Quantum confinement particle in a 2D quadrupole potential

A. Reyes
Departamento de Recuperación de Hidrocarburos
Eje Central Lázaro Cárdenas Norte 152, San Bartolo Atepehuacan, 07730, D.F.

J. Adrian Reyes, G.J. Vázquez, and M. del Castillo-Mussot
Departamentos de Estado Sólido y Física-Química,
Instituto de Física Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México 01000 D.F. México.

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We analytically solve the Hamiltonian for a quantum particle confined in a cylindrical hard-wall well, subject to the action of a two-dimensional quadrupolar potential at the well center. The angular part of the wavefunction is expressed by Mathieu functions whose angular eigenenergies take negative values when the quadrupolar momentum is above a certain threshold. We show that in this case, the radial part of the eigenfunctions is expressed in terms of Bessel functions of an imaginary order which are imaginary-value functions whose phases are not well defined at the origin. However, the density of probability is well defined everywhere and the wave function satisfies hard-wall boundary conditions for any value of the parameters involved. We discuss an alternative criterion for determining the eigenenergies of the system based on the expected value of the symmetrized radial momentum.

Keywords: Quadrupolar potential; quantum confinement.

Se resuelve analíticamente el hamiltoniano para una partícula confinada en un pozo cilíndrico de paredes duras, sujeto a la acción de un potencial cuadrupolar bidimensional en el centro del pozo. La parte angular de la función de onda es escrita en términos de funciones de Mathieu cuyas energías propias toman valores negativos cuando el momento cuadrupolar está encima de cierto umbral. Se demuestra en este caso que la parte radial pede ser expresada en términos de las funciones de Bessel de orden complejo cuyas fases no están bien definidas en el origen. Sin embargo, la densidad de la probabilidad está bien definida en todos lados y la función de onda satisface las condiciones de frontera para cualquier valor de los parámetros involucrados. Se discute un criterio alternativo para determinar las energías propias del sistema basado en el valor esperado del momento radial simetrizado.

Descriptores: Potencial cuadrupolar; confinamiento cuántico.

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1. Introduction

Isotropic central force problems are widely discussed in textbooks of quantum mechanics [1], but the solutions to systems with nonisotropic forces are rarely treated. However, the recent boom in the construction of artificial semiconductor structures of diverse geometries makes it interesting to consider a nonisotropic potential which leads to nonisotropic wavefunctions whose larger density of probability is concentrated in certain given directions, determined by the features of the interaction. For example, the confinement model for a charged particle might suggest an alternative way for designing a cantilever tip, instead of using a specific wall setup in an angular sector geometry [2]. Moreover, most of the interactions in nature are nonisotropic ones, like for example, in the case of water molecules which present a polar interaction whose direction of stronger strength are those of their hydrogen bonds. It is well known that those substances whose polar arrangement are able to adapt and preserve the hydrogen bonds structure can be dissolved in water and are known as polar solvents. In this way, chemists used to classify chemical compounds in polar and non-polar ones. Also, in the same spirit, molecules and parts of molecules are denominated hydrophilic and hydrophobic because of their tendency to stay or not within water. Another example is given by liquid crystals [3] which nowadays are ubiquitous substances whose highly nonisotropic interactions have been usually modeled by electric dipoles. The macroscopic effects of this nonisotropic interaction give rise to many interesting physical properties which have been widely applied in technology.

Analyzing a nonisotropic potential, we find wavefunctions with some unusual characteristics, shared by a certain general class of nonisotropic systems which do not allow us to use the standard formalism to obtain their eigenvalue spectrum. As a consequence, we develop an alternative procedure to obtain the eigenvalues of the system. It is well known that the energy spectrum for a cyclic variable as an angle is normally positive but discrete. Indeed, the particle is granted to cover uniformly the whole space available to the angular coordinate with the only restriction that of being periodic. A different situation is found when some specific directions are biased with respect to other ones. This is the case when we have a nonisotropic potential whose angular dependence is such that its magnitude is much greater for certain given angles. This means that the angular energy of the particle might be negative, as well as the square of its angular momentum. Hence, the effective equivalent one particle potential coming from the angular part, in the radial equation, which normally is repulsive, turns out to be attractive.

In particular, for a two-dimensional system, this leads to a complex eigenfunction whose value is not completely defined but still remains bounded at the origin. It should be recalled that the complexity of the wavefunction is a peculiar feature for bound states, as is mentioned in undergraduate quantum mechanics courses. In fact, it implies that the vanishing linear momentum condition is not automatically fulfilled, as happens for most of the textbook examples.

We stress that even though we shall consider particularly only the nonisotropic potential corresponding to a two-dimensional quadrupole, a similar behavior is to be expected for potentials having the following general features. First, the angular energy, the linear momentum on z, and the Hamiltonian should form a group of mutually commutable operators which makes possible therefore to describe properly the quantum solution in terms of its eigenvalues. Second, its anisotropy degree should be controlled by changing a certain parameter which in our case is the quadrupole momentum.

In Sec. 2 we write the Schrödinger equation for our model and solve it analytically by separating variables. Up to this point, we also discuss the lack of standard criteria to obtain the energy spectrum of our system. In Sec. 3 we calculate the symmetrized radial momentum and exhibit graphically its peculiar behavior. From this, we establish a new kind of criterion to find the allowed eigenenergies. In Sec. 4, we summarize our results.

2. Quadrupole confinement

First consider a particle constrained by an infinite cylindrical well with impenetrable walls located at $\rho = R$. In the center of this well there is a segment of linear quadrupole potential

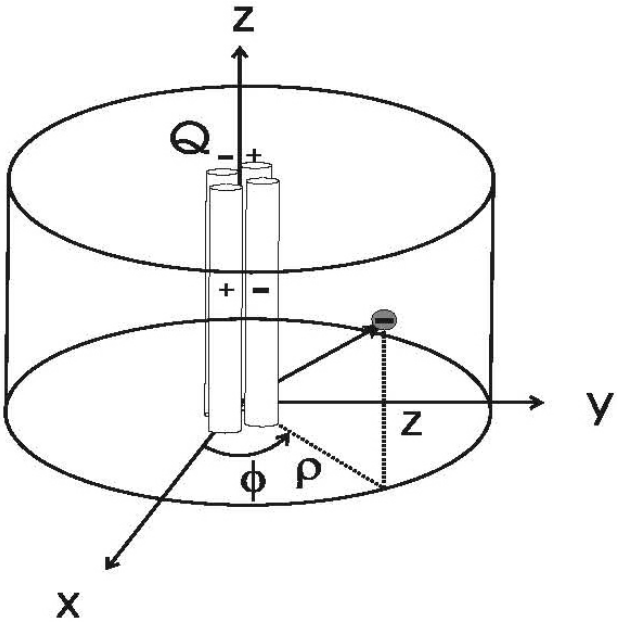

FIGURE 1. Diagram of the system where the positions of the quadrupole and particle coordinates are indicated.

which can be constructed by four charges, two of which are positive and two negative. Of course some more complex charge arrangement could behave predominantly as a quadrupole, as for example a system with a pair of perpendicular electric dipoles. The explicit form for the total potential is given by

$$\begin{aligned} V(\phi,\rho^2) =& Q\frac{\cos 2\phi}{\rho^2}, \quad \text{for} \quad \rho \le R \\ V(\phi,\rho^2) =& \infty, \quad \text{for} \quad \rho > R, \end{aligned} \tag{1}$$

where Q is the quadrupolar magnitude.

In Fig. 1 we schematically show the system including some field lines of the quadrupolar potential together with the hard well.

The Hamiltonian of our system is given by

$$\hat{H} = \frac{-\hbar^2\nabla^2}{2m} + V = \frac{\hat{p}_\rho^2}{2m} + \frac{\hat{H}_\theta}{\rho^2} + \frac{\hat{p}_z^2}{2m} \tag{2}$$

where

$$\hat{p}_\rho = -i\hbar\frac{\partial}{\partial\rho}. \tag{3}$$

This leads to Eq. (2) where $f(\phi) = C\cos 2\phi$,

$$\hat{p}_z = -i\hbar\frac{d}{dZ}$$

and

$$\hat{H}_\theta = -(\hbar^2/2m)d^2/d\phi^2 + f(\phi). \tag{4}$$

The Schrödinger equation for our system can be written in cylindrical coordinates as

$$\begin{aligned} &-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{\partial^2\Psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial Z^2}\right) \\ &+ V(\phi,\rho^2)\Psi = E\Psi, \end{aligned} \tag{5}$$

where E is the energy and $V(\phi,\rho^2)$ is given by the Eq. (1).

Before solving this equation, note that $\hat{H}$, $\hat{H}_\theta$ and $\hat{p}_z$ are a group of mutually commutable operators so that the eigenstates of the Hamiltonian can be described in terms of the eigenvalues of these operators [1]. This is valid for any function $f(\phi)$, but we should consider only functions whose nonisotropic distribution can be controlled by a given parameter as is the case for C.

Equation (5) can be solved by assuming a solution of the form $\Psi(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$:

$$\frac{d^2Z}{dZ^2} + \alpha^2 Z = 0 \tag{6}$$

$$\frac{d^2\Phi}{d\phi^2} + (a - 2q\cos 2\phi)\,\Phi = 0 \tag{7}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\gamma^2 - \frac{\beta^2}{\rho^2}\right)R = 0 \tag{8}$$

where the parameters or separation constants are given by

$$\gamma^2 = \frac{2mE - \hbar^2\alpha^2}{\hbar^2}, \tag{9}$$

$$q = \frac{mC}{\hbar^2} \quad \text{and} \quad a = \beta^2. \tag{10}$$

Equation (7) is the Mathieu equation, which is present in physical situations where harmonic periodic potentials are involved [6]. In this case, the Mathieu functions depend on the parameters a and q. For $q \neq 0$, the Mathieu functions are periodic functions of ϕ only for certain values of a known as Mathieu characteristic values a_n and b_n for even and odd functions, respectively, where n is an integer number. In our case a is the dimensionless angular eigenenergy, q is also a dimensionless parameter measuring the magnitude of the quadrupolar momentum, and n is the corresponding angular quantum number.

A solution to Eq. (7) with period π or 2π, Φ, is the Mathieu function and its Fourier expansion is given in the form:

$$\Phi = \sum_{m=0}^{\infty} (A_m \cos m\phi + B_m \sin m\phi), \tag{11}$$

where B_0 must be taken equal to zero [6]. Here we are considering both even and odd solutions. Another possible Fourier expansion which separates the solutions by parity is given by

$$\Phi_{even} = \sum_{m=0}^{\infty} A_{2m+p} \cos(2m+p)\phi \tag{12}$$

$$\Phi_{odd} = \sum_{m=0}^{\infty} B_{2m+p} \sin(2m+p)\phi. \tag{13}$$

Here p takes the values 1 or 0. For $p = 1$, the solution has a period 2π which will be the only relevant solution for our purpose since our system has to fulfil the boundary conditions

$$\Phi(2\pi) = \Phi(0) \tag{14}$$

and

$$\frac{d\Phi(2\pi)}{d\phi} = \frac{d\Phi(0)}{d\phi}, \tag{15}$$

because for our system, both values ϕ and $\phi + 2\pi$ describe the same physical point.

In the Mathieu theory [7], it is shown that eigenvalues satisfy the following relation:

$$a_0 < b_2 < a_2 < b_4 < a_4 \cdots \tag{16}$$

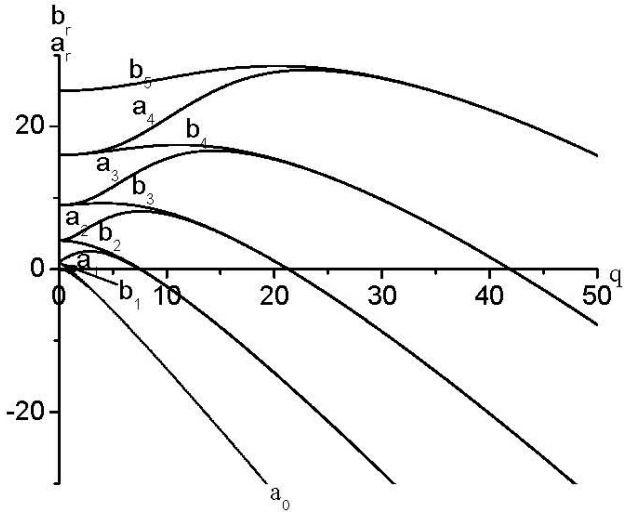

FIGURE 2. Normalized angular eigenenergies as defined by Eq.(10) for even a_r and odd b_r with ($r = 1, ..., 5$) states as function of the dimensionless quadrupolar strength q.

In Fig. 2, these eigenvalues are shown as a function of q. Notice that as q increases, the values of a become negative and the interval for which a is positive is small for a small n. As can be seen, the relation (16) prevents any intersection of the curves. For $q = 0$, we have $a_0(0) = 0$; $b_2(0) = a_2(0) = 2^2$; $b_4(0) = a_4(0) = 4^2$, etc.

For larger values of q, we have the asymptotic behavior [8]

$$a_{2n}(q) \longrightarrow -2q + 2(2n+1)\sqrt{q} - \frac{1}{4}[(n+1)^2 + n^2] \tag{17}$$

$$b_{2n}(q) \longrightarrow -2q + 2(2n-1)\sqrt{q} - \frac{1}{4}[(n-1)^2 + n^2]. \tag{18}$$

In Fig. 3 we have plotted the ground state and the two first excited states corresponding to some values of q of the eigenvalues shown in Fig. 2. It can be seen that the curves change sharply near the origin for large values of q and the number of nodes increases with the order of the state. In the limit of vanishing q, the curves reduce to sinusoidal functions as expected for a planar rotor. Consistently, states with negative energy are associated with more localized wavefunctions.

For positive values of a, β is a real number so that the general solution of Eq. (8) is, as usual, the linear combination of cylindrical Bessel and Neumann functions. However, the Neumann function has to be discarded because it has a singularity at the origin, whereas the zeros of Bessel functions $\chi_{\beta s}$ provide us with the allowed eigenenergies given by

$$E_{\beta s\lambda} = \frac{\hbar^2(\alpha^2 + \gamma^2)}{2m} = \frac{\hbar^2}{2m}\left[\left(\frac{\lambda\pi}{L}\right)^2 + \left(\frac{\chi_{\beta s}}{R}\right)^2\right] \tag{19}$$

where R and L are the radius and height of the confining cylinder.

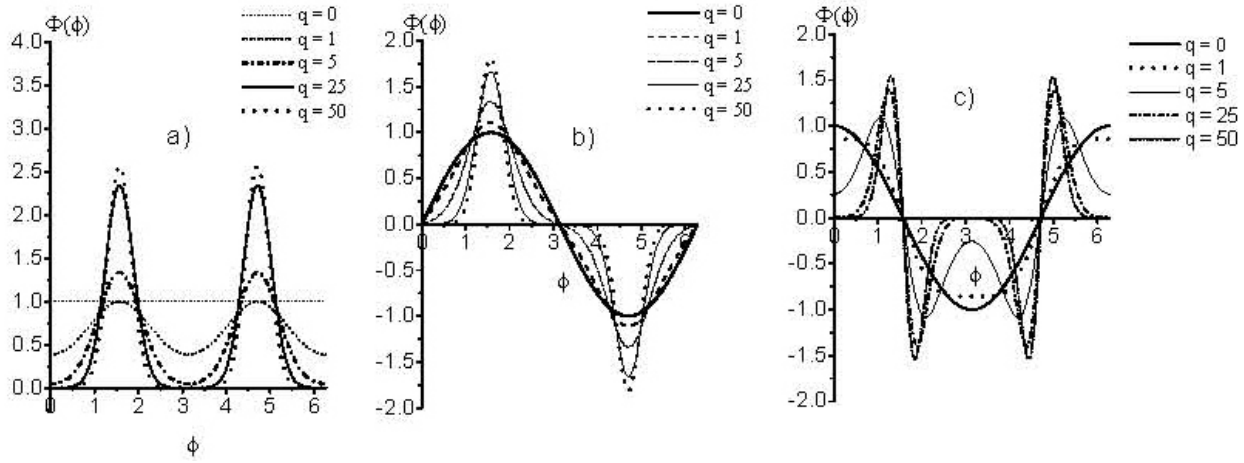

FIGURE 3. Angular eigenfunctions for the a) ground state, b) first excited state, and c) second excited state parametrized by the dimensionless quadrupolar strength q.

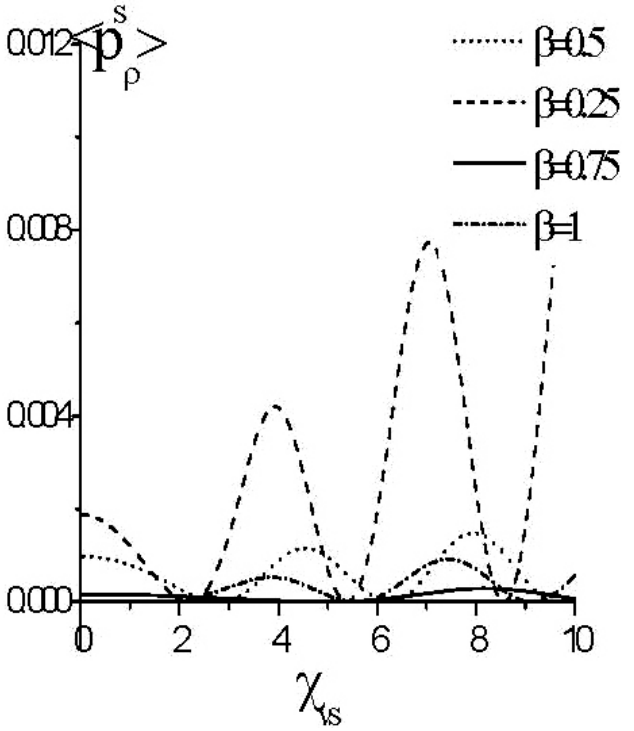

FIGURE 4. Expected value of the radial momentum as a function of $\chi_{\nu s}$ for various values of ν.

Equation (7) exhibits that a negative value which corresponds to a bound state for Φ. If we set $a < 0$ in Eq. (10), we obtain an imaginary value of β. Thus Eq. (8), for the radial part yields

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\gamma^2 + \frac{\beta^2}{\rho^2}\right) R = 0. \qquad (20)$$

which is the Bessel function of complex order as can be checked by replacing β by $-\beta i$. By direct substitution into the first term of the series expansion of the Bessel function [9], we find that near the origin the latter functions behave as

$$\left.\begin{matrix} J_{i\beta} \\ N_{i\beta} \end{matrix}\right\} \approx \frac{\cos(\beta \ln x) \pm i \sin(\beta \ln x)}{(i\beta)!}, \qquad (21)$$

where $x!$ denotes the factorial function of x. Thus, the radial part of the wavefunction is not well defined at the origin. Nevertheless, the density of probability $|R|^2$ which is the quantity having a physical meaning, is completely defined. Additionally, since the phase of the functions defined by Eq. (21) rotates much more rapidly as one gets near to the origin, the phase of the wavefunction will have the same behavior in the vicinity of the origin.

The general solution in this case is given by

$$R_c(\rho) = C_1 J_{i\beta}(\bar{\chi}_{\beta s}\bar{\rho}) + C_2 N_{i\beta}(\bar{\chi}_{i\beta s}\bar{\rho}) \qquad (22)$$

where $\bar{\rho} = \rho/R$ and C_1, C_2 are constants to be determined.

In Eq. (21), we can see that near the origin the lineally independent functions $J_{i\beta}$ and $N_{i\beta}$ have the dependences, $\cos(\beta \ln x) \pm i \sin(\beta \ln x)$, which diverge at the origin. Since the physical restrictions are imposed on the density of probability, whose radial part, $P_\rho = \bar{\rho}\,|R|^2$, where R is given by Eq. (22). Then $\bar{\rho}\cos(\beta \ln \bar{\rho})$ and $\bar{\rho}\sin(\beta \ln \bar{\rho})$ are finite, so that the density of probability exists, even at the origin, as we will be seen in Fig. 5. We can take both solutions, $J_{i\beta}$ and $N_{i\beta}$ as physically valid.

To satisfy the hard wall boundary condition $\Psi(\bar{\rho}{=}1){=}0$, we require

$$\frac{C_1}{C_2} = -\frac{N_{i\beta}(\bar{\chi}_{i\beta s})}{J_{i\beta}(\bar{\chi}_{i\beta s})}. \qquad (23)$$

It is important to stress that with Eq. (22) and (23), the boundary condition can be fulfilled for any value of $\bar{\chi}_{i\beta s}$. That is to say, up to this point we have apparently a continuous spectrum described by Eq. (19), since any value of $\bar{\chi}_{i\beta s}$ is permitted.

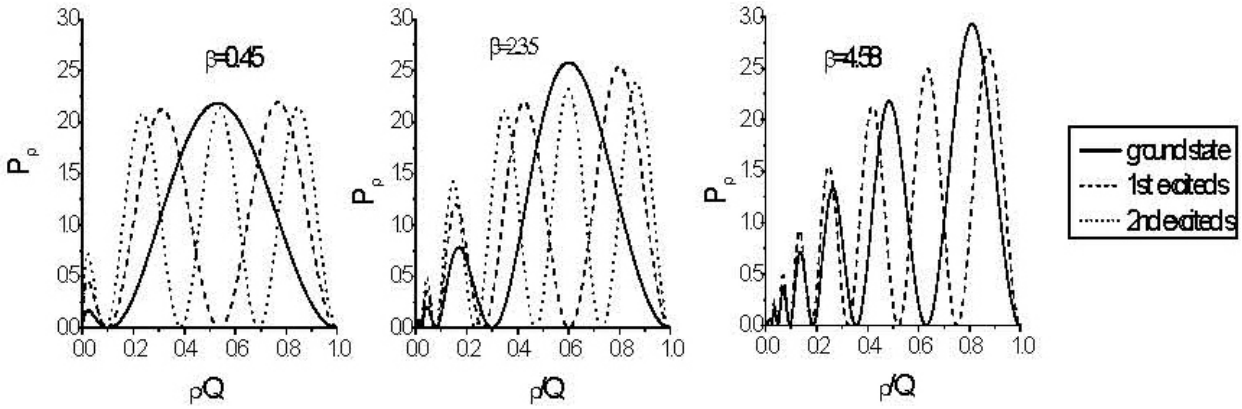

FIGURE 5. Radial part of the density of probability $P_\rho = \bar{\rho}\,|R|^2$ for the three first states for various values of ν.

In 3D the whole wavefunction for our particle can be therefore written in the following form:

$$\Psi(\bar{\rho},\phi,\bar{z})_{\beta s\lambda} = N\sin\lambda\pi\bar{z}\Phi_\beta(\phi) \times\left[-\frac{N_{i\beta}(\bar{\chi}_{i\beta s})}{J_{i\beta}(\bar{\chi}_{i\beta s})}J_{i\beta}(\bar{\chi}_{\beta s}\bar{\rho})+N_{i\beta}(\bar{\chi}_{i\beta s}\bar{\rho})\right] \quad (24)$$

where $\bar{z}=(z/L)$, N is the normalization constant, and $\Phi(\phi)$ is one of the Mathieu functions given by Eqs. (12) and (13). For hard walls located at $z=0$ and $z=L$, λ must be an integer number.

2.1. Radial momentum

The expected value of the radial momentum $\langle\hat{p}_\rho\rangle$ is not necessarily zero for $a<0$, because R is not a real function. Using the bidimensional operator for the radial momentum given by Eq. (3), we get

$$\langle\hat{p}_\rho\rangle = \int_v \Psi^*\hat{p}_\rho\Psi dv. \quad (25)$$

Let us consider first the case when β is real, for which Eq. (24) turns out to be

$$\begin{aligned}\langle\hat{p}_{\bar{\rho}}\rangle &= -N^2\hbar i\int_0^1\int_0^{2\pi}\int_0^1 \sin^2(\lambda\pi\bar{z})\Phi_\beta^2(\phi) \times\left[J_\beta(\chi_{\beta s}\bar{\rho})J_\beta'(\chi_{\beta s}\bar{\rho})\right]d\bar{z}d\phi\bar{\rho}d\bar{\rho}\\ &= -\frac{N^2\hbar i\pi^2}{2}\int_0^1\bar{\rho}J_\beta(\chi_{\beta s}\bar{\rho})J_\beta'(\chi_{\beta s}\bar{\rho})d\bar{\rho}. \quad (26)\end{aligned}$$

To perform the last integral, we integrate by parts to get

$$\int_0^1\bar{\rho}J_\beta(\chi_{\beta s}\bar{\rho})J_\beta'(\chi_{\beta s}\bar{\rho})d\bar{\rho} = \frac{\bar{\rho}}{2}J_\beta^2(\chi_{\beta s}\bar{\rho})|_0^1 \quad (27)$$

to finally obtain

$$\langle\hat{p}_{\bar{\rho}}\rangle = -\frac{N^2\hbar i}{4\lambda}\sin\lambda\pi\bar{z}\pi J_\beta^2(\chi_{\beta s}) = 0, \quad (28)$$

which vanishes, since $\chi_{\beta s}$ denotes the roots of the Bessel functions.

Analogously, we can calculate $\langle\hat{p}_{\bar{\rho}}\rangle$ for an imaginary β. Then by introducing the variable $\nu=i\beta$, Eq. (24) takes the form

$$\Psi(\bar{\rho},\phi,\bar{z})_{\nu s\lambda} = N\sin\lambda\pi\bar{z}\Phi(\phi) \times\left(\frac{N_\nu(\bar{\chi}_{\nu s})}{J_\nu(\bar{\chi}_{\nu s})}J_\nu(\bar{\chi}_{\nu s}\bar{\rho})-N_\nu(\bar{\chi}_{\nu s}\bar{\rho})\right), \quad (29)$$

and its derivative is given by

$$\frac{\partial\Psi}{\partial\bar{\rho}} = N\sin\lambda\pi\bar{z}\Phi(\phi) \times\left(\frac{N_\nu(\bar{\chi}_{\nu s})}{J_\nu(\bar{\chi}_{\nu s})}J_\nu'(\bar{\chi}_{\nu s}\bar{\rho})-N_\nu'(\bar{\chi}_{\nu s}\bar{\rho})\right), \quad (30)$$

and as a consequence $\langle\hat{p}_{\bar{\rho}}\rangle$ is given explicitly by

$$\begin{aligned}\langle\hat{p}_{\bar{\rho}}\rangle = &-\frac{N^2\hbar i}{4}\\ &\times\int_0^1\left(J_{-\nu}(\bar{\chi}_{\nu s}\bar{\rho})-\frac{J_{-\nu}(\bar{\chi}_{\nu s})N_{-\nu}(\bar{\chi}_{\nu s}\bar{\rho})}{N_{-\nu}(\bar{\chi}_{\nu s})}\right)\\ &\times\left(J_\nu'(\bar{\chi}_{\nu s}\bar{\rho})-\frac{J_\nu(\bar{\chi}_{\nu s})N_\nu'(\bar{\chi}_{\nu s}\bar{\rho})}{N_\nu(\bar{\chi}_{\nu s})}\right)\bar{\rho}d\bar{\rho}. \quad (31)\end{aligned}$$

A direct numerical calculation of this expression shows that $|\langle\hat{p}_{\bar{\rho}}\rangle|$ is always positive for values of α and ν within the intervals $[0,20]$ and $[0,100]$, respectively. This result is not consistent with the concept of a confined particle, for which we expect $\langle\hat{p}\rangle=0$. This expression for confined systems described by real-valued wavefunctions, is a consequence of the fact that the momentum is a self-adjoint operator. Nevertheless for the 2D case, $\hat{p}_{\bar{\rho}}$ is not self-adjoint. Using this insight we decide to explore the expectation value of the symmetrized momentum $\hat{p}_{\bar{\rho}}^s = (\hat{p}_{\bar{\rho}}+\hat{p}_{\bar{\rho}}^\top)/2$, which is by construction a self-adjoint operator. Here

$$\hat{p}_{\bar{\rho}}^{\top} = -i\hbar\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)$$

is the adjoint operator to $\hat{p}_{\bar{\rho}}$.

In Fig. 4 we plot $\left|\langle\hat{p}_{\bar{\rho}}^{s}\rangle\right|$ as a function of $\chi_{\nu s}$ for various values of ν. Note that it has oscillatory behavior and even vanishes for specific values of $\chi_{\nu s}$. Since a bound state, as the one we describe here, is physically expected to have a vanishing radial momentum, then, we assume, the condition $\langle\hat{p}_{\bar{\rho}}^{s}\rangle = 0$ that determines the values of $\bar{\chi}_{\nu s}$ which correspond to bound states.

We must point out that the approach to obtain $\langle\hat{p}_{\bar{\rho}}^{s}\rangle = 0$ is purely mathematical. The physical condition $\langle\hat{p}_{\bar{\rho}}\rangle = 0$ does not give us information to find the values of $\chi_{\beta s}$. This mathematical approach is justified when obtaining the results shown in Fig. 4 in that it provides the discrete values of the parameter $\chi_{\beta s}$ for which the approach of the $\langle\hat{p}_{\bar{\rho}}^{s}\rangle = 0$ value is satisfied.

To illustrate our results, we have calculated and shown in Fig. 5 the radial part of the density of probability $P_\rho = \bar{\rho}\,|R|^2$ for the three first eigenstates $r = 0, r = 1$ and $r = 2$, and various values of q. Notice that for the eigenfunctions with smaller values of ν, P_ρ exhibits the number of oscillations corresponding to the order of the eigenstate, whereas for larger values of ν additional oscillations of less amplitude appear near the origin. Moreover, the number of additional oscillations increases for larger values of ν, leading to profiles with many oscillations. One can hardly distinguish them from the curves of different eigenstates, even if these are the ground state and the first excited state. To understand this, notice that from Fig. 2 the density of probability has the form of two angular sectors centered at $\pi/2$ and $-3\pi/2$ and joined at the origin, whose widths diminish as the anisotropy increases. Then, the position of the particle at the origin also tends to be determined while $\langle\hat{p}_{\rho}^{2}\rangle$ gets larger for an increasing anisotropy, as is manifested by the increasing in the number of oscillations near the origin. This is consistent with the Heisenberg principle.

This also can be argued by writing the Schrödinger Eq. (2) in the following way:

$$\hat{H}\psi = \frac{\hat{p}_{\rho}^{2}}{2m}\psi + \frac{\hat{H}_\theta}{\rho^2}\psi + \frac{\hat{p}_{z}^{2}}{2m}\psi = E\psi. \tag{32}$$

Then from Eq. (7) we know that ψ also satisfies

$$\hat{H}_\theta\psi = \frac{\hbar^2 a}{2m}\psi, \tag{33}$$

so Eq. (32) becomes

$$\frac{\hat{p}_{\rho}^{2}}{2m}R + \frac{\hbar^2 a}{2m\rho^2}R = \left(E + \frac{\hbar^2 k_z^2}{2m}\right)R. \tag{34}$$

Hence, for a negative value of a, we have a quite unusual, attractive inertia potential which, in contrast to the potential wall located at $\rho = R$ that just reflects back the particle, attracts the particle to the origin. The effective $R-$associated eigenenergy, $E_{eff} = E - \langle\hbar^2 a/2m\rho^2\rangle + \hbar^2 k_z^2/2m$ also gets larger because R behaves as if the particle was in an excited state. Thus, for this case, the ground state is expected to have many oscillations as shown in the curves of Fig. 5 for large ν.

3. Conclusions

We have written and solved the Schrödinger equation for a particle confined in a cylindrical well with a 2D quadrupole at its center. We showed that the angular part of the wavefunction of this nonisotropic potential can be expressed in terms of periodic Mathieu functions, whereas the radial part is written in terms of both real and imaginary order Bessel and Neumann functions for smaller and larger values of q, respectively. The peculiarities of the solution for larger values of q, written in terms of imaginary order Bessel and Neumann function, are twofold. First they can satisfy the boundary condition without imposing any restriction on the parameters of the solution, and second the expectation-values of their radial momentum are not identically null. Thus, we require a vanishing expectation-value for the symmetrized linear momentum in order to determine the allowed values of energy.

We should point out that the system we consider is original, and has a relatively simple analytical solution, but the features of its solution and solving procedure are not standard. The same behavior is expected for a general class of potentials whose angular energy, linear momentum on z, and Hamiltonian form a group of mutually commutable operators, having a controlling parameter for increasing the interaction anisotropy. We hope that our analysis helps to explain the behavior of some anisotropic potential for which, as far as we know, there are no exact solutions.

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