

The Cauchy problem for a forced harmonic oscillator

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We construct an explicit solution of the Cauchy initial value problem for the one-dimensional Schrödinger equation with a time-dependent Hamiltonian operator for the forced harmonic oscillator. The corresponding Green function (propagator) is derived with the help of the generalized Fourier transform and a relation with representations of the Heisenberg–Weyl group $N(3)$ in a certain special case first, and then is extended to the general case. A three parameter extension of the classical Fourier integral is discussed as a by-product. Motion of a particle with a spin in uniform perpendicular magnetic and electric fields is considered as an application; a transition amplitude between Landau levels is evaluated in terms of Charlier polynomials. In addition, we also solve an initial value problem to a similar diffusion-type equation.

Keywords: The Cauchy initial value problem; the Schrödinger equation; forced harmonic oscillator; Landau levels; the hypergeometric functions; the Hermite polynomials; the Charlier polynomials; Green functions; Fourier transform and its generalizations; the Heisenberg–Weyl group $N(3)$.

En el presente trabajo construimos una solución explícita unidimensional a la ecuación de Schrödinger con condiciones iniciales de Cauchy y con un operador Hamiltoniano dependiente del tiempo para el oscilador armónico forzado. La correspondiente función de Green (propagador) se deriva con aplicaciones de la transformada de Fourier generalizada y con una relación a las representaciones del grupo $N(3)$ de Heisenberg–Weyl, para un caso especial primero y después se extiende al caso general. Estudiamos por medio de un producto una extensión de tres parámetros a la integral clásica de Fourier. Consideramos, como una aplicación, el movimiento de una partícula giratoria en un campo eléctrico y en un campo magnético perpendicularmente uniforme; evaluamos en términos de polinomios de Charlier una transición de amplitud entre los niveles de Landau. Además resolvemos una ecuación similar a la de difusión con valores iniciales.

Descriptores: Problema de valor inicial de Cauchy; ecuación de Schrödinger; oscilador armónico forzado; niveles de Landau; funciones hipergeométricas; polinomios de Hermite; polinomios de Charlier; funciones de Green; transformada de Fourier y sus generalizaciones; el grupo Heisenberg–Weyl.

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1. Introduction

The time-dependent Schrödinger equation for the one-dimensional harmonic oscillator has the form

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \quad (1)$$

where the Hamiltonian is

$$H = \frac{\hbar\omega}{2} \left(-\frac{\partial^2}{\partial x^2} + x^2 \right) = \frac{\hbar\omega}{2} (aa^\dagger + a^\dagger a). \quad (2)$$

Here a^\dagger and a are the creation and annihilation operators, respectively, given by

$$a^\dagger = \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right), \quad a = \frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right); \quad (3)$$

see [21] for another definition. They satisfy the familiar commutation relation

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \quad (4)$$

A natural modification of the Hamiltonian operator (2) is as follows:

$$H \rightarrow H(t) = \frac{\hbar\omega}{2} (aa^\dagger + a^\dagger a) + \hbar(\delta(t)a + \delta^*(t)a^\dagger), \quad (5)$$

where $\delta(t)$ is a complex valued function of time t and the symbol $*$ denotes complex conjugation. This operator is Hermitian, namely, $H^\dagger(t) = H(t)$. It corresponds to the case of the forced harmonic oscillator which is of interest in many advanced problems. Examples include polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other interactions of the modes with external fields. It has particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [9, 20, 23, 34, 35, 45]. Extensively used propagator techniques were originally introduced by Richard Feynman in Refs. 16 to 19.

On this note we construct an exact solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(t)\psi \quad (6)$$

with the Hamiltonian of the form (5), subject to the initial condition

$$\psi(x, t)|_{t=0} = \psi_0(x), \quad (7)$$

where $\psi_0(x)$ is an arbitrary square integrable complex-valued function from $\mathcal{L}^2(-\infty, \infty)$. We shall start with a particular

choice of the time-dependent function $\delta(t)$ given by (22) below, which is later extended to the general case. The explicit form of Eq. (6) is given by (25) and (77) below, and an extension to similar diffusion-type equations is also discussed.

This paper is organized as follows. In Sec. 2 we remind the reader about the textbook solution of the stationary Schrödinger equation for the one-dimensional simple harmonic oscillator. In Sec. 3 we consider the eigenfunction expansion for the time-dependent Schrödinger equation (6) and find its particular solutions in terms of the Charlier polynomials for a certain forced harmonic oscillator. The series solution to the corresponding initial value problem is obtained in Sec. 4. It is further transformed into an integral form in Sec. 7 after discussing two relevant technical tools, namely, the representations of the Heisenberg–Weyl group $N(3)$ and the generalized Fourier transform in Sec. 5 and 6, respectively. An important special case of the Cauchy initial value problem for the simple harmonic oscillator is outlined in Sec. 8 and a three-parameter generalization of the Fourier transform is introduced in Sec. 9 as a by-product. In Secs. 10 and 11 we solve the initial value problem for the general forced harmonic oscillator in terms of the corresponding Green function (or Feynman’s propagator) and the eigenfunction expansion, respectively, by a different method that uses all technical tools developed before in the special case. An extension to the case of time-dependent frequency is given in Sec. 12. Then in Sec. 13, we outline important special and limiting cases of the Feynman propagators. Finally in Sec. 14, the motion of a charged particle with a spin in uniform magnetic and electric fields that are perpendicular to each other is considered as an application; we evaluate a transition amplitude between Landau levels under the influence of the perpendicular electric field in terms of Charlier polynomials and find the corresponding propagator in three dimensions. Solutions to similar diffusion-type equations are discussed in Sec. 15.

The Cauchy initial value problem for a forced harmonic oscillator was originally considered by Feynman in his path integrals approach to the nonrelativistic quantum mechanics [16, 17, 20]. Since then this problem and its special and limiting cases were discussed by many authors [7, 23, 25, 29, 34, 50] the simple harmonic oscillator; [3, 10, 24, 37, 43] the particle in a constant external field; see also references therein. It is worth noting that an exact solution to the n -dimensional time-dependent Schrödinger equation for a certain modified oscillator is found in Ref. 30. These simple exactly solvable models may be of interest in a general treatment of the non-linear time-dependent Schrödinger equation (see Refs. 26, 27, 36, 44, 46, 53, and references therein). They also provide explicit solutions which can be useful for testing numerical methods of solving the time-dependent Schrödinger equation.

2. The simple harmonic oscillator in one dimension

The time-dependent Hamiltonian operator (5) has the following structure:

$$H(t) = H_0 + H_1(t), \quad (8)$$

where

$$H_0 = \frac{\hbar\omega}{2} (aa^\dagger + a^\dagger a) \quad (9)$$

is the Hamiltonian of the harmonic oscillator and

$$H_1(t) = \hbar(\delta(t)a + \delta^*(t)a^\dagger) \quad (10)$$

is the time-dependent “perturbation”, which corresponds to an external time-dependent force that does not depend on the coordinate x (dipole interaction) and a similar velocity-dependent term (see Refs. 20, 23, and 34 for more details).

The solution to the stationary Schrödinger equation for the one-dimensional harmonic oscillator

$$H_0\Psi = E\Psi, \quad H_0 = \frac{\hbar\omega}{2} \left(-\frac{\partial^2}{\partial x^2} + x^2 \right) \quad (11)$$

is a standard textbook problem in quantum mechanics (see Refs. 13, 21, 28, 34, 35, 39, 45, and 52 for example). The orthonormal wave functions are given by

$$\Psi = \Psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x) \quad (12)$$

with

$$\int_{-\infty}^{\infty} \Psi_n^*(x) \Psi_m(x) dx = \delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases} \quad (13)$$

where $H_n(x)$ are the Hermite polynomials, a family of the (very) classical orthogonal polynomials (see Refs. 1, 2, 4, 12, 14, 38, 39, 41, and 49). The corresponding oscillator discrete energy levels are

$$E = E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad (n = 0, 1, 2, \dots). \quad (14)$$

The actions of the creation and annihilation operators (3) on the oscillator wave functions (12) are given by

$$a \Psi_n = \sqrt{n} \Psi_{n-1}, \quad a^\dagger \Psi_n = \sqrt{n+1} \Psi_{n+1}. \quad (15)$$

These “ladder” equations follow from the differentiation formulas

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x) = 2x H_n(x) - H_{n+1}(x), \quad (16)$$

which are valid for the Hermite polynomials.

3. Eigenfunction expansion for the time-dependent Schrödinger equation

In the spirit of Dirac’s time-dependent perturbation theory in quantum mechanics (see Refs. 13, 21, 28, 35, and 45), we are

looking for a solution to the initial value problem in (6)–(7) as an infinite series

$$\psi = \psi(x, t) = \sum_{n=0}^{\infty} c_n(t) \Psi_n(x), \tag{17}$$

where $\Psi_n(x)$ are the oscillator wave functions (12) which depend only on the space coordinate x and $c_n(t)$ are the yet unknown time-dependent coefficients. Substituting this form of solution into the Schrödinger equation (6) with the help of the orthogonality property (13) and the “ladder” relations (15), we obtain the following linear infinite system:

$$i \frac{dc_n(t)}{dt} = \omega \left(n + \frac{1}{2} \right) c_n(t) + \delta(t) \sqrt{n+1} c_{n+1}(t) + \delta^*(t) \sqrt{n} c_{n-1}(t) \quad (n = 0, 1, 2, \dots) \tag{18}$$

of the first-order ordinary differential equations with $c_{-1}(t) \equiv 0$. The initial conditions are

$$c_n(0) = \int_{-\infty}^{\infty} \Psi_n^*(x) \psi_0(x) dx \tag{19}$$

due to the initial data (7) and the orthogonality property (13).

Now we specify the exact form of the function $\delta(t)$ in order to find a particular solution to the system (18) in terms of the so-called Charlier polynomials that belong to the classical orthogonal polynomials of a discrete variable (see Refs. 11, 12, 14, 38, and 39). One can easily verify that the following Ansatz

$$c_n(t) = (-1)^n \frac{\mu^{n/2}}{\sqrt{n!}} \times e^{-i(\omega(n+1/2)-(n+\mu))t} (e^{-i\lambda t} p_n(\lambda)) \tag{20}$$

gives the three-term recurrence relation

$$\lambda p_n(\lambda) = -\mu p_{n+1}(\lambda) + (n + \mu) p_n(\lambda) - n p_{n-1}(\lambda) \tag{21}$$

for the Charlier polynomials $p_n(\lambda) = c_n^\mu(\lambda)$ (see Refs. 38 and 39 for example), when we choose

$$\delta(t) = \sqrt{\mu} e^{i(\omega-1)t}, \quad \delta^*(t) = \sqrt{\mu} e^{-i(\omega-1)t} \tag{22}$$

with the real parameter μ such that $0 < \mu < 1$. Thus with $p_n(\lambda) = c_n^\mu(\lambda)$, Eq. (20) yields a particular solution of the system (18) for any value of the spectral parameter λ .

By the superposition principle the solution to this linear system of ordinary differential equations, which satisfies the initial condition (19), can be constructed as a linear combination

$$c_n(t) = \sum_{m=0}^{\infty} c_{nm}(t) c_m(0), \tag{23}$$

where $c_{nm}(t)$ is a “Green” function, or a particular solution that satisfies the simplest initial conditions

$$c_{nm}(0) = \delta_{nm}. \tag{24}$$

In the next section we will obtain the function $c_{nm}(t)$ in terms of the Charlier polynomials; see Eq. (33) below. In Sec. 5 we establish a relation with the representations of the Heisenberg–Weyl group $N(3)$; see Eq. (46). A generalization to an arbitrary function $\delta(t)$ will be given later.

4. Solution of the Cauchy problem

We can now construct the exact solution to the original Cauchy problem in (6)–(7) for the time-dependent Schrödinger equation with the Hamiltonian of the form (8)–(10) and (22). More explicitly, we will solve the following partial differential equation:

$$i \frac{\partial \psi}{\partial t} = \frac{\omega}{2} \left(-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) + \sqrt{2\mu} \left((\cos(\omega-1)t) x \psi + i (\sin(\omega-1)t) \frac{\partial \psi}{\partial x} \right) \tag{25}$$

subject to the initial condition

$$\psi(x, t)|_{t=0} = \psi_0(x) \quad (-\infty < x < \infty). \tag{26}$$

By (17), (19), and (23) our solution has the form

$$\psi(x, t) = \sum_{n=0}^{\infty} \Psi_n(x) \sum_{m=0}^{\infty} c_{nm}(t) \times \int_{-\infty}^{\infty} \Psi_m(y) \psi_0(y) dy, \tag{27}$$

where

$$c_{nm}(t) = \left(-\mu^{1/2} \right)^{n-m} \sqrt{\frac{m!}{n!}} e^{-i(\omega(n+1/2)-(n+\mu))t} \times \frac{\mu^m}{m!} \sum_{k=0}^{\infty} e^{-ikt} c_n^\mu(k) c_m^\mu(k) e^{-\mu \frac{\mu^k}{k!}} = (-1)^{n-m} e^{-\mu \frac{\mu^{(n+m)/2}}{\sqrt{n!m!}}} e^{-i(\omega(n+1/2)-(n+\mu))t} \times \sum_{k=0}^{\infty} c_n^\mu(k) c_m^\mu(k) \frac{(\mu e^{-it})^k}{k!}, \tag{28}$$

in view of the superposition principle and the orthogonality property

$$\sum_{k=0}^{\infty} c_n^\mu(k) c_m^\mu(k) e^{-\mu \frac{\mu^k}{k!}} = \frac{m!}{\mu^m} \delta_{nm} \quad (0 < \mu < 1) \tag{29}$$

of the Charlier polynomials (see Ref. 38 for example).

The right-hand side of (28) can be transformed into a single sum with the help of the following generating relation for the Charlier polynomials:

$$\sum_{k=0}^{\infty} \frac{(\mu_1 \mu_2 s)^k}{k!} c_n^{\mu_1}(k) c_m^{\mu_2}(k) = e^{\mu_1 \mu_2 s} (1 - \mu_1 s)^m \times (1 - \mu_2 s)^n {}_2F_0 \left(-n, -m; \frac{s}{(1 - \mu_1 s)(1 - \mu_2 s)} \right). \tag{30}$$

(See Refs. 31 to 33, and 22 for more information and Ref. 6 for the definition of the generalized hypergeometric series.) Choosing $\mu_1 = \mu_2 = \mu$ and $s = e^{-it}/\mu$ we obtain

$$c_{nm}(t) = \frac{(-i)^{n+m}}{\sqrt{2^{n+m}n!m!}} e^{-i((\omega-1)n+(n+m)/2)t} e^{-2\mu \sin^2(t/2)} \times e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \left(2\sqrt{2\mu} \sin(t/2)\right)^{n+m} \times {}_2F_0\left(-n, -m; -\frac{1}{4\mu \sin^2(t/2)}\right). \quad (31)$$

The hypergeometric series representation for the Charlier polynomials is

$$c_n^\mu(x) = {}_2F_0\left(-n, -x; -\frac{1}{\mu}\right) \quad (32)$$

(see [38] for example). Thus

$$c_{nm}(t) = e^{-i(\mu \sin t + (\omega/2 - \mu)t)} e^{-i((\omega-1)n+(n+m)/2)t} \times \frac{(-i)^{n+m}}{\sqrt{2^{n+m}n!m!}} e^{-\beta^2/4} \beta^{n+m} c_n^{\beta^2/2}(m) \quad (33)$$

with $\beta = \beta(t) = 2\sqrt{2\mu} \sin(t/2)$ and, as a result, by substitution of this expression into the series (27), we obtain the eigenfunction expansion solution to the original Cauchy problem (25)–(26). We shall be able to find an integral form of this solution in Sec. 7 after discussing representations of the Heisenberg–Weyl group $N(3)$ and a generalization of the Fourier transform in the next two sections. This complete solution to the particular initial value problem (25)–(26) will suggest a correct form of the Green function (propagator) for the general forced harmonic oscillator in Secs. 10 and 11.

5. Relation with the Heisenberg–Weyl group $N(3)$

Let $N(3)$ be the three-dimensional group of the upper triangular real matrices of the form

$$\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} = (\alpha, \beta, \gamma). \quad (34)$$

The map

$$T(\alpha, \beta, \gamma) \Psi(x) = e^{i(\gamma + \beta x)} \Psi(x + \alpha) \quad (35)$$

defines a unitary representation of the Heisenberg–Weyl group $N(3)$ in the space of square integrable functions $\Psi \in \mathcal{L}^2(-\infty, \infty)$ (see Ref. 51, 38, and 47 for more details).

The set $\{\Psi_n(x)\}_{n=0}^\infty$ of the wave functions of the harmonic oscillator (12) forms a complete orthonormal system in $\mathcal{L}^2(-\infty, \infty)$. The matrix elements of the representation (35) with respect to this basis are related to the Charlier polynomials as follows:

$$T(\alpha, \beta, \gamma) \Psi_n(x) = \sum_{m=0}^\infty T_{mn}(\alpha, \beta, \gamma) \Psi_m(x), \quad (36)$$

where

$$T_{mn}(\alpha, \beta, \gamma) = \int_{-\infty}^\infty \Psi_m^*(x) e^{i(\gamma + \beta x)} \Psi_n(x + \alpha) dx = \frac{i^{m-n}}{\sqrt{m!n!}} e^{i(\gamma - \alpha\beta/2)} e^{-\nu/2} \times \left(\frac{i\alpha + \beta}{\sqrt{2}}\right)^m \left(\frac{i\alpha - \beta}{\sqrt{2}}\right)^n c_m^\nu(n) \quad (37)$$

with $\nu = (\alpha^2 + \beta^2)/2$ [38]. A similar integral

$$\int_{-\infty}^\infty H_m(x + y) H_n(x + z) e^{-x^2} dx = \sqrt{\pi} 2^n m! z^{n-m} L_m^{n-m}(-2yz) \quad (38)$$

is evaluated in Ref. 15 in terms of the Laguerre polynomials $L_m^\alpha(\xi)$, whose relation with the Charlier polynomials is

$$c_n^\mu(x) = (-\mu)^{-n} n! L_n^{x-n}(\mu), \quad (39)$$

see Ref. 38. Its special case $y = z$ in the form of

$$\int_{-\infty}^\infty H_m(x) H_n(x) e^{-(x-y)^2} dx = \sqrt{\pi} 2^n m! y^{n-m} L_m^{n-m}(-2y^2) \quad (40)$$

is of particular interest in this paper.

The unitary relation

$$\sum_{n=0}^\infty T_{mn}^*(\alpha, \beta, \gamma) T_{m'n}(\alpha, \beta, \gamma) = \delta_{mm'} \quad (41)$$

holds due to the orthogonality property of the Charlier polynomials (29).

The relevant special case of these matrix elements is

$$T_{mn}(0, \beta, 0) = t_{mn}(\beta) = \frac{i^{m+n}}{\sqrt{2^{m+n}m!n!}} \times e^{-\beta^2/4} \beta^{m+n} c_m^{\beta^2/2}(n), \quad (42)$$

which explicitly acts on the oscillator wave functions as follows:

$$e^{i\beta x} \Psi_n(x) = \sum_{m=0}^\infty t_{mn}(\beta) \Psi_m(x). \quad (43)$$

Relations (30), (32) and (42) imply

$$\sum_{k=0}^\infty t_{mk}(\beta_1) t_{nk}(\beta_2) s^k = \frac{i^{m+n}}{\sqrt{2^{m+n}m!n!}} e^{-(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 s)/4} \times (\beta_1 + \beta_2 s)^m (\beta_2 + \beta_1 s)^n c_m^\lambda(n) \quad (44)$$

with

$$\lambda = (\beta_1^2 + \beta_2^2 + \beta_1\beta_2 (s + s^{-1})) / 2,$$

which is an extension of the addition formula

$$\sum_{k=0}^{\infty} t_{mk}(\beta_1) t_{nk}(\beta_2) = t_{mn}(\beta_1 + \beta_2) \tag{45}$$

for the matrix elements.

In order to obtain functions $c_{nm}(t)$ in terms of the matrix elements $t_{mn}(\beta)$ of the representations of the Heisenberg–Weyl group, we compare (33) and (42). The result is

$$c_{nm}(t) = (-1)^{n+m} e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \times e^{-i((\omega-1)n + (n+m)/2)t} t_{mn}(\beta), \tag{46}$$

where $\beta = 2\sqrt{2\mu} \sin(t/2)$. Our solution (27) takes the form

$$\begin{aligned} \psi(x, t) &= \sum_{n=0}^{\infty} \Psi_n(x) \sum_{m=0}^{\infty} c_{nm}(t) \int_{-\infty}^{\infty} \Psi_m(y) \psi_0(y) dy \\ &= e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \sum_{n=0}^{\infty} (-1)^n e^{-it(\omega-1/2)n} \\ &\times \Psi_n(x) \sum_{m=0}^{\infty} t_{mn}(\beta) \int_{-\infty}^{\infty} (-1)^m e^{-imt/2} \\ &\times \Psi_m(y) \psi_0(y) dy. \end{aligned} \tag{47}$$

In the next section, we will discuss a generalization of the Fourier transformation, which will allow us to transform this multiple series into a single integral form in Sec. 7; see Eqs. (55) and (60)–(62) below.

6. The generalized Fourier transform

The Mehler generating function, or the Poisson kernel for Hermite polynomials, is given by

$$K_r(x, y) = \sum_{n=0}^{\infty} r^n \Psi_n(x) \Psi_n(y) = \frac{1}{\sqrt{\pi(1-r^2)}} \times \exp\left(\frac{4xyr - (x^2 + y^2)(1+r^2)}{2(1-r^2)}\right), \tag{48}$$

where $\Psi_n(z)$ are the oscillator wave functions defined by (12) and $|r| \leq 1, r \neq \pm 1$ (see Refs. 14, 41, 49, and 52 for example). Using the orthogonality property (13) one gets

$$r^n \Psi_n(x) = \int_{-\infty}^{\infty} K_r(x, y) \Psi_n(y) dy, \quad |r| < 1. \tag{49}$$

Thus the wave functions Ψ_n are also eigenfunctions of an integral operator corresponding to the eigenvalues r^n .

We denote

$$\mathcal{K}_\tau(x, y) = K_{e^{i\tau}}(x, y) = \frac{e^{i(\pi/2 - \tau)/2}}{\sqrt{2\pi \sin \tau}} \times \exp\left(i \frac{2xy - (x^2 + y^2) \cos \tau}{2 \sin \tau}\right) \tag{50}$$

with $0 < \tau < \pi$ and use the fact that the oscillator wave functions are the eigenfunctions of the generalized Fourier transform

$$e^{in\tau} \Psi_n(x) = \int_{-\infty}^{\infty} \mathcal{K}_\tau(x, y) \Psi_n(y) dy \tag{51}$$

corresponding to the eigenvalues $e^{in\tau}$. (See Refs. 5, 42, and 48 for more details on the generalized Fourier transform, its inversion formula and their extensions. It is worth noting that the classical Fourier transform corresponds to the particular value $\tau = \pi/2$ [52]. Its three-parameter extension will be discussed in Sec. 9.)

7. An integral form of solution

Now let us transform the series (47) into a single integral form. With the help of the inversion formula for the generalized Fourier transform (see (51) with $\tau \rightarrow -\tau$) and the symmetry property

$$H_n(-x) = (-1)^n H_n(x)$$

we get

$$(-1)^m e^{-imt/2} \Psi_m(y) = \int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) \Psi_m(z) dz. \tag{52}$$

Then by (43) and Fubini’s theorem,

$$\begin{aligned} \sum_{m=0}^{\infty} t_{mn}(\beta) \int_{-\infty}^{\infty} (-1)^m e^{-imt/2} \Psi_m(y) \psi_0(y) dy &= \sum_{m=0}^{\infty} t_{mn}(\beta) \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) \Psi_m(z) dz \right) \psi_0(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) \left(\sum_{m=0}^{\infty} t_{mn}(\beta) \Psi_m(z) \right) \psi_0(y) dy dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) (e^{i\beta z} \Psi_n(z)) \psi_0(y) dy dz. \end{aligned} \tag{53}$$

Now the series (47) takes the form

$$\begin{aligned} \psi(x, t) &= e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \sum_{n=0}^{\infty} (-1)^n e^{-it(\omega - 1/2)n} \Psi_n(x) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) (e^{i\beta z} \Psi_n(z)) \psi_0(y) dy dz \\ &= e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) e^{i\beta z} \left(\sum_{n=0}^{\infty} e^{-it(\omega - 1/2)n} \Psi_n(-x) \Psi_n(z) \right) \psi_0(y) dy dz \\ &= e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathcal{K}_{-t/2}(-y, z) e^{i\beta z} \mathcal{K}_{t(1/2 - \omega)}(-x, z) dz \right) \psi_0(y) dy \end{aligned} \tag{54}$$

in view of the generating relations (48) and (50). Thus

$$\psi(x, t) = e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \int_{-\infty}^{\infty} \mathcal{G}_t(x, y) \psi_0(y) dy, \tag{55}$$

where we define the kernel as

$$\mathcal{G}_t(x, y) := \int_{-\infty}^{\infty} \mathcal{K}_{t(1/2 - \omega)}(-x, z) e^{i\beta z} \mathcal{K}_{-t/2}(-y, z) dz. \tag{56}$$

This can be evaluated with the help of the familiar elementary integrals

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} e^{i(az^2 + 2bz)} dz = \sqrt{\frac{\pi i}{a}} e^{-ib^2/a} \tag{57}$$

(see Refs. 9, 15, and 40 also). Denoting $\tau_1 = t(\omega - 1/2)$ and $\tau_2 = t/2$, from (50) we get

$$\mathcal{K}_{-\tau_1}(-x, z) e^{i\beta z} \mathcal{K}_{-\tau_2}(-y, z) = \frac{e^{i(\omega t - \pi)/2}}{2\pi \sqrt{\sin \tau_1 \sin \tau_2}} e^{i(x^2 \cot \tau_1 + y^2 \cot \tau_2)/2} e^{i(\beta + x/\sin \tau_1 + y/\sin \tau_2)z} e^{i(\cot \tau_1 + \cot \tau_2)z^2/2} \tag{58}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{K}_{-\tau_1}(-x, z) e^{i\beta z} \mathcal{K}_{-\tau_2}(-y, z) dz &= \frac{e^{i(\omega t - \pi)/2}}{2\pi \sqrt{\sin \tau_1 \sin \tau_2}} e^{i(x^2 \cot \tau_1 + y^2 \cot \tau_2)/2} \\ &\times \int_{-\infty}^{\infty} e^{i((\beta + x/\sin \tau_1 + y/\sin \tau_2)z + (\cot \tau_1 + \cot \tau_2)z^2/2)} dz. \end{aligned} \tag{59}$$

As a result

$$\mathcal{G}_t(x, y) = \frac{e^{i(\omega t - \pi/2)/2}}{\sqrt{2\pi \sin \omega t}} e^{i(x^2 \cot \tau_1 + y^2 \cot \tau_2)/2} \exp\left(\frac{\sin \tau_1 \sin \tau_2 (\beta + x/\sin \tau_1 + y/\sin \tau_2)^2}{2i \sin \omega t}\right) \tag{60}$$

with $\tau_1 = t(\omega - 1/2)$, $\tau_2 = t/2$ and $\beta = \beta(t) = 2\sqrt{2\mu} \sin(t/2)$. Thus the explicit form of this kernel is given by

$$\mathcal{G}_t(x, y) = \frac{e^{i(\omega t - \pi/2)/2}}{\sqrt{2\pi \sin \omega t}} \exp\left(\frac{(x^2 + y^2) \sin \omega t - (x^2 - y^2) \sin(\omega - 1)t}{2i(\cos \omega t - \cos(\omega - 1)t)}\right) \exp\left(\frac{ik_t^2(x, y)}{\sin \omega t (\cos \omega t - \cos(\omega - 1)t)}\right), \tag{61}$$

where

$$\begin{aligned} k_t(x, y) &= (x + y) \sin(\omega t/2) \cos((\omega - 1)t/2) \\ &\quad - (x - y) \cos(\omega t/2) \sin((\omega - 1)t/2) - \sqrt{2\mu} \sin(t/2) (\cos \omega t - \cos(\omega - 1)t). \end{aligned} \tag{62}$$

The last expression can be transformed into a somewhat more convenient form

$$\mathcal{G}_t(x, y) = \mathcal{K}_{\omega t}^*(x, y) \exp\left(\frac{\sin((\omega - 1/2)t) \sin(t/2) \beta^2}{2i \sin \omega t}\right) \exp\left(\frac{(x \sin(t/2) + y \sin((\omega - 1/2)t)) \beta}{i \sin \omega t}\right) \tag{63}$$

with $\beta = 2\sqrt{2\mu} \sin(t/2)$ in terms of the kernel of the generalized Fourier transform (50). Our formulas (55) and (60)–(63) provide an integral form to the solution of the Cauchy initial value problem (25)–(26) in terms of a Green function.

By choosing $\psi_0(x) = \delta(x - x_0)$, where $\delta(x)$ is the Dirac delta function, we formally obtain

$$\psi(x, t) = G(x, x_0, t) = e^{-i(\mu \sin t + (\omega/2 - \mu)t)} \mathcal{G}_t(x, x_0), \quad (64)$$

which is the fundamental solution to the time-dependent Schrödinger equation (25). One can show that

$$\lim_{t \rightarrow 0^+} \psi(x, t) = \psi_0(x) \quad (65)$$

by methods of Refs. 5, 42, and 52. The details are left to the reader.

The time evolution operator for the time-dependent Schrödinger equation (6) can formally be written as

$$U(t, t_0) = T \left(\exp \left(-\frac{i}{\hbar} \int_{t_0}^t H(t') dt' \right) \right), \quad (66)$$

where T is the time ordering operator which orders operators with larger times to the left [9], [21]. Namely, this unitary operator takes a state at time t_0 to a state at time t , so that

$$\psi(x, t) = U(t, t_0) \psi(x, t_0) \quad (67)$$

and

$$U(t, t_0) = U(t, t') U(t', t_0), \quad (68)$$

$$U^{-1}(t, t_0) = U^\dagger(t, t_0) = U(t_0, t). \quad (69)$$

We have constructed this time evolution operator explicitly as the following integral operator

$$U(t, t_0) \psi(x, t_0) = e^{-i(\mu \sin(t-t_0) + (\omega/2 - \mu)(t-t_0))} \times \int_{-\infty}^{\infty} \mathcal{G}_{t-t_0}(x, y) \psi(y, t_0) dy \quad (70)$$

with the kernel given by (60)–(63), for the particular form of the time-dependent Hamiltonian in (8)–(10) and (22). The Green function (propagator) for the general forced harmonic oscillator is constructed in Sec. 10; see Eqs. (79)–(84).

8. The Cauchy problem for the simple harmonic oscillator

In an important special case $\mu = 0$, the initial value problem

$$i \frac{\partial \psi}{\partial t} = \frac{\omega}{2} \left(-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right),$$

$$\psi(x, t)|_{t=0} = \psi_0(x) \quad (-\infty < x < \infty) \quad (71)$$

has the following explicit solution:

$$\psi(x, t) = \sum_{n=0}^{\infty} e^{-i\omega(n+1/2)t} \Psi_n(x)$$

$$\times \int_{-\infty}^{\infty} \Psi_n(y) \psi_0(y) dy = \frac{1}{\sqrt{2\pi i \sin(\omega t)}}$$

$$\times \int_{-\infty}^{\infty} \exp \left(i \frac{(x^2 + y^2) \cos(\omega t) - 2xy}{2 \sin(\omega t)} \right) \psi_0(y) dy. \quad (72)$$

The last relation is valid when $0 < t < \pi/\omega$. Analytic continuation in a larger domain is discussed in Ref. 29 and 50.

Equation (72) gives the time evolution operator (66) for the simple harmonic oscillator in terms of the generalized Fourier transform. This result and its extension to a general forced harmonic oscillator without the velocity-dependent term in the Hamiltonian are well-known (see Refs. 7, 17, 20, 23, 25, 29, 34, 50, and references therein; further generalizations are given in Secs. 10–12; more special cases will be discussed in Sec. 13).

9. Three parameter generalization of the Fourier transform

The properties of the time evolution operator in (67)–(70) suggest the following extension of the classical Fourier integral:

$$f(x) = \int_{-\infty}^{\infty} \mathcal{L}_t(x, y) g(y) dy, \quad (73)$$

where the kernel is given by

$$\mathcal{L}_t(x, y) = \mathcal{K}_{\omega t}(x, y)$$

$$\times \exp \left(i \frac{\sin((\omega - 1/2)t) \sin^3(t/2)}{2 \sin \omega t} \varepsilon^2 \right)$$

$$\times \exp \left(i \frac{(x \sin(t/2) + y \sin((\omega - 1/2)t)) \sin(t/2)}{\sin \omega t} \varepsilon \right) \quad (74)$$

depends on the three free parameters t , ω and ε . If $\varepsilon = 0$ and $\omega t = \tau$ we arrive at the kernel of the generalized Fourier transform (50). The formal inversion formula is given by

$$g(y) = \int_{-\infty}^{\infty} \mathcal{L}_t^*(x, y) f(x) dx. \quad (75)$$

The details are left to the reader. Note that, in terms of a

distribution,

$$\int_{-\infty}^{\infty} \mathcal{L}_t^*(x, y) \mathcal{L}_t(x, z) dx = e^{i \cot(\omega t)(y^2 - z^2)/2} e^{i\varepsilon(z-y) \sin((\omega-1/2)t) \sin(t/2)/\sin \omega t} \times \frac{1}{2\pi \sin \omega t} \int_{-\infty}^{\infty} e^{ix(z-y)/\sin \omega t} dx = \delta(y - z), \tag{76}$$

which gives the corresponding orthogonality property of the \mathcal{L} -kernel. These results admit further generalizations with the help of the time evolution operators found in Secs. 10 and 12.

10. The general forced harmonic oscillator

Our solution to the initial value problem (25)–(26) obtained in the previous sections admits a generalization. The Cauchy problem for the general forced harmonic oscillator

$$i \frac{\partial \psi}{\partial t} = \frac{\omega}{2} \left(-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) - f(t) x \psi + i g(t) \frac{\partial \psi}{\partial x}, \tag{77}$$

where $f(t)$ and $g(t)$ are two arbitrary real valued functions of time only (such that the integrals in (82)–(84) below converge and $a(0) = 0$), with the initial data

$$\psi(x, t)|_{t=0} = \psi_0(x) \quad (-\infty < x < \infty), \tag{78}$$

has the following explicit solution:

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi_0(y) dy. \tag{79}$$

Here the Green function (or Feynman’s propagator [17], [20], [34]) is given by

$$G(x, y, t) = G_0(x, y, t) e^{i(a(t)x + b(t)y + c(t))} \tag{80}$$

with

$$G_0(x, y, t) = \frac{1}{\sqrt{2\pi i \sin \omega t}} \times \exp \left(i \frac{(x^2 + y^2) \cos \omega t - 2xy}{2 \sin \omega t} \right) \tag{81}$$

and

$$a(t) = \frac{1}{\sin \omega t} \int_0^t (f(s) \sin \omega s + g(s) \cos \omega s) ds, \tag{82}$$

$$b(t) = \int_0^t \frac{\omega a(s) - g(s)}{\sin \omega s} ds, \tag{83}$$

$$c(t) = \int_0^t \left(g(s) a(s) - \frac{\omega}{2} a^2(s) \right) ds \tag{84}$$

provided $a(0) = b(0) = c(0) = 0$. The case $g(t) \equiv 0$ is discussed in Refs. 17, 20, and 34, but the answers for $b(t)$ and $c(t)$ are given in different forms; we shall elaborate on this later.

Indeed, the previously found solution (63)–(64) in the special case of the forced oscillator (25) suggests to look for a general Green function in the form (80), namely,

$$\psi = u e^{iS}, \tag{85}$$

where $u = G_0(x, y, t)$ is the fundamental solution of the Schrödinger equation for the simple harmonic oscillator (71) and $S = a(t)x + b(t)y + c(t)$. Its substitution into (77) gives

$$\left(\frac{da}{dt} x + \frac{db}{dt} y + \frac{dc}{dt} \right) u = \left(ag + xf - \frac{\omega}{2} a^2 \right) u + i(a\omega - g) \frac{\partial u}{\partial x}, \tag{86}$$

where by (81)

$$\frac{\partial u}{\partial x} = i \frac{x \cos \omega t - y}{\sin \omega t} u. \tag{87}$$

As a result

$$\frac{da}{dt} x + \frac{db}{dt} y + \frac{dc}{dt} = ag + xf - \frac{\omega}{2} a^2 - (a\omega - g) \frac{x \cos \omega t - y}{\sin \omega t}, \tag{88}$$

and equating the coefficients of x , y and 1, we obtain the following system of ordinary differential equations

$$\frac{d}{dt} (\sin \omega t a(t)) = f(t) \sin \omega t + g(t) \cos \omega t, \tag{89}$$

$$\frac{d}{dt} b(t) = \frac{\omega a(t) - g(t)}{\sin \omega t}, \tag{90}$$

$$\frac{d}{dt} c(t) = g(t) a(t) - \frac{\omega}{2} a^2(t), \tag{91}$$

whose solutions are (82)–(84), respectively, if the integrals converge. This method is equivalent to solving the quantum mechanical Hamilton–Jacobi equation for the general forced harmonic oscillator [34].

Equation (83) can be rewritten as

$$b(t) = - \int_0^t (\sin \omega s a(s)) d \cot \omega s - \int_0^t \frac{g(s)}{\sin \omega s} ds$$

and, integrating by parts,

$$b(t) = - \cos \omega t a(t) + \int_0^t (f(s) \cos \omega s - g(s) \sin \omega s) ds \tag{92}$$

by (89). With the help of (82) and the addition formulas for trigonometric functions we finally arrive at

$$b(t) = -\frac{1}{\sin \omega t} \int_0^t (f(s) \sin \omega(s-t) + g(s) \cos \omega(s-t)) ds, \tag{93}$$

which is equivalent to the form obtain in Refs. 17, 20, and 34, when $g(t) \equiv 0$.

In a similar fashion,

$$c(t) = \int_0^t g(s) a(s) ds + \frac{1}{2} \int_0^t (\sin \omega s a(s))^2 d \cot \omega s$$

and as a result

$$c(t) = \frac{1}{2} \sin \omega t \cos \omega t a^2(t) + \int_0^t \sin \omega s a(s)$$

$$\times (-f(s) \cos \omega s + g(s) \sin \omega s) ds. \tag{94}$$

This can be transformed into the form given in Refs. 17 and 20 when $g(t) \equiv 0$. The details are left to the reader.

Evaluation of elementary integrals results in (63) again in the special case (25). The simple case $f(t) = 2 \cos \omega t$ and $g(t) \equiv 0$ gives

$$a(t) = \frac{\sin \omega t}{\omega}, \quad b(t) = t, \tag{95}$$

$$c(t) = \frac{1}{8\omega^2} \sin 2\omega t - \frac{1}{4\omega} t.$$

The corresponding propagator in (80) does satisfy the Schrödinger equation (77), which can be verified by a direct differentiation with the help of a computer algebra system. The details are left to the reader. A case of the forced modified oscillator is discussed in Ref. 30.

11. Eigenfunction expansion for the general forced harmonic oscillator

Separation of the x and y variables in Feynman’s propagator (80)–(84) with the help of the Mehler generating function (48) written as

$$G_0(x, y, t) = \sum_{k=0}^{\infty} e^{-i\omega(k+1/2)t} \Psi_k(x) \Psi_k(y) \tag{96}$$

gives

$$G(x, y, t) = G_0(x, y, t) e^{i(ax+by+c)} = e^{i(c-\omega t/2)} \sum_{k=0}^{\infty} e^{-i\omega kt} (e^{iax} \Psi_k(x)) (e^{iby} \Psi_k(y))$$

$$= e^{i(c-\omega t/2)} \sum_{k=0}^{\infty} e^{-i\omega kt} \left(\sum_{n=0}^{\infty} t_{nk}(a) \Psi_n(x) \right) \left(\sum_{m=0}^{\infty} t_{mk}(b) \Psi_m(y) \right)$$

$$= e^{i(c-\omega t/2)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Psi_n(x) \Psi_m(y) \left(\sum_{k=0}^{\infty} e^{-i\omega kt} t_{nk}(a) t_{mk}(b) \right) \tag{97}$$

by (43). The last series can be summed by using the addition formula (44) in the form

$$\sum_{k=0}^{\infty} e^{-i\omega kt} t_{nk}(a) t_{mk}(b) = \frac{i^{m+n}}{\sqrt{2^{m+n} m! n!}} e^{i(ab \sin \omega t)/2} e^{-\chi^2/4} (a+bz)^n (b+az)^m c_m^{\chi^2/2}(n), \tag{98}$$

with $z = e^{-i\omega t}$ and $\chi^2 = a^2 + b^2 + 2ab \cos \omega t$. As a result we arrive at the following eigenfunction expansion of the forced harmonic oscillator propagator:

$$G(x, y, t) = e^{i(c-(\omega t-ab \sin \omega t)/2)} e^{-\chi^2/4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Psi_n(x) \Psi_m(y) \frac{i^{n+m}}{\sqrt{2^{n+m} n! m!}} (a+bz)^n (b+az)^m c_m^{\chi^2/2}(n) \tag{99}$$

in terms of the Charlier polynomials. The special case $g(t) \equiv 0$ is discussed in Ref. 20 but the connection with the Charlier polynomials is not emphasized.

The solution (79) takes the form

$$\psi(x, t) = \sum_{n=0}^{\infty} \Psi_n(x) \sum_{m=0}^{\infty} c_{nm}(t) \int_{-\infty}^{\infty} \Psi_m(y) \psi_0(y) dy, \tag{100}$$

where

$$c_{nm}(t) = e^{i(c-\omega t-ab \sin \omega t)/2} e^{-\chi^2/4} \frac{i^{n+m}}{\sqrt{2^{n+m}n!m!}} (a+bz)^n (b+az)^m c_m^{\chi^2/2}(n) \tag{101}$$

with $z = e^{-i\omega t}$ and $\chi^2 = a^2 + b^2 + 2ab \cos \omega t$. Functions $a = a(t)$, $b = b(t)$ and $c = c(t)$ here are given by the integrals (82)–(84), respectively, and $\lim_{t \rightarrow 0^+} c_{nm}(t) = \delta_{nm}$.

If $\psi_0(x) = \psi(x, t)|_{t=0} = \Psi_m(x)$, Eq. (100) becomes

$$\psi(x, t) = \sum_{n=0}^{\infty} c_{nm}(t) \Psi_n(x). \tag{102}$$

Thus function $c_{nm}(t)$ gives explicitly the quantum mechanical amplitude that the oscillator initially in state m is found at time t in state n [20]. An application to the motion of a charged particle with a spin in uniform perpendicular magnetic and electric fields is considered in Sec. 14.

As a by-product we found the fundamental solution $c_{nm}(t)$ of the system (18) in terms of the Charlier polynomials for an arbitrary complex valued function $\delta(t) = (-f(t) + ig(t))/\sqrt{2}$. The explicit solution of the corresponding initial value problem is given by (23).

12. Time-dependent frequency

An extension of the Schrödinger equation to the case of the forced harmonic oscillator with the time-dependent frequency is as follows:

$$i \frac{\partial \psi}{\partial t} = \frac{\omega(t)}{2} \left(-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) - f(t)x\psi + ig(t) \frac{\partial \psi}{\partial x}, \tag{103}$$

where $\omega(t) > 0$, $f(t)$ and $g(t)$ are arbitrary real valued functions of time only. It can be easily solved by the substitution

$$\tau = \tau(t) = \int_0^t \omega(s) ds, \quad \frac{d\tau}{dt} = \omega(t), \tag{104}$$

which transforms this equation into a familiar form

$$i \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \left(-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) - f_1(\tau)x\psi + ig_1(\tau) \frac{\partial \psi}{\partial x}, \tag{105}$$

see the original Eq. (77) with respect to the new time variable τ , where $\omega = 1$ and

$$f_1(\tau) = \frac{f(t)}{\omega(t)}, \quad g_1(\tau) = \frac{g(t)}{\omega(t)}. \tag{106}$$

Therefore by (80)–(81) the propagator has the form

$$G(x, y, \tau) = G_0(x, y, \tau) e^{i(a(\tau)x+b(\tau)y+c(\tau))} \tag{107}$$

with

$$G_0(x, y, \tau) = \frac{1}{\sqrt{2\pi i \sin \tau}} \times \exp \left(i \frac{(x^2 + y^2) \cos \tau - 2xy}{2 \sin \tau} \right) \tag{108}$$

and the system (89)–(91) becomes

$$\frac{d}{d\tau} (\sin \tau a(\tau)) = f_1(\tau) \sin \tau + g_1(\tau) \cos \tau, \tag{109}$$

$$\frac{d}{d\tau} b(\tau) = \frac{a(\tau) - g_1(\tau)}{\sin \tau}, \tag{110}$$

$$\frac{d}{d\tau} c(\tau) = g_1(\tau) a(\tau) - \frac{1}{2} a^2(\tau). \tag{111}$$

Thus

$$a(\tau) = \frac{1}{\sin \tau} \int_0^t (f(s) \sin \tau(s) + g(s) \cos \tau(s)) ds, \tag{112}$$

$$b(\tau) = \int_0^t \frac{\omega(s) a(\tau(s)) - g(s)}{\sin \tau(s)} ds, \tag{113}$$

$$c(\tau) = \int_0^t \left(g(s) a(\tau(s)) - \frac{1}{2} \omega(s) a^2(\tau(s)) \right) ds, \tag{114}$$

which is an extension of equations (82)–(83) to the case of the forced harmonic oscillator with the time-dependent frequency. The solution to the Cauchy initial-value problem is given by

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, \tau) \psi_0(y) dy \tag{115}$$

with

$$\tau = \int_0^t \omega(s) ds.$$

The details are left to the reader.

13. Some special cases

The time-dependent Schrödinger equation for the forced harmonic oscillator is usually written in the form

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \tag{116}$$

with the following Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 - F(t)x - G(t)p, \quad p = \frac{\hbar}{i} \frac{\partial}{\partial x}, \tag{117}$$

where \hbar is the Planck constant, m is the mass of the particle, ω is the classical oscillation frequency, $F(t)$ is a uniform in space external force depending on time, function $G(t)$ represents a similar velocity-dependent term, and p is the linear momentum operator. The initial value problem is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{m\omega^2}{2}x^2\Psi - F(t)x\Psi + i\hbar G(t) \frac{\partial \Psi}{\partial x} \tag{118}$$

with

$$\Psi(x, t)|_{t=t_0} = \Psi(x, t_0). \tag{119}$$

Among important special cases are: the free particle, when $\omega = F = G = 0$; a particle in a constant external field, where $\omega = G = 0$ and $F = \text{constant}$; the simple harmonic oscillator with $F = G = 0$. In this section for the benefits of the reader we provide explicit forms for the corresponding propagators by taking certain limits in the general solution.

The usual change of the space variable

$$\Psi(x, t) = \psi(\xi, t), \quad \xi = \sqrt{\frac{m\omega}{\hbar}} x \tag{120}$$

reduces Eq. (118) to the form (77) with respect to ξ with

$$f(t) = \frac{F(t)}{\sqrt{\hbar\omega m}}, \quad g(t) = \sqrt{\frac{m\omega}{\hbar}} G(t). \tag{121}$$

The time evolution operator is

$$\Psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t, t_0) \Psi(y, t_0) dy \tag{122}$$

with the propagator of the form

$$G(x, y, t, t_0) = G_0(x, y, t, t_0) \times e^{i(a(t,t_0)x + b(t,t_0)y + c(t,t_0))}. \tag{123}$$

Here

$$G_0(x, y, t, t_0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega(t-t_0)}} \times \exp\left(\frac{im\omega}{2\hbar \sin \omega(t-t_0)} [(x^2 + y^2) \times \cos \omega(t-t_0) - 2xy]\right), \tag{124}$$

$$a(t, t_0) = \frac{m\omega}{\hbar \sin \omega(t-t_0)} \int_{t_0}^t \left(F(s) \frac{\sin \omega(s-t_0)}{m\omega} + G(s) \cos \omega(s-t_0) \right) ds, \tag{125}$$

$$b(t, t_0) = -a(t_0, t) \tag{126}$$

and

$$c(t, t_0) = \int_{t_0}^t \left(G(s) a(s, t_0) - \frac{\hbar}{2m} a^2(s, t_0) \right) ds. \tag{127}$$

The simple harmonic oscillator propagator, when $F = G = 0$, is given by Eq. (124); see Refs. 7, 17, 20, 23, 25, 29, 34, 50, and references therein for more details. In the limit $\omega \rightarrow 0$ we obtain

$$G_0(x, y, t, t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left(\frac{im(x-y)^2}{2\hbar(t-t_0)}\right) \tag{128}$$

as the free particle propagator [20].

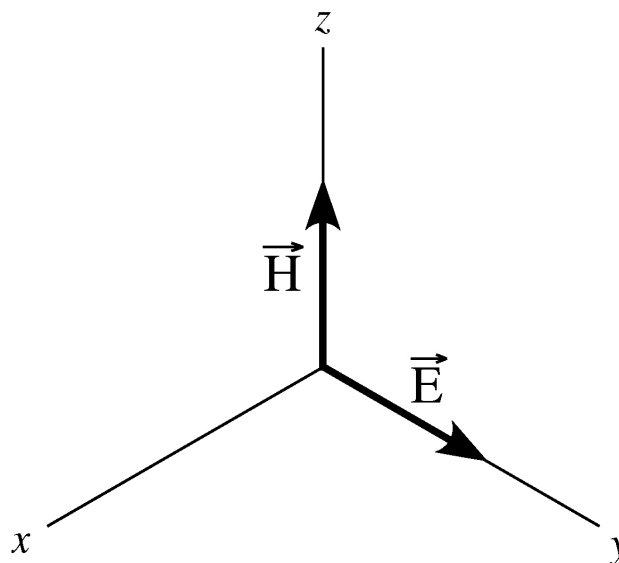


FIGURE 1. Magnetic and electric fields in R^3 .

For a particle in a constant external field $\omega = G = 0$ and $F = \text{constant}$. The corresponding propagator is given by

$$G(x, y, t, t_0) = \sqrt{\frac{m}{2\pi i \hbar (t - t_0)}} \exp\left(\frac{im(x - y)^2}{2\hbar(t - t_0)}\right) \times \exp\left(\frac{iF(x + y)}{2\hbar}(t - t_0) - \frac{iF^2}{24\hbar m}(t - t_0)^3\right). \quad (129)$$

This case was studied in detail in Refs. 3, 10, 20, 24, 37, and 43. We have corrected a typo in Ref. 20.

14. Motion in uniform perpendicular magnetic and electric fields

14.1. Solution of a particular initial value problem

A particle with a spin s has also an intrinsic magnetic momentum μ with the operator

$$\hat{\mu} = \mu \hat{s} / s, \quad (130)$$

where \hat{s} is the spin operator and μ is a constant characterizing the particle, which is usually called the magnitude of the magnetic momentum. For the motion of a charged particle in uniform magnetic \mathbf{H} and electric \mathbf{E} fields, which are perpendicular to each other (Fig. 1), the corresponding three-dimensional time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad (131)$$

has the Hamiltonian of the form [28]

$$\hat{H} = \frac{1}{2m} \left(\hat{p}_x + \frac{eH}{c} y \right)^2 + \frac{1}{2m} \hat{p}_y^2 + \frac{1}{2m} \hat{p}_z^2 - \frac{\mu}{s} \hat{s}_z H - yF, \quad (132)$$

where $\hat{p} = -i\hbar \nabla$ is the linear momentum operator, functions H and F/e are the magnitudes of the uniform magnetic and electric fields in z and y directions, respectively. The corresponding vector potential $\mathbf{A} = -yH \mathbf{e}_x$ is defined up to a gauge transformation. Here we follow the original choice of Ref. 28 (see a remark at the end of this section).

Since (132) does not contain the other components of the spin, the operator \hat{s}_z commutes with the Hamiltonian \hat{H} and the z -component of the spin is conserved. Thus the operator \hat{s}_z can be replaced by its eigenvalue $s_z = \sigma$ in the Hamiltonian

$$\hat{H} = \frac{1}{2m} \left(\hat{p}_x + \frac{eH}{c} y \right)^2 + \frac{1}{2m} \hat{p}_y^2 + \frac{1}{2m} \hat{p}_z^2 - \frac{\mu\sigma}{s} H - yF \quad (133)$$

with $\sigma = -s, -s + 1, \dots, s - 1, s$. Then the spin dependence of the wave function becomes unimportant and the

wave function in the Schrödinger equation (131) can be taken as an ordinary coordinate function $\Psi = \Psi(\mathbf{r}, t, \sigma)$.

The Hamiltonian (133) does not contain the coordinates x and z explicitly. Therefore the operators \hat{p}_x and \hat{p}_z also commute with the Hamiltonian and the x and z components of the linear momentum are conserved. The corresponding eigenvalues p_x and p_z take all values from $-\infty$ to ∞ ; see [28] for more details. In this paper we consider the simplest case when the magnetic field H is a constant and the electric force F is a function of time t (see Fig. 1); a more general case will be discussed elsewhere. Then the substitution

$$\Psi(\mathbf{r}, t) = e^{i(xp_x + zp_z - S(t, t_0))/\hbar} \psi(y, t), \quad (134)$$

where

$$S(t, t_0) = \left(\frac{p_z^2}{2m} - \frac{\mu\sigma}{s} H \right) (t - t_0) + \frac{cp_x}{eH} \int_{t_0}^t F(\tau) d\tau, \quad (135)$$

results in the one-dimensional time-dependent Schrödinger equation of the harmonic oscillator driven by an external force in the y -direction

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y^2} + \frac{m\omega_H^2}{2} (y - y_0)^2 \psi - F(t) (y - y_0) \psi \quad (136)$$

with

$$\omega_H = \frac{|e|H}{mc}, \quad y_0 = -\frac{cp_x}{eH}. \quad (137)$$

The Cauchy initial-value problem subject to special data

$$\Psi(\mathbf{r}, t)|_{t=t_0} = e^{i(xp_x + zp_z)/\hbar} \psi(y, t_0) = e^{i(xp_x + zp_z)/\hbar} \varphi(y - y_0) \quad (138)$$

has the following solution:

$$\Psi(\mathbf{r}, t) = \Psi(\mathbf{r}, t, p_x, p_z) = e^{i(xp_x + zp_z - S(t, t_0))/\hbar} \times \int_{-\infty}^{\infty} G(y - y_0, \eta, t, t_0) \varphi(\eta) d\eta, \quad (139)$$

where the propagator takes the form

$$G(y, \eta, t, t_0) = G_1(y, \eta, t - t_0) \times e^{i(a(t, t_0)y + b(t, t_0)\eta + c(t, t_0))} \quad (140)$$

with

$$G_1(y, \eta, t) = \sqrt{\frac{m\omega_H}{2\pi i\hbar \sin \omega_H t}} \times \exp\left(\frac{im\omega_H}{2\hbar \sin \omega_H t} ((y^2 + \eta^2) \cos \omega_H t - 2y\eta)\right), \quad (141)$$

$$a(t, t_0) = \frac{1}{\hbar \sin \omega_H (t - t_0)} \times \int_{t_0}^t F(\tau) \sin \omega_H (\tau - t_0) d\tau, \quad (142)$$

$$b(t, t_0) = -a(t_0, t) \quad (143)$$

and

$$c(t, t_0) = -\frac{\hbar}{2m} \int_{t_0}^t a^2(\tau, t_0) d\tau. \quad (144)$$

See Eqs. (122)–(127) with $G \equiv 0$. Function $c(t, t_0)$ can be written in several different forms.

14.2. Landau levels

In an absence of the external force $F \equiv 0$, Eq. (136) is formally identical to the time-dependent Schrödinger equation for a simple harmonic oscillator with the frequency ω_H . The standard substitution

$$\psi(y, t) = e^{-i\varepsilon(t-t_0)/\hbar} \chi(y) \quad (145)$$

gives the corresponding stationary Schrödinger equation as follows [28]:

$$\chi'' + \frac{2m}{\hbar^2} \left(\varepsilon - \frac{1}{2} m\omega_H^2 (y - y_0)^2 \right) \chi = 0, \quad (146)$$

which has the square integrable solutions only when

$$\varepsilon = \hbar\omega_H \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \quad (147)$$

The eigenfunctions are

$$\chi_n(y) = \frac{1}{\sqrt{2^n n! a_H \sqrt{\pi}}} \times \exp\left(-\frac{(y - y_0)^2}{2a_H^2}\right) H_n\left(\frac{y - y_0}{a_H}\right), \quad a_H = \sqrt{\frac{\hbar}{m\omega_H}}, \quad (148)$$

where $H_n(\eta)$ are the Hermite polynomials.

Thus the total energy levels of a particle in a uniform magnetic field have the form

$$E_n = E_n(p_z, \sigma) = \hbar\omega_H \left(n + \frac{1}{2} \right) + \frac{p_z^2}{2m} - \frac{\mu\sigma}{s} H \quad (n = 0, 1, 2, \dots). \quad (149)$$

The first term here gives the discrete energy values corresponding to motion in a plane perpendicular to the field. They are called Landau levels. The expression (149) does not contain the quantity p_x , which takes all real values. Therefore the total energy levels are continuously degenerate. For an electron, $\mu/s = -|e|\hbar/mc$, and formula (149) becomes

$$E_n = E_n(p_z, \sigma) = \hbar\omega_H \left(n + \frac{1}{2} + \sigma \right) + \frac{p_z^2}{2m}. \quad (150)$$

In this case, there is an additional degeneracy: the levels with $n, \sigma = 1/2$ and $n + 1, \sigma = -1/2$ coincide:

$$E_n(p_z, 1/2) = E_{n+1}(p_z, -1/2).$$

The three-dimensional wave functions corresponding to the energy levels (149) are given by

$$\Psi_n(\mathbf{r}, t, \sigma) = \Psi_n(\mathbf{r}, t, p_x, p_z, \sigma) = e^{-iE_n(p_z, \sigma)(t-t_0)/\hbar} e^{i(xp_x + zp_z)/\hbar} \chi_n(y). \quad (151)$$

They are the eigenfunctions of the following set of commuting operators $\hat{p}_x, \hat{p}_z, \hat{s}_z$, and \hat{H} with $F \equiv 0$:

$$\begin{aligned} \hat{H}\Psi_n &= E_n\Psi_n, & \hat{s}_z\Psi_n &= \sigma\Psi_n, \\ \hat{p}_x\Psi_n &= p_x\Psi_n, & \hat{p}_z\Psi_n &= p_z\Psi_n. \end{aligned} \quad (152)$$

The orthogonality relation in \mathbf{R}^3 is

$$\begin{aligned} \int_{\mathbf{R}^3} \Psi_n^*(\mathbf{r}, t, p_x, p_z, \sigma) \Psi_m(\mathbf{r}, t, p'_x, p'_z, \sigma') dx dy dz \\ = (2\pi\hbar)^2 \delta_{nm} \delta_{\sigma\sigma'} \delta(p_x - p'_x) \delta(p_z - p'_z), \end{aligned} \quad (153)$$

where

$$\delta(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha\xi} d\xi \quad (154)$$

is the Dirac delta function.

14.3. Transition amplitudes

In the presence of external force, the quantum mechanical amplitude of a transition between Landau's levels under the influence of the perpendicular electric field can be explicitly found as a special case of our formulas (100)–(102). Indeed, solution (139) takes the form

$$\begin{aligned} \Psi(\mathbf{r}, t, \sigma) &= e^{-iS(t, t_0)/\hbar} \sum_{n=0}^{\infty} \Psi_n(\mathbf{r}, t_0, \sigma) \\ &\times \sum_{m=0}^{\infty} c_{nm}(t, t_0) \int_{-\infty}^{\infty} \chi_m(\eta) \psi(\eta, t_0) d\eta \end{aligned} \quad (155)$$

in view of the bilinear generating relation (99). If $\psi(y, t_0) = \chi_m(y)$, this equation becomes

$$\Psi(\mathbf{r}, t, \sigma) = e^{-iS(t, t_0)/\hbar} \times \sum_{n=0}^{\infty} c_{nm}(t, t_0) \Psi_n(\mathbf{r}, t_0, \sigma), \quad (156)$$

where coefficients $c_{nm}(t, t_0)$ are given by (101) in terms of Charlier polynomials as follows:

$$c_{nm}(t, t_0) = e^{i(c - (\omega_H(t-t_0) - ab \sin \omega_H(t-t_0))/2)} e^{-\gamma^2/4} \times \frac{i^{n+m}}{\sqrt{2^{n+m} n! m!}} (a + b\delta)^n (b + a\delta)^m c_m^{\gamma^2/2}(n) \quad (157)$$

with $\delta = e^{-i\omega_H(t-t_0)}$ and $\gamma^2 = a^2 + b^2 + 2ab \cos \omega_H(t-t_0)$. Functions $a = a(t, t_0)$, $b = b(t, t_0)$ and $c = c(t, t_0)$ are evaluated by the integrals (142)–(144), respectively. The last two formulas (156)–(157) and (135) give us the quantum mechanical amplitude that the particle initially in Landau state m is found at time t in state n . For the particle initially in the ground state $m = 0$, the probability of occupying state n at time t is given by the Poisson distribution

$$|c_{n0}(t, t_0)|^2 = e^{-\mu} \frac{\mu^n}{n!}, \quad \mu = \frac{1}{2} (a^2 + b^2 + 2ab \cos \omega_H(t-t_0)) < 1. \quad (158)$$

The details are left to the reader.

14.4. Propagator in three dimensions

Our particular solutions (139) subject to special initial data (138) have been constructed above as eigenfunctions of the operators \hat{p}_x and \hat{p}_z , whose continuous eigenvalues p_x and p_z vary from $-\infty$ to ∞ . By the superposition principle, one can look for a general solution in \mathbf{R}^3 as a double Fourier integral of the particular solution

$$\begin{aligned} \Psi(\mathbf{r}, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(p_x, p_z) \Psi(\mathbf{r}, t, p_x, p_z) dp_x dp_z \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp_x dp_z a(p_x, p_z) e^{i(xp_x + zp_z)/\hbar} e^{-iS(t, t_0)/\hbar} \\ &\times \int_{-\infty}^{\infty} G(y - y_0, \eta, t, t_0) \varphi(\eta) d\eta, \end{aligned} \quad (159)$$

where the Green function (propagator) is given as a double Fourier integral

$$G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i((x-\xi)p_x + (z-\zeta)p_z)/\hbar} e^{-iS(t, t_0)/\hbar} G(y - y_0, \eta - y_0, t, t_0) dp_x dp_z \quad (166)$$

with the help of the Fubini theorem.

where functions $a(p_x, p_z)$ do not depend on time t and $S(t, t_0)$ is given by (135). Now we replace the special initial data (138) in \mathbf{R}^3 by the general one

$$\Psi(\mathbf{r}, t)|_{t=t_0} = \phi(x, y, z), \quad (160)$$

which is independent of p_x (and y_0). Letting $t \rightarrow t_0$ in (159) and using the fundamental property of the Green function,

$$\lim_{t \rightarrow t_0^+} \int_{-\infty}^{\infty} G(y - y_0, \eta, t, t_0) \varphi(\eta) d\eta = \varphi(y - y_0), \quad (161)$$

one gets

$$\begin{aligned} \phi(x, y, z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(p_x, p_z) \varphi(y - y_0) \\ &\times e^{i(xp_x + zp_z)/\hbar} dp_x dp_z, \end{aligned} \quad (162)$$

where y_0 is a function of p_x in view of (137). Thus

$$\begin{aligned} a(p_x, p_z) \varphi(y - y_0) &= \frac{1}{(2\pi\hbar)^2} \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\xi, y, \zeta) e^{-i(\xi p_x + \zeta p_z)/\hbar} d\xi d\zeta \end{aligned} \quad (163)$$

by the inverse of the Fourier transform. Its substitution into (159) gives

$$\begin{aligned} \Psi(\mathbf{r}, t) &= \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp_x dp_z e^{i(xp_x + zp_z - S(t, t_0))/\hbar} \\ &\times \int_{-\infty}^{\infty} d\eta G(y - y_0, \eta - y_0, t, t_0) \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\xi, \eta, \zeta) e^{-i(\xi p_x + \zeta p_z)/\hbar} d\xi d\zeta \end{aligned} \quad (164)$$

as a solution of our initial value problem. A familiar integral form of this solution is as follows:

$$\Psi(\mathbf{r}, t) = \int_{\mathbf{R}^3} G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) \phi(\xi, \eta, \zeta) d\xi d\eta d\zeta, \quad (165)$$

This integral can be evaluated in terms of elementary functions as follows. Integration over p_z gives the free particle propagator of a motion in the direction of magnetic field

$$G_0(z - \zeta, t - t_0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar} \left((z - \zeta) p_z - \frac{p_z^2}{2m} (t - t_0) \right)\right) dp_z = \sqrt{\frac{m}{2\pi i\hbar(t - t_0)}} \exp\left(\frac{im(z - \zeta)^2}{2\hbar(t - t_0)}\right) \quad (167)$$

by the integral (57). Thus

$$\begin{aligned} G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) &= \exp\left(\frac{i\mu\sigma H}{\hbar s} (t - t_0)\right) G_0(z - \zeta, t - t_0) \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar} (x - \xi) p_x\right) \exp\left(-\frac{icp_x}{\hbar eH} \int_{t_0}^t F(\tau) d\tau\right) \\ &\times G(y - y_0, \eta - y_0, t, t_0) dp_x = \exp\left(\frac{i\mu\sigma H}{\hbar s} (t - t_0)\right) G_0(z - \zeta, t - t_0) G_1(y, \eta, t - t_0) \\ &\times e^{i(a(t,t_0)y + b(t,t_0)\eta + c(t,t_0))} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{ip_x}{\hbar} \left(x - \xi - \frac{c}{eH} \int_{t_0}^t F(\tau) d\tau\right)\right) \\ &\times \exp\left(\frac{i(a(t,t_0) + b(t,t_0))cp_x}{eH}\right) \exp\left(\frac{-i}{\hbar} \left(\frac{p_x^2}{m\omega_H} + \frac{|e|}{e} (y + \eta) p_x\right) \tan(\omega_H(t - t_0)/2)\right) dp_x. \end{aligned} \quad (168)$$

In view of (57), the last integral is given by

$$\begin{aligned} &\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{ip_x}{\hbar} \left(x - \xi - \frac{c}{eH} \int_{t_0}^t F(\tau) d\tau\right)\right) \exp\left(\frac{i(a(t,t_0) + b(t,t_0))cp_x}{eH}\right) \\ &\times \exp\left(\frac{-i}{\hbar} \left(\frac{p_x^2}{m\omega_H} + \frac{e}{|e|} (y + \eta) p_x\right) \tan(\omega_H(t - t_0)/2)\right) dp_x = \sqrt{\frac{m\omega_H \cot(\omega_H(t - t_0)/2)}{4\pi i\hbar}} \\ &\times \exp\left(\frac{im\omega_H \cot(\omega_H(t - t_0)/2)}{4\hbar} \beta^2\right), \end{aligned} \quad (169)$$

where

$$\beta = x - \xi - \frac{e}{|e|} (y + \eta) \tan(\omega_H(t - t_0)/2) + d(t, t_0) \quad (170)$$

with

$$d(t, t_0) = \frac{c}{eH \sin \omega_H(t - t_0)} \int_{t_0}^t F(\tau) (\sin \omega_H(\tau - t_0) - \sin \omega_H(\tau - t) - \sin \omega_H(t - t_0)) d\tau. \quad (171)$$

Here we have used (143)–(142). As a result, we arrive at the following factorization of our propagator:

$$G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) = G_0(z - \zeta, t - t_0) G_1(y, \eta, t - t_0) e^{i(a(t,t_0)y + b(t,t_0)\eta + c(t,t_0))} G_2(x, \xi, y, \eta, t, t_0), \quad (172)$$

where $G_0(z - \zeta, t - t_0)$ is the free particle propagator in (167), $G_1(y, \eta, t - t_0)$ is the simple harmonic oscillator propagator in (141), and

$$G_2(x, \xi, y, \eta, t, t_0) = \exp\left(\frac{i\mu\sigma H}{\hbar s} (t - t_0)\right) \sqrt{\frac{m\omega_H \cot(\omega_H(t - t_0)/2)}{4\pi i\hbar}} \exp\left(\frac{im\omega_H \cot(\omega_H(t - t_0)/2)}{4\hbar} \beta^2\right) \quad (173)$$

with $\beta = \beta(x, \xi, y, \eta, t, t_0)$ given by (170)–(171).

Our propagator can be simplified to a somewhat more convenient form as follows:

$$G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) = G_0(z - \zeta, t - t_0) G_H(x, \xi, y, \eta, t - t_0) G_F(x, \xi, y, \eta, t, t_0). \quad (174)$$

Here $G_0(z, t)$ is the free particle propagator in the direction of magnetic field. The function

$$\begin{aligned} G_H(x, \xi, y, \eta, t) &= \exp\left(\frac{i\mu\sigma H t}{\hbar s}\right) \frac{m\omega_H}{4\pi i\hbar \sin(\omega_H t/2)} \\ &\times \exp\left(\frac{im\omega_H}{4\hbar} \left(\left((x - \xi)^2 + (y - \eta)^2\right) \cot(\omega_H t/2) - 2\frac{e}{|e|} (x - \xi)(y + \eta)\right)\right) \end{aligned} \quad (175)$$

is the propagator corresponding to a motion in a plane perpendicular to the magnetic field in the absence of an electric field (compare our expression with one in Ref. 20, where $F = \mu = 0$, and see a remark below in order to establish an identity of two results). The third factor

$$G_F(x, \xi, y, \eta, t, t_0) = e^{iW_F(t, t_0)/\hbar} \tag{176}$$

with

$$W_F(t, t_0) = \hbar(a(t, t_0)y + b(t, t_0)\eta + c(t, t_0)) + \frac{1}{4}m\omega_H d(t, t_0) \times \left((d(t, t_0) + 2(x - \xi)) \cot(\omega_H(t - t_0)/2) - 2\frac{e}{|e|}(y + \eta) \right) \tag{177}$$

is a contribution from the electric field. When $F = 0$, $W_F = 0$ and $G_F = 1$.

The solution to the Cauchy initial value-problem in \mathbf{R}^3 subject to the general initial data

$$\Psi(\mathbf{r}, t)|_{t=t_0} = \Psi(\mathbf{r}, t_0) = \phi(x, y, z) \tag{178}$$

has the form

$$\Psi(\mathbf{r}, t) = \int_{\mathbf{R}^3} G(\mathbf{r}, \boldsymbol{\rho}, t, t_0) \Psi(\boldsymbol{\rho}, t_0) d\xi d\eta d\zeta, \tag{179}$$

which gives explicitly the time evolution operator for a motion of a charged particle in uniform perpendicular magnetic and electric fields with a given projection of the spin $s_z = \sigma$ in the direction of the magnetic field. By choosing $\Psi(\mathbf{r}, t_0) = \delta(\mathbf{r} - \mathbf{r}_0)$, where $\delta(\mathbf{r})$ is the Dirac delta function in three dimensions, we formally obtain

$$\Psi(\mathbf{r}, t) = G(\mathbf{r}, \mathbf{r}_0, t, t_0) \tag{180}$$

as the wave function at time t of the particle initially located at a point $\mathbf{r} = \mathbf{r}_0$. Then Eq. (179) gives a general solution by the superposition principle.

Remark. The vector potential of the uniform magnetic field in the z -direction is defined up to a gauge transformation [28]

$$\begin{aligned} \mathbf{A} &= -yH \mathbf{e}_x \rightarrow \mathbf{A}' = \mathbf{A} + \nabla f \\ &= -\frac{1}{2}yH \mathbf{e}_x + \frac{1}{2}xH \mathbf{e}_y = \frac{1}{2}\mathbf{H} \times \mathbf{r} \end{aligned} \tag{181}$$

with $f(x, y) = xyH/2$. The corresponding transformation of the wave function is given by

$$\begin{aligned} \Psi \rightarrow \Psi' &= \Psi \exp\left(\frac{ief(x, y)}{\hbar c}\right) \\ &= \Psi \exp\left(\frac{im\omega_H}{4\hbar} \left(2\frac{e}{|e|}xy\right)\right) \end{aligned} \tag{182}$$

and in view of (179)

$$\begin{aligned} G_H &\rightarrow \exp\left(\frac{ief(x, y)}{\hbar c}\right) G_H \exp\left(-\frac{ief(\xi, \eta)}{\hbar c}\right) \\ &= \frac{m\omega_H}{4\pi i\hbar \sin(\omega_H t/2)} \exp\left(\frac{im\omega_H}{4\hbar} \left([(x - \xi)^2 + (y - \eta)^2 \right] \right. \\ &\quad \left. \times \cot(\omega_H t/2) - 2\frac{e}{|e|}(x\eta - \xi y)\right), \end{aligned} \tag{183}$$

which is, essentially, Eq. (3-64) on page 64 of [20], where we have corrected a typo. The details are left to the reader.

15. Diffusion-type equation

15.1. Special case

A formal substitution of $t \rightarrow -it$ and $\psi \rightarrow u$ into Eq. (25) with $\omega = 2\kappa$ and $\sqrt{2\mu} = \varepsilon$ yields the following time-dependent diffusion-type equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \left(\frac{\partial^2 u}{\partial x^2} - x^2 u \right) \\ &- \varepsilon \left((\cosh(2\kappa - 1)t) xu + (\sinh(2\kappa - 1)t) \frac{\partial u}{\partial x} \right), \end{aligned} \tag{184}$$

where the initial condition is

$$u(x, t)|_{t=0} = u_0(x) \quad (-\infty < x < \infty). \tag{185}$$

As in the case of the time-dependent Schrödinger equation, in order to solve this initial value problem, we use the eigenfunction expansion method. Hence the solution is given by

$$u(x, t) = \sum_{n=0}^{\infty} \Psi_n(x) \sum_{m=0}^{\infty} c_{nm}(t) \int_{-\infty}^{\infty} \Psi_m(y) u_0(y) dy, \tag{186}$$

where

$$c_{nm}(t) = (-1)^{n-m} \frac{\varepsilon^{n+m}}{\sqrt{2^{n+m}n!m!}} e^{-(\varepsilon^2/2)(1-e^{-t})} e^{-((2\kappa-1)n+\kappa-\varepsilon^2/2)t} (1-e^{-t})^{m+n} {}_2F_0$$

$$\times \left(-n, -m; \frac{2e^{-t}}{\varepsilon^2(1-e^{-t})^2} \right) \tag{187}$$

by analytic continuation $t \rightarrow -it$ with $\omega = 2\kappa$ and $\mu = \varepsilon^2/2 < 1$ in (28) and (31). One can easily verify that

$$\lim_{t \rightarrow 0^+} c_{nm}(t) = \delta_{nm}$$

and that if $0 < \varepsilon < \sqrt{2}$ and $\kappa \geq 1/2, \kappa > \varepsilon^2/2$,

$$\lim_{t \rightarrow \infty} c_{nm}(t) = 0 \quad (m, n = 0, 1, 2, \dots).$$

Thus the limiting distribution is

$$\lim_{t \rightarrow \infty} u(x, t) \equiv 0 \quad (-\infty < x < \infty), \tag{188}$$

which is independent of the initial data (185). Relation (47) becomes

$$u(x, t) = e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \sum_{n=0}^{\infty} (-1)^n e^{-t(2\kappa-1/2)n} \Psi_n(x) \sum_{m=0}^{\infty} t_{mn}(\beta) \int_{-\infty}^{\infty} (-1)^m e^{-mt/2} \Psi_m(y) u_0(y) dy \tag{189}$$

with $\beta = \beta(t) = -2i\varepsilon \sinh(t/2)$. With the help of (49), (43) and the Fubini theorem we transform

$$\sum_{m=0}^{\infty} t_{mn}(\beta) \int_{-\infty}^{\infty} e^{-mt/2} \Psi_m(-y) u_0(y) dy = \iint_{-\infty}^{\infty} K_{e^{-t/2}}(-y, z) \left(\sum_{m=0}^{\infty} t_{mn}(\beta) \Psi_m(z) \right) u_0(y) dy dz$$

$$= \iint_{-\infty}^{\infty} K_{e^{-t/2}}(-y, z) (e^{\gamma z} \Psi_n(z)) u_0(y) dy dz, \tag{190}$$

where $\gamma = i\beta = 2\varepsilon \sinh(t/2)$. The series (189) becomes

$$u(x, t) = e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \sum_{n=0}^{\infty} e^{-t(2\kappa-1/2)n} \Psi_n(-x) \iint_{-\infty}^{\infty} K_{e^{-t/2}}(-y, z) (e^{\gamma z} \Psi_n(z)) u_0(y) dy dz$$

$$= e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \iint_{-\infty}^{\infty} K_{e^{-t/2}}(-y, z) e^{\gamma z} \left(\sum_{n=0}^{\infty} e^{-t(2\kappa-1/2)n} \Psi_n(-x) \Psi_n(z) \right) u_0(y) dy dz$$

$$= e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K_{e^{-t/2}}(-y, z) e^{\gamma z} K_{e^{-t(2\kappa-1/2)}}(-x, z) dz \right) u_0(y) dy \tag{191}$$

in view of the generating relation (48). Therefore, the integral form of the solution (186)–(187) is

$$u(x, t) = e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \int_{-\infty}^{\infty} \mathcal{H}_t(x, y) u_0(y) dy, \tag{192}$$

where by the definition

$$\mathcal{H}_t(x, y) := \int_{-\infty}^{\infty} K_{e^{-t(2\kappa-1/2)}}(-x, z) e^{\gamma z} K_{e^{-t/2}}(-y, z) dz. \tag{193}$$

Denoting $r_1 = e^{-t(2\kappa-1/2)}$ and $r_2 = e^{-t/2}$ we obtain by (48) that

$$\int_{-\infty}^{\infty} K_{r_1}(-x, z) e^{\gamma z} K_{r_2}(-y, z) dz = \frac{1}{\pi \sqrt{(1-r_1^2)(1-r_2^2)}} \exp\left(-\frac{(1+r_1^2)(1-r_2^2)x^2 + (1-r_1^2)(1+r_2^2)y^2}{2(1-r_1^2)(1-r_2^2)}\right) \times \int_{-\infty}^{\infty} \exp\left(\frac{(1-r_1^2)(1-r_2^2)\gamma - 2r_1(1-r_2^2)x - 2r_2(1-r_1^2)y}{(1-r_1^2)(1-r_2^2)} z\right) \exp\left(-\frac{1-r_1^2r_2^2}{(1-r_1^2)(1-r_2^2)} z^2\right) dz \tag{194}$$

and the integral can be evaluated with the help of an elementary formula

$$\int_{-\infty}^{\infty} e^{-az^2+2bz} dz = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0. \tag{195}$$

As a result, an analog of the heat kernel in (192)–(193) is given by

$$\mathcal{H}_t(x, y) = \frac{1}{\sqrt{\pi(1-r_1^2r_2^2)}} \exp\left(-\frac{(1-r_1^2r_2^2)(x^2+y^2) + (r_1^2-r_2^2)(x^2-y^2)}{2(1-r_1^2)(1-r_2^2)}\right) \times \exp\left(\frac{[(1-r_1^2)(1-r_2^2)\gamma - (r_1+r_2)(1-r_1r_2)(x+y) - (r_1-r_2)(1+r_1r_2)(x-y)]^2}{4(1-r_1^2)(1-r_2^2)(1-r_1^2r_2^2)}\right) \tag{196}$$

with $r_1 = e^{-t(2\kappa-1/2)}$, $r_2 = e^{-t/2}$ and $\gamma = 2\varepsilon \sinh(t/2)$, $t > 0$. The last expression can be simplified to a somewhat more convenient form

$$\mathcal{H}_t(x, y) = \exp\left(-\left(\frac{r_1+r_2}{1+r_1r_2}(x+y) + \frac{r_1-r_2}{1-r_1r_2}(x-y)\right) \frac{\gamma}{2}\right) \exp\left(\frac{(1-r_1^2)(1-r_2^2)}{1-r_1^2r_2^2} \frac{\gamma^2}{4}\right) K_{r_1r_2}(x, y) \tag{197}$$

in terms of the Mehler kernel (48). One can show that tion

$$\lim_{t \rightarrow 0^+} u(x, t) = u_0(x) \tag{198}$$

by methods of Ref. 52. The details are left to the reader.

A formal substitution of $u_0(x) = \delta(x-x_0)$ into (192) gives

$$u(x, t) = H(x, x_0, t) = e^{-(\varepsilon^2/2) \sinh t - (\kappa - \varepsilon^2/2)t} \mathcal{H}_t(x, x_0) \tag{199}$$

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} - x^2 u \right) + f(t) x u - g(t) \frac{\partial u}{\partial x}, \tag{201}$$

as the fundamental solution of the diffusion Eq. (184).

In the limit $\varepsilon \rightarrow 0$ we obtain

$$u(x, t) = e^{-\kappa t} \int_{-\infty}^{\infty} K_{e^{-2\kappa t}}(x, y) u_0(y) dy \tag{200}$$

where $f(t)$ and $g(t)$ are real-valued functions of time, subject to the initial condition

as the exact solution of the corresponding initial value problem in terms of the Mehler kernel (48). This kernel gives also a familiar expression in statistical mechanics for the density matrix for a system consisting of a simple harmonic oscillator [20].

$$u(x, t)|_{t=0} = u_0(x) \quad (-\infty < x < \infty). \tag{202}$$

15.2. Generalization

A formal substitution of $t \rightarrow -it$ and $\psi \rightarrow u$ into (77) with $\omega = 2\kappa$ and $f \rightarrow f, g \rightarrow -ig$ yields a diffusion-type equa-

The exact solution is

$$u(x, t) = \int_{-\infty}^{\infty} H(x, y, t) u_0(y) dy \quad (203)$$

and the Green function can be found in the form

$$H(x, y, t) = H_0(x, y, t) e^{a(t)x+b(t)y+c(t)}, \quad (204)$$

where

$$H_0(x, y, t) = \sqrt{\frac{r}{\pi(1-r^2)}} \times \exp\left(\frac{4xyr - (x^2 + y^2)(1+r^2)}{2(1-r^2)}\right) \quad (205)$$

with $r = e^{-2\kappa t}$. Indeed, substitution of (204) into (201) gives the system of equations

$$\frac{d}{dt}(\sinh(2\kappa t) a(t)) = f(t) \sinh(2\kappa t) + g(t) \cosh(2\kappa t), \quad (206)$$

$$\frac{d}{dt} b(t) = \frac{2\kappa a(t) - g(t)}{\sinh(2\kappa t)}, \quad (207)$$

$$\frac{d}{dt} c(t) = \kappa a^2(t) - g(t) a(t) \quad (208)$$

and the solutions are

$$a(t) = \frac{1}{\sinh(2\kappa t)} \times \int_0^t (f(s) \sinh(2\kappa s) + g(s) \cosh(2\kappa s)) ds, \quad (209)$$

$$b(t) = \int_0^t \frac{2\kappa a(s) - g(s)}{\sinh(2\kappa s)} ds, \quad (210)$$

$$c(t) = \int_0^t (\kappa a^2(s) - g(s) a(s)) ds \quad (211)$$

provided $a(0) = b(0) = c(0) = 0$.

An analog of the expansion (99) is

$$H(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm}(t) \Psi_n(x) \Psi_m(y) \quad (212)$$

with

$$c_{nm}(t) = \frac{1}{\sqrt{2^{n+m} n! m!}} e^{c-\kappa t - (ab/2) \sinh(2\kappa t) + \lambda^2/4} \times (a + br)^n (b + ar)^m F_0\left(-n, -m; \frac{2}{\lambda^2}\right). \quad (213)$$

Here $r = e^{-2\kappa t}$, $\lambda^2 = a^2 + b^2 + 2ab \cosh(2\kappa t)$ and functions $a(t)$, $b(t)$ and $c(t)$ are given by the integrals (209)–(211), respectively. This can be derived by expanding the kernel (204) in the double series in the same fashion as in Sec. 11, or by the substitution $t \rightarrow -it$, $a \rightarrow -ia$, $b \rightarrow -ib$, and $c \rightarrow -ic$ in (99). The coefficients $c_{nm}(t)$ are positive when $t > 0$.

The solution (203) takes the form

$$u(x, t) = \sum_{n=0}^{\infty} \Psi_n(x) \sum_{m=0}^{\infty} c_{nm}(t) \times \int_{-\infty}^{\infty} \Psi_m(y) u_0(y) dy. \quad (214)$$

These results can be extended to the case when parameter κ is a function of time in Eq. (201). The details are left to the reader.

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