3}$$

TABLE I. In these expressions, m_o, r_o, h are the initial mass, initial radius of the drop and initial height, respectively, and $h_o = h + r_o/\beta$. The last solution gives y implicitly, and after an inversion of the series representation gives y as a function of t.

$Accretion\ Law$	$Body\ Force$	$Equation\ of\ Motion$	$Solution$
$\frac{dm}{dt} = bt$	$kv - mg$	$\frac{dv}{dt} + \frac{(b+k)v}{m} = -g$	$y(t) = h - \frac{m_0^2\, g}{2b^2(-\lambda+1)} - \frac{m^2\, g}{2b^2(\lambda+1)} + \frac{m_0^{\lambda+1}\, m^{-\lambda+1}\, g}{b^2(-\lambda^2+1)}$
$\frac{dm}{dt} = \alpha 4\pi r^2$	$-mg$	$\frac{dv}{dt} + \frac{1}{m}\left(\alpha 4\pi r^2\right) v = -g$	$y(t) = h - \frac{g}{8\alpha^2}\left[(r_0 + \alpha t)^2 + \frac{r_0^4}{(r_0 + \alpha t)^2} - 2r_0^2\right]$
$\frac{dm}{dt} = \beta 4\pi r^2 v$	$-mg$	$\frac{dv}{dt} - \frac{3\beta v^2}{r_0 + \beta(h - y(t))} = -g$	$t = \sqrt{\frac{2\left(\frac{r_0}{\beta}\right)}{7g}} \left((h_0 - y)^7 \left(\frac{\beta}{r_0}\right)^7 - 1\right)^{\frac{1}{2}}$ ${}_2F_1\left[\frac{3}{7}, \frac{1}{2}, \frac{3}{2}, -\left((h_0 - y)^7 \left(\frac{\beta}{r_0}\right)^7 - 1\right)\right]$

where now $\mathbf{p}$ is the momentum, $\mathbf{F}_b$ is the body force, which in our case is gravity and friction, and $\mathbf{\Phi}$ is the momentum flux within the volume given by the first integral in Eq. (3). Because the assumed spherical symmetry of the drop, $\mathbf{\Phi}$, and the surface force are zero, our equation of motion is

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F}_b = m\mathbf{g} - k\mathbf{v}. \tag{4}$$

This equation of motion, for the three specific laws of accretion, gives the results summarized in Table I. The derivation of these results can be found in Ref. 13.

We hope that the present work will be usefull for advanced undergraduate and beginning graduate students as well as for teachers of this level. The interested reader will find in these examples a useful illustration of the applications of canonical methods to open systems. In this way, he or she goes beyond the usual problems: the harmonic oscillator, and the Kepler problem. These problems are rather simple since the Hamiltonians are time-independent and can be solved with the generating function $S(p, Q{=}E, t){=}W(p, E){-}Et$. On the other hand, since the end of the 19th, century we have known from Helmholtz [17-19] that not every second order differential equation can be derived from a variational principle. One example of this situation is the equation of motion of a harmonic oscillator with friction linear in the velocity. However, we can deal with some of these cases by introducing non-natural (not of the form $L{=}T{-}V$), S-equivalent Lagrangians [17-20]. These Lagrangians lead to equations of motion that may differ from the equation of interest, but somehow both equations have the same solutions and permit the canonical treatment of systems with friction and other open systems [21-33]. The Lagrangian can be obtained by solving the inverse problem of the calculus of variations [17-19] or by guessing, as is usually done in physics. This is the technique that will allow us to use the canonical treatment of a falling drop with accretion. We also show an example where the Poisson method can be applied to an open system. The few examples that illustrate the Poisson method are the harmonic oscillator and the particle in free fall.

This work continues the treatment by canonical methods of open systems, such as the rocket and falling rope, given in previous work [12,29-33].

2. Accretion proportional to time

The development of the canonical approach requires a Lagrangian from which the equation of motion of our dynamical system can be derived. Then a Hamiltonian can be obtained and from it the Hamilton-Jacobi equation can be formed. In the case of a falling drop with accretion proportional to time, it is easy to verify that the Lagrangian,

$$L = \frac{1}{2} m^{\lambda}(t)\, v^2 - m^{\lambda}(t) g y, \tag{5}$$

leads immediately to the equation

$$m^{\lambda}(t)\left(\frac{dv}{dt} + \frac{\lambda b}{m(t)} v + g\right) = 0, \tag{6}$$

whose solutions are the same as those of Eq. (10). For $\lambda = 1$, this Lagrangian is an instantaneous natural Lagrangian.

From this Lagrangian we obtain the Hamiltonian,

$$H = \frac{p^2}{2m^{\lambda}} + m^{\lambda}(t)\, g y, \tag{7}$$

with which we can follow the Hamilton-Jacobi method and describe the motion as a flux in phase space.

Since the Hamiltonian is linear in y, it is convenient to use a generating function of type F_3 that gives

$$y = -\frac{\partial F_3}{\partial p}. \tag{8}$$

Then the corresponding Hamilton-Jacobi equation is

$$\frac{p^2}{2m^\lambda (t)} - m^\lambda (t)\, g \left(\frac{\partial F_3}{\partial p}\right) + \left(\frac{\partial F_3}{\partial t}\right) = 0. \tag{9}$$

Thanks to the use of F_3, the H-J equation is linear, and therefore is easier to solve even if it depends on time. The standard method for solving this type of equation is the method of characteristics, but we shall solve it first by some guessing, as we did in a previous work [12,30-33].

2.1. Guessing a solution

First we guess that F_3 may be of the form

$$F_3 (p, Q, t) = \gamma_1 (t)\, p^2 + \gamma_2 (t)\, p + \xi (t) + \phi(Q, p, t). \tag{10}$$

We choose $\gamma_1 (t)$ in such a way that the first term on the left side of Eq. (9) is eliminated. This first term of Eq. (10) will also produce a term linear in p after deriving with respect to p, which we try to cancel by choosing an appropriate $\gamma_2 (t)$. Then there will be an extra function of time that will be eliminated with an adequate choice of $\xi (t)$. What remains is a simple equation for $\phi(Q, p, t)$.

Thus, if the following conditions are satisfied,

$$\frac{1}{2m^\lambda} + \frac{d\gamma_1}{dt} = 0, \tag{11}$$

$$-2g\gamma_1 m^\lambda + \frac{d\gamma_2}{dt} = 0, \tag{12}$$

$$-g\gamma_2 m^\lambda + \frac{d\xi}{dt} = 0, \tag{13}$$

then the H-J equation becomes

$$\frac{\partial \phi}{\partial t} - m^\lambda g \frac{\partial \phi}{\partial p} = 0, \tag{14}$$

which can be solved by a separation of variables, thus obtaining

$$\phi = \exp[-Q \left(p + \frac{g m^{\lambda+1}}{b (\lambda + 1)}\right). \tag{15}$$

Here Q is the constant of separation, and the solutions to Eqs. (11) to (13) lead to the results

$$\gamma_1 (t) = -\frac{1}{2b} \frac{m^{-\lambda+1}}{(-\lambda + 1)}, \tag{16}$$

$$\gamma_2 (t) = -\frac{g m_0 t}{b (-\lambda + 1)} - \frac{1}{2} \frac{g t^2}{(-\lambda + 1)}, \tag{17}$$

and

$$\xi (t) = \frac{g^2}{b^2 (-1 + \lambda)} \frac{m^{\lambda+1}}{(3 + \lambda)} \times \left[-\frac{m_0^2}{b (1 + \lambda)} + m_0 t + \frac{b}{2} t^2\right]. \tag{18}$$

The substitution of Eqs. (16) to (18) into Eq. (10) leads to the result we are looking for:

$$\begin{aligned} F_3 (p, Q, t) = {} & \left(-\frac{1}{2b} \frac{m^{-\lambda+1}}{(-\lambda + 1)}\right) p^2 \\ & + \left(-\frac{g m_0 t}{b (-\lambda + 1)} - \frac{1}{2} \frac{g t^2}{(-\lambda + 1)}\right) p \\ & - \frac{g^2}{b^2 (-1 + \lambda)} \frac{m^{\lambda+1}}{(3 + \lambda)} \\ & \times \left[\frac{m_0^2}{b (1 + \lambda)} - m_0 t - \frac{b}{2} t^2\right] \\ & + \exp[-Q \left(p + \frac{g m^{\lambda+1}}{b (\lambda + 1)}\right). \end{aligned} \tag{19}$$

Before proceeding to find $y(t)$ from the generating function F_3, it is convenient to rewrite the solution found by the Newtonian method in terms of γ_1 and γ_2.

First we note that

$$\gamma_1 (0) = \gamma_{10} = -\frac{1}{2b} \frac{m_0^{-\lambda+1}}{(-\lambda + 1)}, \tag{20}$$

and

$$\gamma_2 (0) = \gamma_{20} = 0. \tag{21}$$

It is also useful to define

$$\eta (t) = \frac{g}{b} \frac{m^{\lambda+1}}{(\lambda + 1)}, \tag{22}$$

so that

$$\eta (0) = \eta_0 = \frac{g}{b} \frac{m_0^{\lambda+1}}{(\lambda + 1)}. \tag{23}$$

Using these expressions, we can write

$$\begin{aligned} y (t) = {} & h - 2\eta (0)\, \gamma_1 (t) \\ & + 2\eta (0)\, \gamma_1 (0) + \gamma_2 (t) \frac{(-\lambda + 1)}{(\lambda + 1)}. \end{aligned} \tag{24}$$

This result will simplify the calculations that follow.

Now, from Eqs. (8) and (10), we find that

$$\begin{aligned} y (t) = {} & -\frac{\partial F_3}{\partial p} = -2\gamma_1 (t)\, p \\ & - \gamma_2 (t) + Q \exp[-Q (p + \eta)], \end{aligned} \tag{25}$$

which after taking into account the initial conditions, $y(0)=h$, $p(0)=0$, results in

$$h = Q\exp[-Q\eta_0]. \tag{26}$$

Besides, H-J theory says that

$$P = -\frac{\partial F_3}{\partial Q}. \tag{27}$$

Thus we have that

$$P = (p+\eta)\exp\left[-Q\left(p+\eta\right)\right], \tag{28}$$

and for $t=0$ this expression must be

$$P = \eta_0\exp\left[-Q\eta_0\right]. \tag{29}$$

It seems very complicated to get $p(t)$ from Eq. (25) and (26), but we can proceed in the following way.

Since P and Q are constant in time, then from Eq. (28) we conclude that

$$p+\eta = \theta = \text{constant}. \tag{30}$$

But since $p(0)=0$, then

$$\eta_0 = \theta, \tag{31}$$

and so we have

$$p\left(t\right) = -\eta\left(t\right) + \eta_0. \tag{32}$$

On the other hand, from Eq. (26) we have that

$$Q = h\exp[Q\eta_0]. \tag{33}$$

Then using Eq. (32) in (28), we find that

$$\exp[-Q(p+\eta)] = \exp[-Q\eta_0], \tag{34}$$

so that Eq. (25) can be written as

$$y\left(t\right) = h + 2\gamma_1\left(t\right)\eta\left(t\right) - 2\gamma_1\left(t\right)\eta_0 - \gamma_2\left(t\right). \tag{35}$$

Also, it is not hard to show that

$$2\gamma_1\left(t\right)\eta\left(t\right) - \gamma_2\left(t\right) = 2\gamma_{10}\eta_0 + \frac{(-\lambda+1)}{(\lambda+1)}\gamma_2, \tag{36}$$

so that Eq. (35) becomes

$$y\left(t\right) = h - 2\eta_0\gamma_1\left(t\right) + 2\eta_0\gamma_{10} + \gamma_2\left(t\right)\frac{(-\lambda+1)}{(\lambda+1)}, \tag{37}$$

which after substitution of $\eta_0, \gamma_{10}, \gamma_1\left(t\right), \gamma_2\left(t\right)$, gives the known result,

$$y\left(t\right) = h - \frac{m_0^2 g}{2b^2\left(-\lambda+1\right)} - \frac{m^2 g}{2b^2\left(\lambda+1\right)} + \frac{m_0^{\lambda+1}m^{-\lambda+1}g}{b^2\left(-\lambda^2+1\right)}. \tag{38}$$

2.2. The method of characteristics

Now we shall solve the H-J equation by the method of characteristics, which is the standard method for solving this type of equation [34,35]. Besides, this may be useful since most examples in texts involve Hamiltonians not depending on time and use the generating function F_2.

The H-J Eq. (9) is of, type

$$P\left(x,y,z\right)\frac{\partial z}{\partial x} + Q\left(x,y,z\right)\frac{\partial z}{\partial y} = R\left(x,y,z\right). \tag{39}$$

The solution to this equation is obtained from the associated Lagrange system of equations,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{40}$$

The corresponding identifications between Eq. (9) and Eq. (39) are

$$\begin{array}{ccccc} x & \rightarrow & p & & P=-m^{\lambda}g \\ y & \rightarrow & t & & Q=1 \\ z & \rightarrow & F_3 & & R=\frac{-p^2}{2m^{\lambda}} \end{array} \tag{41}$$

with which the system, Eq. (40), becomes

$$\frac{dp}{-m^{\lambda}g} = \frac{dt}{1} = \frac{dF_3}{-\dfrac{p^2}{2m^{\lambda}}}. \tag{42}$$

From the first equality we find immediately that

$$\begin{aligned} p\left(t\right) &= -\int m^{\lambda}g dt + C_1 \\ &= -\frac{m^{\lambda+1}g}{\left(\lambda+1\right)b} + C_1. \end{aligned} \tag{43}$$

It must be emphasized that here C_1, and C_2 below, are constants of motion. So, they must be regarded as parameters, rather than true constants [36].

The next equality can be written, using the identification in Eq. (41), as

$$-\frac{1}{2m^{\lambda}}\left(-\frac{m^{\lambda+1}g}{\left(\lambda+1\right)b} + C_1\right)^2 dt = dF_3\left(p,Q,t\right), \tag{44}$$

which can be integrated immediately, giving as a result

$$\begin{aligned} F_3\left(p,Q,t\right) = &-\frac{m^{\lambda+3}g^2}{2\left(\lambda+1\right)^2\left(\lambda+3\right)b^3} \\ &+ \frac{C_1 m^2 g}{2\left(\lambda+1\right)b^2} - \frac{C_1^2 m^{-\lambda+1}}{2\left(-\lambda+1\right)b} + C_2. \end{aligned} \tag{45}$$

The integration constant C_1 can be obtained from Eq. (43); that is,

$$C_1 = p + \frac{m^{\lambda+1}g}{\left(\lambda+1\right)b}. \tag{46}$$

Now, according to the method of characteristics the solution is a function such that

$$f\left(C_1, C_2\right) = 0. \tag{47}$$

We can choose

$$C_2 = QC_1, \tag{48}$$

thus obtaining

$$C_2 = Q\left(p + \frac{m^{\lambda+1}g}{(\lambda+1)\,b}\right). \tag{49}$$

By substituting Eqs. (46) and (49) in Eq. (45), we obtain

$$\begin{aligned} F_3\left(p, Q, t\right) = &-\frac{g^2 m^{\lambda+3}}{2\left(\lambda+1\right)^2\left(\lambda+3\right)b^3} \\ &+ \frac{m^2 g}{2\left(\lambda+1\right)b^2}\left(p + \frac{m^{\lambda+1}g}{(\lambda+1)\,b}\right) \\ &- \frac{m^{-\lambda+1}}{2b\left(-\lambda+1\right)}\left(p + \frac{m^{\lambda+1}g}{(\lambda+1)\,b}\right)^2 \\ &+ Q\left(p + \frac{m^{\lambda+1}g}{(\lambda+1)\,b}\right), \end{aligned} \tag{50}$$

which is the generating function we were looking for that permits us to solve the dynamical problem according to the H-J method.

Since

$$P = -\frac{\partial F_3}{\partial Q}, \tag{51}$$

then

$$P = -\left(p + \frac{m^{\lambda+1}g}{(\lambda+1)\,b}\right). \tag{52}$$

Since P is a constant of motion, the solution satisfying the initial condition $p(0) = 0$ is obviously

$$p\left(t\right) = -\frac{m^{\lambda+1}g}{(\lambda+1)\,b} + \frac{m_0^{\lambda+1}g}{(\lambda+1)\,b}. \tag{53}$$

On the other hand, we have that

$$\begin{aligned} y = -\frac{\partial F_3}{\partial p} = &\frac{m^{-\lambda+1}}{b\left(-\lambda+1\right)}p + \frac{m^2 g}{b^2\left(-\lambda^2+1\right)} \\ &- \frac{m^2 g}{2b^2\left(\lambda+1\right)} - Q. \end{aligned} \tag{54}$$

Substituting now the expression for p already found, Eq. (53), we obtain, after some simplifying,

$$y\left(t\right) = -\frac{m^2 g}{2b^2\left(\lambda+1\right)} + \frac{m^{-\lambda+1}m_0^{\lambda+1}g}{b^2\left(-\lambda^2+1\right)} - Q. \tag{55}$$

Q is specified through the initial condition, $y(0) = h$, obtaining

$$Q = -h + \frac{m_0^2 g}{2b^2\left(-\lambda+1\right)}. \tag{56}$$

With this we find the solution to the dynamical problem, that is,

$$y\left(t\right) = h - \frac{m_0^2 g}{2b^2\left(-\lambda+1\right)} - \frac{m^2 g}{2b^2\left(\lambda+1\right)} + \frac{m^{-\lambda+1}m_0^{\lambda+1}g}{b^2\left(-\lambda^2+1\right)},$$

which is in accord with the solution previously found, Eq. (38).

2.3. Solution with a generating function $\mathbf{F}_2$

Since we are exploring the solution to open systems with canonical methods, we now solve the problem using the more familiar generating function $F_2\left(q, P, t\right) = S$, the action itself.

The H-J equation in this case is

$$\frac{1}{2m^\lambda}\left(\frac{\partial S}{\partial y}\right)^2 + m^\lambda g y + \frac{\partial S}{\partial t} = 0. \tag{57}$$

In order to solve it, we follow the Charpitt method [34,35], according to which we have to solve the associated system

$$\begin{aligned} \frac{dp}{\dfrac{\partial f}{\partial y} + p\dfrac{\partial f}{\partial S}} &= \frac{dq}{\dfrac{\partial f}{\partial t} + q\dfrac{\partial f}{\partial S}} = \frac{dy}{-\dfrac{\partial f}{\partial p}} \\ &= \frac{dt}{-\dfrac{\partial f}{\partial q}} = \frac{dS}{-\left(p\dfrac{\partial f}{\partial p} + q\dfrac{\partial f}{\partial q}\right)}, \end{aligned} \tag{58}$$

together with the equation

$$dS = pdy + qdt, \tag{59}$$

where

$$f = \frac{p^2}{2m^\lambda} + m^\lambda g y + q = 0, \tag{60}$$

and

$$q = \frac{\partial S}{\partial t}. \tag{61}$$

Then we obtain,

$$\begin{aligned} \frac{dp}{m^\lambda g} &= \frac{dq}{-\lambda b\left(\dfrac{p^2}{2m^{\lambda+1}} - g y m^{\lambda-1}\right)} \\ &= \frac{dy}{-\dfrac{p}{m^\lambda}} = \frac{dt}{-1} = \frac{dS}{-\left(\dfrac{p^2}{m^\lambda} + q\right)}. \end{aligned} \tag{62}$$

From the equality between the first and fourth terms in Eq. (62) we have that

$$\frac{dp}{m^\lambda g} = \frac{dt}{-1}, \tag{63}$$

from which it is immediately found that

$$p = C_1 - \frac{m^{\lambda+1} g}{(\lambda+1)\, b}. \tag{64}$$

And from Eq. (60) we obtain

$$q = -m^{\lambda} g y - \frac{p^2}{2m^{\lambda}}. \tag{65}$$

This, after substitution of p from Eq. (64), becomes

$$q = -m^{\lambda} g y - \frac{1}{2m^{\lambda}} \left(C_1 - \frac{m^{\lambda+1} g}{b\,(\lambda+1)} \right)^2. \tag{66}$$

We now have the results necessary to find the generating function S. Substituting the results, Eq. (64) and (66), into Eq. (59) we obtain

$$dS\,(q,P,t) = \left(C_1 - \frac{m^{\lambda+1} g}{b\,(\lambda+1)} \right) dy + \left[-m^{\lambda} g y - \frac{1}{2m^{\lambda}} \left(C_1 - \frac{m^{\lambda+1} g}{b\,(\lambda+1)} \right)^2 \right] dt. \tag{67}$$

This is an exact differential equation whose solution is

$$S\,(q,P,t) = Py - \frac{m^{\lambda+1} g}{b\,(\lambda+1)} y + \frac{P m^2 g}{2b^2\,(\lambda+1)} - \frac{P^2 m^{-\lambda+1}}{2b\,(-\lambda+1)} - \int dt \frac{m^{\lambda+2} g^2}{2b^2\,(\lambda+1)^2}. \tag{68}$$

If we identify C_1 with P, this is precisely the result from which we can proceed to find the solution to the dynamical problem.

From the theory, we have that

$$p = \frac{\partial S}{\partial y} = P - \frac{m^{\lambda+1} g}{b\,(\lambda+1)}. \tag{69}$$

Since P is a constant of motion, then from the initial condition $p(0) = 0$ we find that

$$P = \frac{m_0^{\lambda+1} g}{b\,(\lambda+1)}. \tag{70}$$

On the other hand we also know that

$$Q = \frac{\partial S}{\partial P} = y + \frac{m^2 g}{2b^2\,(\lambda+1)} - \frac{P m^{-\lambda+1}}{b\,(-\lambda+1)}. \tag{71}$$

Also from the initial condition $y(0) = h$ we obtain

$$Q = h - \frac{m_0^2 g}{2b^2\,(-\lambda+1)}, \tag{72}$$

which, substituted in Eq. (71) gives the known result, Eq. (38).

$$y(t) = h - \frac{m_0^2 g}{2b^2\,(-\lambda+1)} - \frac{m^2 g}{2b^2\,(\lambda+1)} + \frac{m_0^{\lambda+1} m^{-\lambda+1} g}{b^2\,(-\lambda^2+1)}.$$

As we can see, the S-equivalent Lagrangian and the Hamiltonian derived from it are adequate to solve this problem by canonical methods.

3. Accretion proportional to the surface of the drop

It is easy to show that the Lagrangian,

$$L = \frac{1}{2} (r_0 + \alpha t)^3 v^2 - (r_0 + \alpha t)^3 g x, \tag{73}$$

leads immediately to the equation

$$(r_0 + \alpha t)^3 \left(\frac{dv}{dt} + \frac{3\alpha v}{(r_0 + \alpha t)} + g \right) = 0, \tag{74}$$

whose solutions are the same as those of Eq. (falta)

$$\frac{dv}{dt} + \frac{3\alpha v}{(r_0 + \alpha t)} = 0. \tag{75}$$

The corresponding Hamiltonian is

$$H = \frac{p^2}{2\,(r_0 + \alpha t)^3} + (r_0 + \alpha t)^3 g x, \tag{76}$$

which makes it possible to find the solution of the problem by the H-J method. The change

$$m^{\lambda} \to (r_0 + \alpha t)^3 \tag{77}$$

makes the problem analogous to the case of accretion proportional to time, so we shall not discuss further it here.

4. Accretion proportional to the surface times the velocity

The appropriate Lagrangian and corresponding Hamiltonian are

$$L = \frac{1}{2} (h_0 - y)^6 v^2 + \frac{1}{7} g\,(h_0 - y)^7 \tag{78}$$

and

$$H = \frac{1}{2} \frac{p^2}{(h_0 - y)^6} - \frac{1}{7} g\,(h_0 - y)^7. \tag{79}$$

It is straightforward to show that this Lagrangian leads to the equation

$$(h_0 - y)^6 \left(\frac{dv}{dt} - \frac{6v^2}{(h_0 - y)} + g \right) = 0, \tag{80}$$

whose solutions are the same as those of equation

$$\frac{dv}{dt} - \frac{3\beta v^2}{r_0 + \beta\,(h - y\,(t))} = -g, \tag{81}$$

where

$$h_0 = h + \frac{r_o}{\beta}.$$

It is convenient to define $z \equiv h_0 - y$. Then the Hamiltonian becomes

$$H = \frac{p^2}{2z^6} - \frac{1}{7} g z^7. \tag{82}$$

This leads immediately to the Hamilton-Jacobi equation

$$\frac{1}{2z^6}\left(\frac{\partial S}{\partial z}\right)^2 - \frac{1}{7} g z^7 + \frac{\partial S}{\partial t} = 0. \tag{83}$$

Since the time does not appear explicitly, then we propose, as usual,

$$S(z, \mathcal{E}, t) = W(z, \mathcal{E}) - \mathcal{E} t, \tag{84}$$

where $\mathcal{E}$ is the value of the Hamiltonian as constant of motion. Then Eq. (82) becomes

$$\frac{1}{2z^6}\left(\frac{dW}{dz}\right)^2 - \frac{1}{7} g z^7 = \mathcal{E}\,. \tag{85}$$

From this we obtain

$$\frac{dW}{dz} = \sqrt{\frac{2g}{7}}\, z^3 \sqrt{z^7 + \frac{7\mathcal{E}}{g}}. \tag{86}$$

Then

$$W = \sqrt{\frac{2g}{7}} \int dz\; z^3 \sqrt{z^7 + \frac{7\mathcal{E}}{g}}. \tag{87}$$

It is important to note that we can write

$$\int dz\; z^3 \sqrt{z^7 + \frac{7\mathcal{E}}{g}} = \int_{z_o}^{z} dz'\; (z')^3 \sqrt{(z')^7 + \frac{7\mathcal{E}}{g}}, \tag{88}$$

where $z_o = r_o/\beta$ and so Eq. (84) can be expressed as

$$S = \sqrt{\frac{2g}{7}} \int_{z_o}^{z} dz'\; (z')^3 \sqrt{(z')^7 + \frac{7\mathcal{E}}{g}} - \mathcal{E} t. \tag{89}$$

Proceeding in the usual way, we obtain, from

$$Q = \frac{\partial S}{\partial \mathcal{E}}, \tag{90}$$

$$Q = \sqrt{\frac{7}{2g}} \int_{z_o}^{z} dz' \frac{(z')^3}{\sqrt{(z')^7 + \frac{7\mathcal{E}}{g}}} - t. \tag{91}$$

Since $\mathcal{E}$ is the value of the Hamiltonian and $p(0)=0$, then

$$t = \sqrt{\frac{7}{2g}} \int_{z_o}^{z} dz' \frac{(z')^3}{\sqrt{(z')^7 - z_0^7}},$$

which in terms of the original variable is

$$t = -\sqrt{\frac{7}{2g}} \int_{h}^{y} dy' \frac{(h_o - y')^3}{\sqrt{(h_o - y')^7 - \left(\frac{r_o}{\beta}\right)^7}}; \tag{92}$$

this integral is not immediate but can be found in standard tables [36] and is given as

$$t = \sqrt{\frac{2\left(\frac{r_0}{\beta}\right)}{7g}} \left((h_0 - y)^7 \left(\frac{\beta}{r_0}\right)^7 - 1\right)^{\frac{1}{2}}$$
$$\times\ {}_2F_1\left[\frac{3}{7}, \frac{1}{2}, \frac{3}{2}, -\left((h_0 - y)^7 \left(\frac{\beta}{r_0}\right)^7 - 1\right)\right].$$

To obtain a more physical results we can proceed as follows. With a Taylor expansion around $y{=}h$, we obtain

$$t = -\sqrt{\frac{2}{g}}\,(h-y)^{\frac{1}{2}} + \frac{(h-y)^{\frac{3}{2}}}{\sqrt{2g}}\left(\frac{\beta}{r_0}\right)$$
$$+ \frac{(h-y)^{\frac{5}{2}}}{\sqrt{32g}}\left(\frac{\beta}{r_0}\right)^2 + O\,(h-y)^{\frac{7}{2}}\,. \tag{93}$$

Finally, inverting this series we find

$$(h-y)^{\frac{1}{2}} = -\sqrt{\frac{g}{2}}\,t - \frac{g^{\frac{3}{2}}}{4\sqrt{2}}\left(\frac{\beta}{r_0}\right) t^3$$
$$- \frac{7 g^{\frac{5}{2}}}{32\sqrt{2}}\left(\frac{\beta}{r_0}\right)^2 t^5 + O\,(t)^7\,, \tag{94}$$

which after squaring gives $y(t)$ as

$$y(t) = h - \frac{g}{2} t^2 + \frac{g^2}{4}\left(\frac{\beta}{r_0}\right) t^4 + O\left(t^6\right). \tag{95}$$

This time it is obvious that we get the free fall case as $\beta \to 0$.

5. Poisson method for the solution of this case

In the last case, the canonical treatment was possible using a Hamiltonian not depending explicitly on time. Then we can try the Poisson method and see if it is easier to find the solution, Eq. (95).

According to the Poisson method [37], the path is given by the expansion

$$y(t) = y_0 + t\,[y, H]|_{t=0} + \frac{t^2}{2!}\,[[y, H]\,, H]|_{t=0}$$
$$+ \frac{t^3}{3!}\,[[[y, H]\,, H]\,, H]_{t=0} + \cdots\,. \tag{96}$$

Using the Hamiltonian, Eq. (79) we obtain

$$\varphi_1 = [y, H]\rfloor_{t=0} = \left(\frac{\partial y}{\partial y}\frac{\partial H}{\partial p} - \frac{\partial y}{\partial p}\frac{\partial H}{\partial y}\right)\rfloor_{t=0} = \frac{p}{(h_0 - y)^6}\rfloor_{t=0} = 0, \tag{97}$$

$$\varphi_2 = \left(\frac{\partial \varphi_1}{\partial y}\frac{\partial H}{\partial p} - \frac{\partial \varphi_1}{\partial p}\frac{\partial H}{\partial y}\right)\rfloor_{t=0} = \left(\frac{6p^2}{(h_0 - y)^{13}} + \frac{3p^2}{(h_0 - y)^{13}} - g\right)\rfloor_{t=0} = -g, \tag{98}$$

$$\varphi_3 = \left(\frac{\varphi_2}{\partial y}\frac{\partial H}{\partial p} - \frac{\varphi_2}{\partial p}\frac{\partial H}{\partial y}\right)\rfloor_{t=0} = \left(\frac{39p^2}{(h_0 - y)^{20}} + \frac{6pg}{(h_0 - y)^7} - \frac{3p^2}{(h_0 - y)^{20}}\right)\rfloor_{t=0} = 0, \tag{99}$$

$$\begin{aligned}\varphi_4 = {}& \left(\frac{\varphi_3}{\partial y}\frac{\partial H}{\partial p} - \frac{\varphi_3}{\partial p}\frac{\partial H}{\partial y}\right)\rfloor_{t=0} = \frac{1}{(h_0 - y)^{27}}\left(6p^2\left\{\begin{array}{c} -130p^2 + \left(\frac{21p^2}{(h_0-y)^8} - 6g\,(h_0 - y)^5\right)(h_0 - y)^8 \\ +13\,(h_0 - y)^7\left(g\,(h_0 - y)^6 + \frac{3p^2}{(h_0-y)^7}\right)\end{array}\right\}\right)\rfloor_{t=0} \\ & - \frac{1}{(h_0 - y)^{20}}\left(3\left(g\,(h_0 - y)^6 + \frac{3p^2}{(h_0 - y)^7}\right)\right. \\ & \left.\times\left(27p^2 - 2\,(h_0 - y)^7\left(g\,(h_0 - y)^6 + \frac{3p^2}{(h_0 - y)^7}\right)\right)\right)\rfloor_{t=0} = 6g^2\left(\frac{\beta}{r_0}\right).\end{aligned} \tag{100}$$

Substituting these results in Eq. (96) gives

$$y = h - \frac{gt^2}{2} + \frac{g^2t^4}{4}\left(\frac{\beta}{r_0}\right) + O\,(t)^6\,,$$

which is the solution already known.

This is another of the few examples [39] in which the method of Poisson leads to an approximate solution in an easier way that the Newtonian method. Of course this depends on having the appropriate Lagrangian and corresponding Hamiltonian.

6. Conclusions

We have solved, by canonical methods, the problem of the motion of a falling drop whose mass grows by accretion according to a specific law of accretion. We have considered three specific laws of accretion, and have solved the problem finding the same solutions obtained by the Newtonian method. The canonical method requires the consideration of Lagrangians more general than the usual natural Lagrangians, those of the form $L = T - V$. Thus we have given another examples of the use of the so-called S-equivalent Lagrangians, which make it possible to find the solution by canonical methods of systems with friction and systems of variable mass. We hope that these examples will illustrate the treatment of some open systems using these methods. After all, the canonical treatment of any systems is a prerequisite to its relativistic statistical mechanics or quantum mechanical formulation.

Acknowledgments

J.L. Jiménez gratefully acknowledges the hospitality of Facultad de Ciencias, where part of this work was prepared during his sabbatical period.

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