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# Gravitational pocket billiards with Mathematica™

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Playing pocket billiards with two gravitational attracting balls and a non-interacting hole requires one to know the trajectories of the balls and therefore to be an "artisan" in the so-called two-body problem, one of the milestones for undergraduate students of Classical Mechanics.

*Keywords:* Two body problem; Kepler´s problem; Mathematica; pocket billiards.

Jugar al billar con dos bolas sometidas a interacción gravitatoria y un hoyo no interactuante precisa conocer las trayectorias de las bolas de billar y, además, ser un "artesano" en el bien conocido Problema de Dos Cuerpos, uno de los más complicados para los estudiantes de Mecánica Clásica.

*Descriptores:* Problema de dos cuerpos; problema de Kepler; Mathematica; billar.

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## 1. Introduction

The gravitational two-body problem has been completely solved since the time of Kepler and Newton. This problem is discussed in most mechanics textbooks [1-4] and together with other problems concerned with gravitation continues to be a subject of lively interest in papers that meet the needs and intellectual interests of college and university teachers and students [5-12]. Conventionally its treatment involves the description of the problem in terms of the relative orbit and of the centre of mass motion, both treated separately, giving way in each case to the equation of motion for a single particle moving under a given force. In the case of the centre of mass, the force is zero for isolated systems.

However, although undergraduate students can solve the above-mentioned equations and therefore obtain the relative orbit and the motion of the centre of mass, they usually have some difficult in envisioning the motion of the two individual bodies from the point of view of a fi ed observer. These difficultie in plotting the individual trajectories of each body are probably due, in our opinion, to a misunderstanding of the concept of relative motion and we must point out that, although the use of the centre of mass frame is often a considerable simplificatio for a two-particle system, we emphasize in this paper the complete point of view of the fi ed observer because in our opinion it gives students a better appreciation of the problem.

As an example of a two-body problem we propose in this paper a simple pocket billiards game, with two gravitational attracting balls and one non-interacting hole; although *Mathematica* is firs and foremost a computer algebra system, we suggest several opportunities to practise with it as a computer program [13,14] to overcome some of the difficultie in obtaining the individual orbits of the balls for the fi ed observer. The proposal has a triple purpose: to practise using an indispensable didactic tool, namely the program, to do something amusing, namely playing the game, and finall and as a consequence of both, to master all the concepts related to the two-body problem.

The present paper is organized as follows:

In Sec. 2 we review the basic equations for the two-body problem.

In Sec. 3 we propose the game with its corresponding notation. We envision graphically how the centre of mass position and the relative position vector must be related to put one of the balls into the hole, and in Sec. 4 we outline a working scheme to obtain the necessary initial conditions to achieve our aim if we know how to solve the relative motion.

In Sec. 5 we analyse how to solve the relative motion as a function of time in the case of closed relative orbit (Kepler's problem) and so how to apply the outlined scheme of Sec. 4. In Sec. 6 we present some results obtained with *Mathematica*. Although the results presented in this section can be obtained numerically with other reliable methods, we use *Mathematica* because it is a valuable tool for research and teaching [15,16].

In Sec. 7 we focus on obtaining straightforward analytical solutions for specially chosen hole positions and in Sec. 8 we write some fina comments.

Finally we include a *Mathematica* document for reproducing some results appearing in this paper and some others concerning the proposed game.

## 2. Basic equations for the two-body problem

The essential information for obtaining the movement of two isolated balls with gravitational interaction between them, can be copied from any mechanics textbook [1]. If we defin the position of the centre of mass ($CM$),

$$\vec{R} = \frac{m_1\,\vec{r}_1 + m_2\,\vec{r}_2}{m_1 + m_2} \tag{1}$$

$\vec{r}_1, \vec{r}_2$ and $m_1, m_2$ being the positions and masses of the balls and the relative position

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \tag{2}$$

without any gravitational external field we get

$$\ddot{\vec{R}} = \vec{0}, \tag{3}$$

and

$$\mu\,\ddot{\vec{r}} = -G\frac{M\,\mu}{r^3}\vec{r} \tag{4}$$

the dots denoting differentiation with respect to the time, $G$ the gravitational constant, $\mu$ the so called *reduced mass* define as $\mu = m_1\,m_2/(m_1 + m_2)$ and $M$ the total mass $m_1 + m_2$. It is worth noting that the equation of motion (3) is identical to the equation of free motion, and the equation of motion (4) is identical to the equation for a single ball of mass $\mu$ moving under the gravitational fiel created by a fi ed mass $M$. Once $\vec{R}$ and $\vec{r}$ are found as functions of time, we can obtain the positions of both balls by solving the simultaneous Eqs. (1) and (2) giving

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\,\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\,\vec{r} \tag{5}$$

and so we can plot the orbit of every ball for the fi ed observer.

Now it is worth emphasizing that:

- **a**) when we refer to the *relative orbit*, the vector $\vec{r}\,(t)$, we mean the orbit of the ball of mass $m_1$ as viewed from a reference system with origin at the position of the ball $m_2$, but with non-rotating axes. These axes are always parallel to those of the fi ed inertial reference system,
- **b**) there are just two masses in our problem: $m_1$ and $m_2$; there is no other ball of mass $\mu$,
- **c**) the problem of a ball under the influenc of a central force with fi ed centre is known by the students before they study the chapter corresponding to the two-body problem [1] and so they are at home in obtaining the relative motion in this problem, by solving Eq. (4).

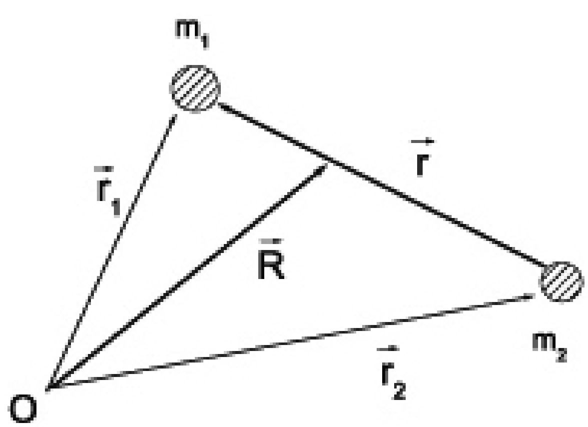


FIGURE 1. The two-body problem.

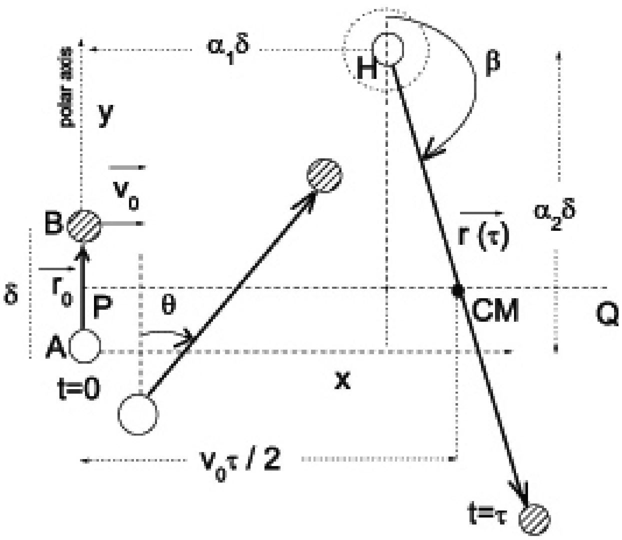


FIGURE 2. The scheme of the proposed pocket billiards game.

## 3. The proposed game

We propose, in this paper, the following modifie game of pocket billiards as indicated in Fig. 2. Suppose we have an isolated system of two equal billiard balls $A$ and $B$ of mass $m$ with gravitational interaction between them and we want to fin the initial conditions required to put ball $A$ into one fi ed hole with the centre at point $H$ somewhere on the pool table. The origin of the fi ed $x$,$y$-axes of our inertial reference system is at the initial position of ball $A$.

At the initial instant, $t = 0$, the balls are separated by a distance $\delta$ and ball $A$ is at rest. Ball $B$ is given an initial velocity $\vec{v}_0$ perpendicular to the initial relative position vector, $\vec{r}_0 = \vec{r}\,(0)$, with origin at the position of $A$ and its modulus is the only unknown quantity in the problem to achieve our aim. So for a given value of $\delta$, $\alpha_1$, $\alpha_2$ we require the value of the initial velocity modulus $v_0$ to put ball $A$ inside the hole $H$ with $(\alpha_1\,\delta\,,\,\alpha_2\,\delta)$ being the Cartesian coordinates of its centre.

Graphically, the general solution to our game is also illustrated in the same Fig. 2. The centre of mass velocity is constant and equal to $v_0/2$, its trajectory being the straight line $PQ$; so to obtain a solution to our problem there must be an instant, $t = \tau$, for which the $CM$ position and the relative position vector at the same instant, $\vec{r}\,(\tau)$, must be coordinated as indicated in the figure

## 4. The general scheme to obtain the solution

We describe, in this section, the scheme for solving the proposed problem that, as has been said, is to put ball $A$ into the hole. By introducing polar co-ordinates $r$ , $\vartheta$ for the relative position vector in the plane of motion, with the Cartesian $y$-axis being the polar axis and $r$ the distance between balls, see again Fig. 2. The Cartesian coordinates of the position of the ball $A$ at any instant are given by

$$x_A = \frac{v_0}{2}\, t - \frac{r}{2}\, \sin\vartheta, \qquad y_A = \frac{\delta}{2} - \frac{r}{2}\, \cos\vartheta \tag{6}$$

so that by knowing $r\ (t)$ and $\vartheta\ (t)$ we obtain $x_A\ (t)$, $y_A\ (t)$ and thus to get ball $A$ inside the hole there must be a time $t = \tau$ for which the equations

$$x_A\ (\tau) = \alpha_1\, \delta, \qquad y_A\ (\tau) = \alpha_2\, \delta \tag{7}$$

are satisfied Here, besides the value, or values, of $v_0$ necessary to satisfy Eq. (7), we obtain as additional information the fligh time $\tau$, the direction of the relative position vector at $t = \tau$, that is to say $\vartheta\ (\tau) \equiv \beta$ and the distance $r\ (\tau)$ between the balls at the same time.

But obtaining the relative motion is a difficul subject. The orbital equation for motion in a central inverse-square force law can be solved in a fairly straightforward manner with results that can be stated in simple closed expressions [1], but describing the motion of the particle in time as it traverses the orbit, that is to say the obtaining of the functions $r\ (t)$ and $\vartheta\ (t)$, is a much more involved matter. That is why this is a good opportunity for the students to practise with the program *Mathematica* and we encourage them at this point to run the program by following the procedure given in the next section to obtain the relative motion.

## 5. The relative motion

In order to fin $r\ (t)$ and $\vartheta\ (t)$, that is to say the relative position vector as a function of time, one must solve Kepler's equation for bounded relative orbits, a different one known as Barker's equation in the parabolic case, and a still different one in the hyperbolic case. In this paper we limit our calculus to elliptical relative orbits and so we can proceed in the following way [2]:

By introducing the auxiliary variable $\psi$, denoted as the *eccentric anomaly*, and define by the relation

$$r = a\,(1 \pm e\, \cos\psi) \tag{8}$$

where $a$ is the semi-major axis of the relative orbit and $e$ its eccentricity, the relation

$$\sqrt{\frac{G\,M}{a^3}}\, t = \psi \pm e\, \sin\psi \tag{9}$$

known as *Kepler's equation* is obtained. In both equations the upper sign refers to the case when the perihelion occurs at $\psi = \pi$ (where $\vartheta = \pi$) and the lower to the case when the perihelion occurs at $\psi = 0$ (where $\vartheta = 0$). So while Eqs. (8) and (9) yield the radial distance $r\ (t)$, the polar angle $\vartheta$ can be expressed in terms of $\psi$ through

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\ \tan\frac{\psi}{2} \tag{10}$$

and so Eqs. (9) and (10) yield $\vartheta\ (t)$.

We consider it worth definin the following dimensionless variables,

$$r^* = \frac{r}{\delta}, \qquad t^* = \frac{t\ (G\,m)^{1/2}}{\delta^{3/2}} \tag{11}$$

and the following dimensionless magnitudes

$$v_0^* = \left(\frac{\delta}{G\,m}\right)^{1/2} v_0, \qquad a^* = \frac{a}{\delta}, \qquad l^* = \frac{l}{\delta},$$
$$E^* = \frac{E\,\delta}{G\,m^2}, \qquad T^* = \frac{T\ (G\,m)^{1/2}}{\delta^{3/2}} \tag{12}$$

$\delta$ being, as stated above, the initial distance between the balls and $v_0, a, l, E, T$ the initial velocity, semi-major axis, semi-latus rectum, energy and period of the relative elliptical orbit, respectively. As these magnitudes are related by [1]

$$E = \frac{1}{2}\,\mu\, v_0^2 - \frac{G\,M\,\mu}{\delta}$$
$$l = \frac{\mu\,\delta^2 v_0^2}{G\,M\,\mu}$$
$$\left(\frac{T}{2\,\pi}\right)^2 = \frac{\mu\, a^3}{G\,M\,\mu} = \frac{a^3}{2\,G\,m}$$
$$2\,a = -\frac{G\,M\,\mu}{E} \tag{13}$$

we obtain for the dimensionless magnitudes define in Eq. (12)

$$2a = -\frac{1}{E} = \frac{4}{4 - v_0^2} \tag{14a}$$

$$l = \frac{v_0^2}{2} \tag{14b}$$

$$E = \frac{1}{4}\, v_0^2 - 1 \tag{14c}$$

$$T = \sqrt{2}\,\pi\, a^{3/2} \tag{14d}$$

where the stars have been dropped for convenience and it must be pointed out that all the magnitudes appearing in the rest of the paper must be considered to be dimensionless.

In their dimensionless form the equations we have to deal with obtained from Eqs. (8), (9) and (10) become

$$r = a\ (1 \pm e\ \cos\psi) \tag{15}$$

$$\sqrt{\frac{2}{a^3}}\, t = \psi \pm e\, \sin\psi \tag{16}$$

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\ \tan\frac{\psi}{2} \tag{17}$$

where the eccentricity $e$, as a function of the modulus of the relative velocity $v_0$, is given by

$$e\ (v_0) = \sqrt{1 + 2\,E\,l} = \left|1 - \frac{v_0^2}{2}\right|. \tag{18}$$

It is worth remarking again that in Eqs. (15) and (16) the upper sign refers to the case $0 < v_0^2 < 2$ and the lower to the case $2 < v_0^2 < 4$, where the maximum limit according Eq. (14a) corresponds to $E = 0$, that is to say for parabolic relative motion.

Once the relative motion is obtained, the balls positions for the fi ed inertial observer are given in their dimensionless form by

$$x_A = \frac{v_0}{2}\,t - \frac{r}{2}\,\sin\vartheta \qquad y_A = \frac{1}{2} - \frac{r}{2}\,\cos\vartheta \tag{19a}$$

$$x_B = \frac{v_0}{2}\,t + \frac{r}{2}\,\sin\vartheta \qquad y_B = \frac{1}{2} + \frac{r}{2}\,\cos\vartheta \tag{19b}$$

and, as has been said in Sec. 4, at the instant ball $A$ falls into the hole the dimensionless equations

$$x_A\,(\tau) = \alpha_1$$

$$y_A\,(\tau) = \alpha_2 \tag{20}$$

must be satisfied

In summary, the whole procedure followed in this section to obtain the motion in time is the following: for a given value of $v_0$ we calculate $e$ and $a$ with Eqs. (18) and (14a). Then we start with any value of $\psi$ and from Eq. (16) we obtain the corresponding value of the dimensionless time $t$. For this time $t$, that is to say for the selected value of $\psi$, we obtain the corresponding value of $r$ from Eq. (15) and the corresponding value of $\vartheta$ from Eq. (17). Once the values of $r$and $\vartheta$ are obtained, we calculate from Eq. (19) the positions for both balls at time $t$.

As an example of the procedure followed we plot in Fig. 3 the trajectories of ball $A$ for the fi ed observer and different values of $v_0 < 2$ in the interval $0 < \psi < 2\pi$. That means according to (16) different fina times because the values of $a$ and $e$ depend on the value of $v_0$. In Fig. 4 the trajectories of the ball $B$ are plotted in the same range of $v_0 < 2$ and $0 < \psi < 2\pi$. Note in both figure that at $t = 0$, $x_B = 0$, $y_B = 1$ and $x_A = 0$, $y_A = 0$ according to Fig. 2

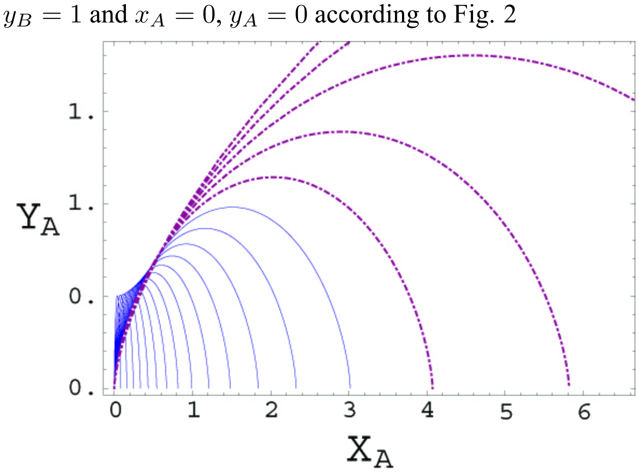


FIGURE 3. The trajectories of ball $A$ plotted with *Mathematica* for different values of $v_0$ in the interval $0 < \psi < 2\pi$. Full line $v_0 = 0.1, 0.2, ...1.3, 1.4$. Dot-dashed line $v_0 = 1.5, 1.6, ..., 1.9$.

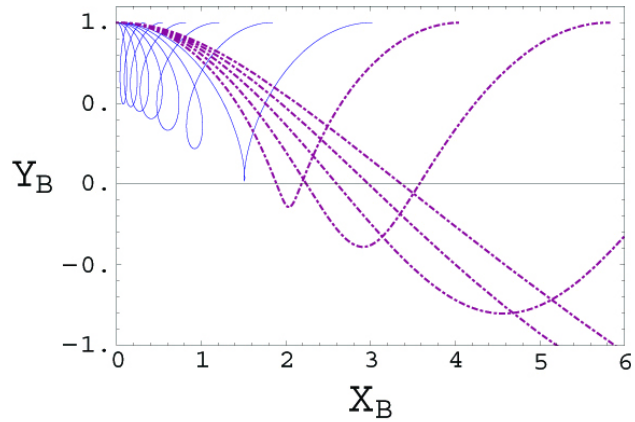


FIGURE 4. The trajectories of the ball $B$ plotted with *Mathematica* for different values of $v_0$ in the interval $0 < \psi < 2\pi$. Full line $v_0 = 0.2, 0.4, ...1.2, 1.4$. Dot-dashed line $v_0 = 1.5, 1.6, ..., 1.9$.

## 6. The pocket billiards game with Mathematica

In this section we use *Mathematica* to solve our proposed problem, that is to say to fin the initial velocity $v_0$ to put ball $A$ into the hole.

a) First we limit our calculus not just to the elliptical relative orbits, $0 < v_0 < 2$, but to the case where the hole is somewhere on the straight line $y = x$ at a distance $d_A$ from the origin.

   By following the procedure given in the last section, after giving a value of $v_0$ we calculate the position of ball $A$ at different values of $\psi$, that is to say according to Eq. (16) at different values of $t$, and then we use the program routine [13]

   FindRoot $[x_A\,[\psi] == y_A\,[\psi]\,, \{\psi, \psi_{start}, \psi_{\min}, \psi_{\max}\}]$

   which searches for a solution of $x_A\,(\psi) = y_A\,(\psi)$ where $\psi_{start}, \psi_{\min}, \psi_{\max}$are chosen suitably after a random search.

   In this way we obtain the value of $\psi_\tau$ which satisfie $x_A\,(\psi_\tau) = y_A\,(\psi_\tau)$ and then with Eq. (16) we obtain the corresponding value of $\tau$, that is to say the firs time for which $x_A\,(\tau) = y_A\,(\tau)$. At this time $\tau$, ball $A$ touches the straight line $y = x$, $d_A = \sqrt{2}\,x_A\,(\tau)$ being the distance from the crossing point to the origin, see Fig. 5. So, obviously, if we place the centre of the hole $H$ with coordinates$(\alpha_1, \alpha_2)$ at this crossing point, that is to say if we make $\alpha_1$=$\alpha_2$=$x_A\,(\tau)$, we achieve our aim.

   Figure 6 shows, for different values of $v_0$, ranging from 0.1 to 1.99, the trajectories of ball $A$ obtained and plotted with *Mathematica*, from the origin $(t = 0)$ to the point where its trajectory crosses the line $y = x$ for the firs time $(t = \tau)$. The function $v_0\ (d_A)$, as define above, is plotted in Fig. 7 and it is worth remarking that this is the form chosen in this paper to give the solutions to our pocket billiards game when the hole is somewhere on the $y = x$ line. That means that after choosing the position of the hole somewhere on the line $y = x$, that is to say for a given value of the distance $d_A$, we obtain from Fig. 7 the necessary value of $v_0$.

We can also obtain several other interesting correspondences concerning the trajectories of Fig. 6. For instance the fligh time function $\tau\ (v_0)$ plotted in Fig. 8 which could also be plotted as $\tau\ (d_A)$. It is worth observing that this function has a minimum for $v_0 \approx 0.5$, this result not being easy to grasp from an analytical point of view.

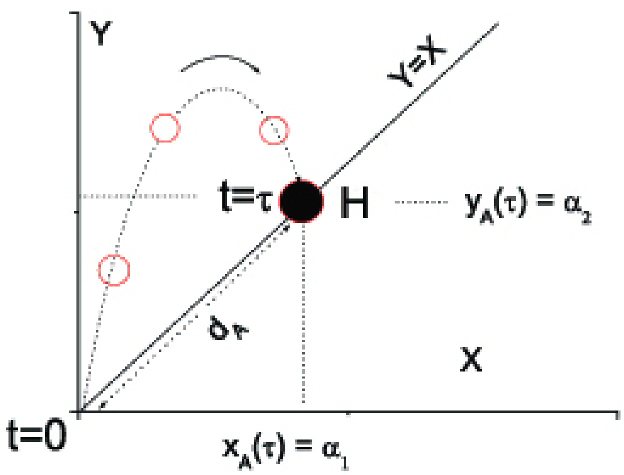


FIGURE 5. The scheme of the trajectory of ball $A$ with the hole on the line $y = x$.

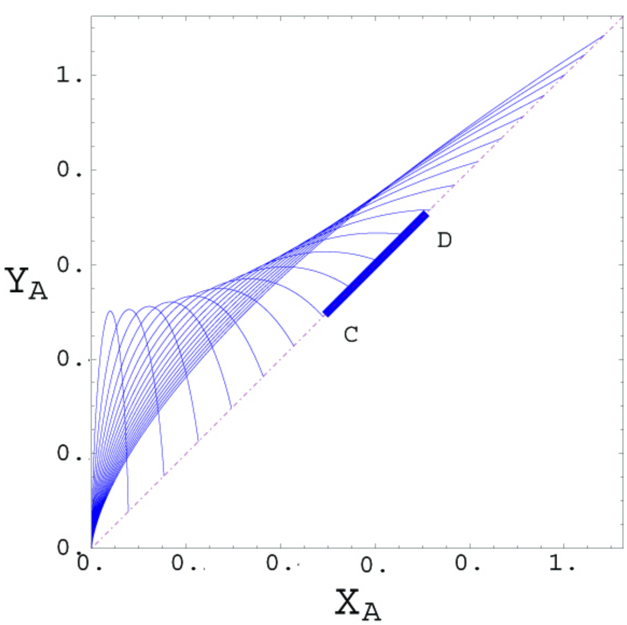


FIGURE 6. First crossing of ball $A$ with the line $y = x$ for $v_0 = 0.1,\ 0.2, ...1.8,\ 1.9$, plotted with *Mathematica*.

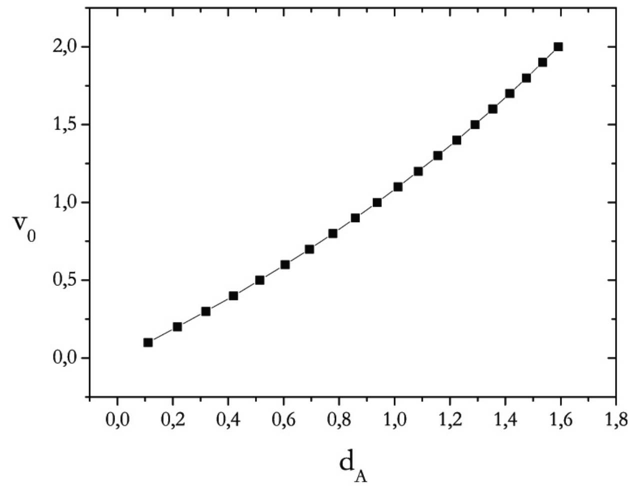


FIGURE 7. $v_0\ (d_A)$ plot where the hole is on the line $y = x$.

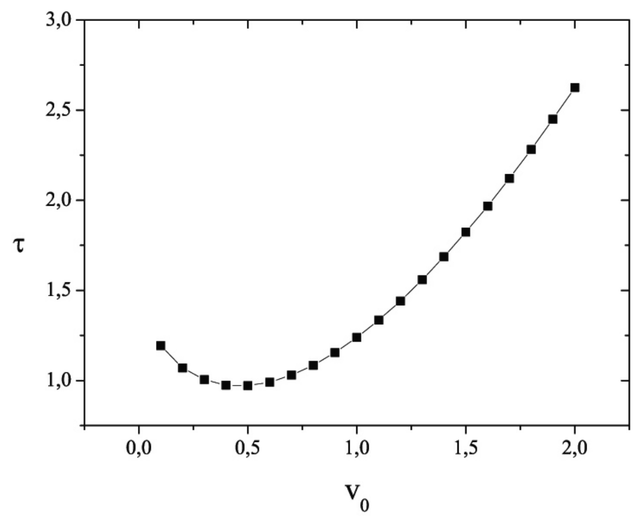


FIGURE 8. $\tau\ (v_0)$ plot $\tau$ where the hole is on the line $y = x$.

b) Second, by still considering elliptical relative orbits, $0 < v_0 < 2$, we extend our calculus to the general case where the hole is somewhere on the straight line $y = p\,x$ with $p \neq 1$, that is to say for any position of the hole. Some of the results also plotted and obtained with *Mathematica* are shown in Figs. 9 and 10 again with $v_0$ ranging from 0.1 to 1.9.

The solution in this case is given in Fig. 9 in the sense that for a given position of the hole, that is to say for given values of $p$ and $d_A$, the corresponding value of $v_0$ can be obtained from the figure The thickest line is identical to the line shown in Fig. 7 and it is worth observing that when the hole is near the origin $(d_A < 0.6)$, the velocity $v_0$ is little dependent on the hole position $(p, d_A)$ in the plotted range $0.6 < p < 2.2$.

For convenience we show in Fig. 10 the fligh time in a different manner. We plot the isochronous curves for constant values of $\tau$ starting from $\tau = 0.9$. To obtain the fligh time in any particular case we must proceed in the following way: the point of Cartesian coordinates $(p, v_0)$ characteristic of the position of the hole corresponds to a determined isochronous curve and so we can read on the curve the corresponding fligh time.

## 7. Straightforward solutions for $\beta = \pi(2\,n+1)$

Straightforward solutions can be obtained analytically in some particular cases. So in this section and without the help of the computer we focus our attention on the $\alpha_1 = \alpha_2 \equiv \alpha$ case, that is to say with the hole somewhere on the straight line $y = x$, $(p = 1)$, and the condition $\beta = \pi(2\,n+1)$, that is to say

$$\tau = (2\,n+1)\,\frac{T}{2} \tag{21}$$

for $n = 0, 1, 2, 3...$, $T$ being the period of the relative orbit, and therefore that means that we are concerned again with elliptical relative orbits, that is to say the condition $0 < v_0 < 2$ still holds.

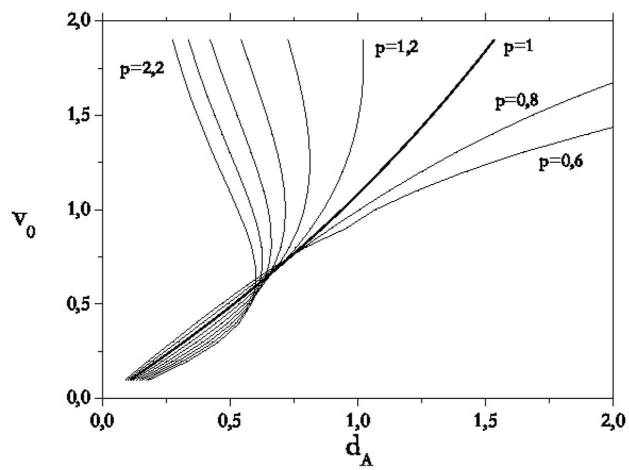


FIGURE 9. $v_0$ $(d_A)$ plot for any position of the hole.

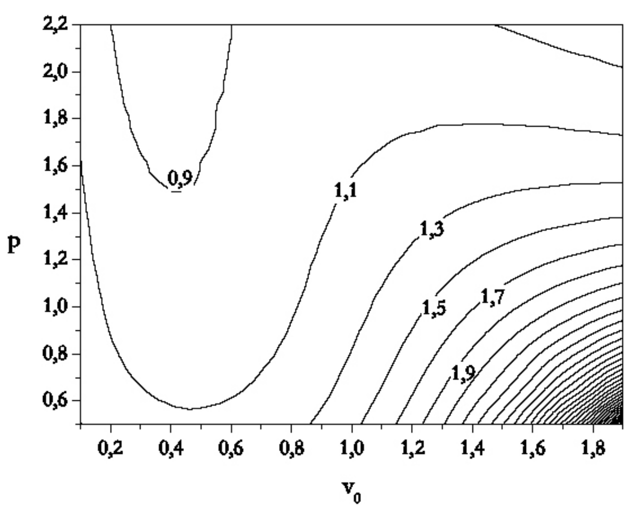


FIGURE 10. $\tau$ $(p, v_0)$ plot for any position of the hole.

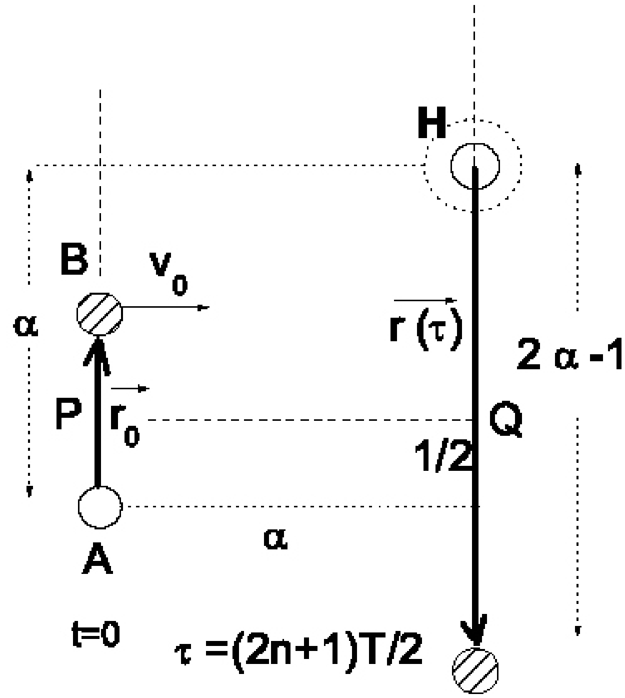


FIGURE 11. Scheme of the straightforward solutions with the hole on the line $y = x$ and the condition$\beta = \pi(2\,n+1)$.

In this case, see Fig. 11, the relative position vector $\vec{r}\ (\tau)$, at the instant $t = \tau$, is antiparallel to the initial relative position vector $\vec{r_0}$ and as $r_{\min} = |\,\vec{r_0}\,|$ and $r_{\max} = |\,\vec{r}\ (\tau)\,|$, we obtain for the major semi-axis of the relative orbit

$$a = \frac{r_{\max} + r_{\min}}{2} = \frac{(2\,\alpha - 1)\,+1}{2} = \alpha\,. \tag{22}$$

where again dimensionless magnitudes are considered.

Now the following relationships must hold: first as the major semi-axis $a = \alpha$ we firs obtain from Eq. (14a)

$$v_0^2 = \frac{2\,(2\,\alpha - 1)}{\alpha} \tag{23}$$

and second, as the distance travelled by the centre of mass moving at velocity $v_0/2$ must be $\alpha$ in a time equal to $\tau$, we obtain with Eq. (21)

$$\frac{(2\,n+1)}{4}\,v_0\,T = \alpha \tag{24}$$

Now the substitution of $T$ from Eq. (14d) and $v_0$ from Eq. (23) yields

$$\alpha = \frac{1}{2}\Big[1 + \frac{4}{\pi^2(2n+1)^2}\Big] \tag{25}$$

and so the hole position $(\alpha\,,\ \alpha)$ as a function of $n$ is obtained, $\alpha$ fulfillin the condition

$$\frac{1}{2} < \alpha < \frac{1}{2} + \frac{2}{\pi^2} = 0.7026 \tag{26}$$

which limits the hole position in the bold line segment $CD$ shown in Fig. 6.

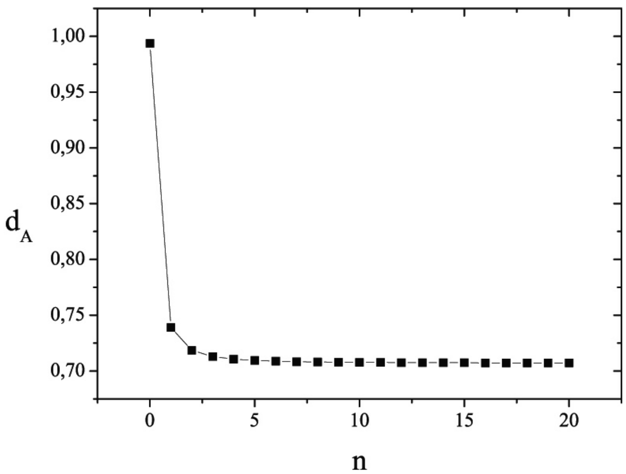


FIGURE 12. $d_A\ (n)$ plot for the straightforward solutions.

As a consequence of Eqs. (25) and (23) there is just one position of the hole on the $CD$ segment for each value of $n$ and one determined value of $v_0$ to put ball $A$ inside the hole after a time $\tau\ (n)$ with the condition $\beta = (2\,n + 1)\ \pi$. The substitution into Eq. (21) of $T$ given in Eq. (14d) and $a = \alpha\ (n)$ given in Eq. (25) yields

$$\tau = \frac{\left[4 + \pi^2\ (2\,n + 1)^2\right]^{3/2}}{4\,\pi^2\,(2\,n + 1)^2} \tag{27}$$

and by substitution of Eq. (25) into Eq. (24) we obtain

$$v_0^2\ (n) = \frac{16}{4 + \pi^2\,(2\,n + 1)^2} \tag{28}$$

and whence the maximum value of $v_0$,$(n = 0)$, corresponds to the maximum value of $\alpha$ and so in this case the hole must be at the extreme $D$. This maximum value is obtained from Eq. (28) yielding

$$v_{0,MAX} = \sqrt{\frac{16}{\pi^2 + 4}} = 1.07 \tag{29}$$

and so as the velocity $v_{0C}$ for circular relative orbit of radius unity is

$$v_{0C}^2 = 2 \tag{30}$$

the following condition

$$v_0\ (n) < v_{0C} \tag{31}$$

is fulfille  and so $v_0$ is always smaller than the velocity corresponding to circular relative orbit.

It is worth remarking that owing to the strong $n$ dependence of $\alpha$ given in Eq. (25) most of the values of $\alpha$ are near $0.5$. That means that for most values of $n$, the hole must be near $C$, that is to say at a distance from the origin $d_A$ near $1/\sqrt{2}$. This fact is shown in Fig. 12 where the function $d_A\ (n)$ is plotted.

## 8. Conclusions

First of all we encourage the use of *Mathematica* in this problem and we hope that we have been persuasive enough in this paper to show how to have an amusing time with this program and our proposed pocket billiards game. The students could extend the problem to other situations than those described in this paper, for instance when the relative orbit is open.

As for the conclusions obtained with the results presented in this paper, we point out some of them here. For instance, as we have shown, there are no unique initial conditions for a determined positioning of the hole. If the hole is placed somewhere in the segment $CD$ on the line $y = x$, we can obtain a solution for $\beta$ being a multiple of $\pi$ or for $\beta$ not fulfillin  this condition. In the firs  case the solution can be straightforwardly obtained with our mechanics textbook, but in the second the help of *Mathematica* is invaluable.

Also we would like to emphasize the benefit  that can be obtained with *Mathematica* in the sense that different and somewhat unpredictable facts can appear as in Fig. 8 where there is an interesting minimum in the $\tau\ (v_0)$ plot, or in Fig. 9 where the cases for hole positions near the origin could be studied in more detail.

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## A How to reproduce the calculations presented in this paper

There is much scope for developing the problem proposed in this paper as a student project. A *Mathematica* document has been included as a complement to this paper[i]. This document shows how to reproduce some important results which have already been discussed in the preceding sections. In addition, it includes the code for visualizing the trajectories of the balls for a fi ed observer as a fil  and other graphics. Basic knowledge of *Mathematica* will be required in order to understand and use the document. We strongly encourage students to follow the instructions given in order to make the most of it.

$i$ Click here to download *Mathematica* document.

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