Critical strings and analyticity of the ζ function analyticity

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In this paper we study a simple analytic continuation of the Riemann ζ function, using Bernoulli numbers and an analytic continuation of the Γ function in the complex plane. We use our results to study the critical condition in bosonic string theory. The approach is simple and gives the student an alternative point of view of the subject. We also show that the mathematical basis needed to understand the critical condition is based on well known properties of the Dirichlet series and the theory of entire functions, and is within reach of the average graduate student.

Keywords: Mathematical techniques in atomic physics; group theory.

En este trabajo estudiamos una continuación analítica simple de la función ζ de Riemann, usando los números de Bernoulli y una continuación analítica de la función Γ en el plano complejo. Utilizamos nuestros resultados para estudiar la condición crítica en teoría bosonica de cuerdas. El desarrollo es simple y da a estudiante un punto de vista alternativo del tema. También demostramos que la base matemática necesaria para entender la condición crítica está basada en las características bien conocidas de la serie de Dirichlet y de la teoría de funciones enteras, lo cual está al alcance de un estudiante de posgrado.

Descriptores: Métodos matemáticos en física atómica; teoría de grupos.

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1. Introduction

String theory is one of the most fascinating developments in Physics in the last twenty years, and one of its most interesting proposals is the existence of hidden extra dimensions in space time. To arrive at this conclusion, string theory focuses on the properties of the Hamiltonian deduced from the theory and analyzes the vacuum energy of the set of creation operators arranged in what is called *normal order*. As is well known, this order is defined in such a way as to obtain precisely a zero contribution to the energy from the vacuum of the theory. When this procedure is carried out in string theory, the following sum must be evaluated:

$$\sum_{n=1}^{\infty} n. \tag{1}$$

Of course, this sum diverges clearly, but following the regularization process that we present here, we shall see that we can assign to this sum a finite value, and what is even stranger, a negative one: -1/12.

Mathematicians have studied such sums in the past (Ref. 1). In the 18th century, Leonhard Euler discovered a relationship that was so curious that he called it paradoxical: 1 - 2 + 3 - 4 + 5 - $\cdots$ = 1/4. One way to study this relationship informally is the following: let us define

$$s = 1 - 2 + 3 - 4 + 5 - \cdots ,$$

so we must have that

$$s = 1 - 2 + 3 - \cdots = (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - \cdots) = h - s,$$

where *h*, called *Grandi's Series*, is the sum of the series

$$h = 1 - 1 + 1 - 1 + \cdots = 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - h.$$

Solving these equations we have: $h = 1 - h$ and $s = h - s$. We obtain $h = 1/2$ and then $s = 1/4$. The series 1 -2 + 3 -4 $\cdots$ is in fact a divergent series, but Euler realized that assigning the value 1/2 was a natural choice. Grandi sent a copy of the 1703 work to Leibniz, who had already considered the divergent alternating series 1 - 2 + 4 - 8 + 16 -$\cdots$ several years before, in 1673. In 1675, as a result of his work, Leibniz formulated the first convergence test in the history of mathematics, the alternating series test. At the end of his life Leibniz granted the value of 1/2 to Grandi's Series, because one could obtain 0 or 1 depending on how we manipulate the series, so that argued Leibnitz, the law of justice dictates that the value of this series should be an intermediate one: 1/2. Strange as it may seem, modern treatments give justice to this result. Later Jacob Bernoulli, and Jacopo Ricatti made some contributions in this area, but it was Leonhard Euler and Riemann who finally found a modern, rigorous treatment to this type of alternating series (Ref. 2).

We can use Grandi's series to calculate the value of Eq. (1) as follows. Let us put

$$(1-4)\sum_{n=1}^{\infty} n = (1+2+3+4+5+6+\cdots)$$
$$-(4+8+12+16+\cdots)$$
$$= 1-2+3-4+5-6+\cdots$$
$$= \sum_{n=1}^{\infty} n(-1)^{n-1},$$

but the last step is precisely Grandi's series

$$\sum_{n=1}^{\infty} n(-1)^{n-1} = \frac{1}{4},$$

which in turn means that

$$\sum_{n=1}^{\infty} n = \frac{-1}{12}.$$

In modern mathematical language the sum [1] is the Riemann $\zeta(s)$ function defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbb{C},$$
$$s = \sigma + it \quad \text{with} \quad \sigma > 1, \tag{2}$$

analytically continued to the value of $s=-1$.

In this paper we study the analytical continuation of Riemann ζ function in a simple, accessible way. We show that critical strings are closely related to the analyticity properties of meromorphic functions and represent a natural choice among all possible string theories available. This point is important since the choice of central charge in critical strings is dictated by two physical considerations. First, to avoid the existence of terms that could violate Lorentz invariance, and second, to minimize the existence of negative norm states or ghosts in the spectrum. But here we see that there is a third point related to the behavior of the analytic continuation of the Riemann ζ function. This last point is somewhat obscured in standard treatments, frequently giving the impression of being too technical to the average reader, which it is not, as we discuss here.

This paper is divided in two parts: the first one tries to remind the reader of the physical basis of the problem, focused mainly on open string theory, since this system illustrates very well the physical background and is simpler to present. The second part consists of the mathematical analytical extension of the ζ function to negative values.

2. Bosonic string theory

We discuss only the bosonic case, which is sufficient for our purposes. Let us call $\gamma_{ab}(\sigma,\tau)$ the auxiliary metric on the world sheet, where $a,b=1,2$. $0 \leqslant \sigma \leqslant \pi$ and τ plays the role of a parametric time; we put $X^\mu(\sigma,\tau)$ for the string amplitudes, where $\mu = 0,1,2,\cdots,D-1$. We start with the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$
$$= -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\gamma}\gamma^{ab}h^{ab}. \tag{3}$$

Here $\gamma = \det(\gamma_{ab})$ is the determinant of the metric,

$$h_{ab} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

and $d^2\sigma = d\tau d\sigma$. Also

$$\eta_{\mu\nu} = \text{diag}(-1,+1,\cdots).$$

The universal Regge slope is the constant α' and is related to the *intrinsic length* of the string $l_s^2 = \alpha'$, (Ref. 4).

2.1. Symmetries

To proceed further we need to study the symmetries present in action (3). First of all we have the global space-time Lorentz-Poincaré symmetry. Under a Lorentz transformation, action (3) remains unchanged. Second, the action (3) is also covariant in the world-sheet indices $a=1,2$ so, under an infinitesimal world-sheet reparametrization $\sigma^a \to \sigma'^a = \sigma^a + \epsilon^a(\sigma,\tau)$, the action remains unchanged. That is the reason we have introduced the world-sheet volume invariant element $d^2\sigma\sqrt{-\gamma}$. For δX^μ, we get (remember X^μ doesn't have world-sheet indexes):

$$\delta X^\mu = \epsilon^a \partial_a X^\mu \tag{4}$$

for the string amplitudes, and for the metric tensor itself we must add the tensorial contribution of the reparametrization changes

$$\delta\gamma^{ab} = \epsilon^c\partial_c\gamma^{ab} - \partial_c\epsilon^a\gamma^{cb} - \partial_c\epsilon^b\gamma^{ac}. \tag{5}$$

But the string action has still more symmetries, which we shall use to simplify it. In particular, we have what is called Weyl invariance

$$\gamma_{ab} \to \gamma'_{ab} = e^{2\omega}\gamma_{ab}, \tag{6}$$

where $\omega(\tau,\sigma)$ is an arbitrary function of τ and σ (but well behaved). So we can rescale the metric and still end up with the same theory. We notice then that there are as many parameters needed to specify the local symmetries (three parameters) as there are independent components of the symmetric world-sheet metric. We shall use this fact to simplify the action in a moment.

2.2. String equations of motion

We are now going to vary the action with respect to the $X^\mu(\sigma,\tau)$ or, more precisely with respect to their derivative:

$$\delta S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \partial_a \left[\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu\right]\delta X^\mu \eta_{\mu\nu} - \frac{1}{2\pi\alpha'}\int d\tau \sqrt{-\gamma}\,\partial_\sigma X_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi}. \quad (7)$$

The last term in the above equation defines the boundary condition and we can use it to define different types of strings. We put Neumann boundary conditions to define *open strings*

$$X'^\mu(\tau,0) = 0$$
$$X'^\mu(\tau,\pi) = 0, \quad (8)$$

or periodic boundary conditions to define *closed strings*

$$X'^\mu(\tau,0) = X'^\mu(\tau,\pi)$$
$$X^\mu(\tau,\pi) = X^\mu(\tau,\pi)$$
$$\gamma_{ab}(\tau,0) = \gamma_{ab}(\tau,\pi). \quad (9)$$

We can now take advantage of the symmetries of the theory. First of all, recall that we have the reparametrization invariance symmetry, Eqs. (4, 5). This means that we have two gauge symmetries, the two 2-dimensional reparametrization $\sigma,\, \tau \rightarrow \tilde{\sigma}(\sigma,\tau),\, \tilde{\tau}(\sigma,\tau)$. On the other hand, the 2- dimensional symmetric metric tensor has three arbitrary functions, so we can use these two gauge degrees of freedom to fix the metric to what is called a *conformal flat space* which is of the form

$$\gamma_{ab} = \eta_{ab} e^\phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^{\phi(\sigma,\tau)}. \quad (10)$$

That is, we have a flat space multiplied by a positive function called the *conformal factor.* Here ϕ is a well behaved function of σ and τ. In this conformal gauge the equations of motion are simply

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\sigma,\tau) = 0. \quad (11)$$

We are going to simplify our approach and deal only with open strings, but we can easily extend our treatment to include closed strings also. Besides, the formulas we present here are quite common and can be found elsewhere. So, for open strings the solution to the equations of movement (11) can be written as

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \quad (12)$$

The coefficients α^μ can be interpreted as the amplitude oscillations of each harmonic when the string oscillates. If the oscillations in the string become zero, we still have a non-null contribution to X^μ, and we can then identify p^μ with the zero mode of the expansion as $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$. We can see already the resemblance of the last expression (12) with Quantum Field Theory and the mode expansion of a vectorial field. In fact, these amplitude coefficients are promoted to creation and annihilation operator in the quantized version of string theory.

2.3. Hamiltonian dynamics

The Lagrangian density in the conformal flat gauge becomes simply

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\left(\partial_\sigma X^\mu \partial_\sigma X_\mu - \partial_\tau X^\mu \partial_\tau X_\mu\right), \quad (13)$$

from which we can derive the canonical conjugate momentum

$$\Pi^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\tau X^\mu)} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \quad (14)$$

where the point means derivative with respect to time τ. From this last expression we can now construct as usual, the Hamiltonian density

$$\mathcal{H} = \dot{X}\mu\Pi_\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\partial_\sigma X^\mu \partial_\sigma X_\mu + \partial_\tau X^\mu \partial_\tau X_\mu\right), \quad (15)$$

from which we can construct in turn the Hamiltonian H by integrating over σ from 0 to π:

$$H = \int_0^\pi d\sigma \mathcal{H}(\sigma) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{-n}\cdot\alpha_n, \quad (16)$$

where $\alpha_{-n}\cdot\alpha_n = \alpha_{-n}^\mu \alpha_{\mu\, n}$. Now, we point out that we are dealing with an invariant reparametrization theory (Eqs. 4-5), so we know that in this case the Hamiltonian must be zero (or, more correctly, weakly zero). We can easily verify this point by explicitly calculating the classical Hamiltonian from Eq. (3), but remember that we are dealing now with flat space. The consequence of the reparametrization invariance is in fact a much richer condition, because it is possible to define an infinite set of operators L_m, called *Virasoro operators*, that satisfy the so-called Virasoro algebra (Ref. 3) as a result of this symmetry. Here, the Hamiltonian is only one of this infinite set of operators, the $L_0 = H$ operator. Similar results are obtained for closed strings. Please see Ref. 4 for details.

The L_0 constraint, related to the Hamiltonian, is important since we shall use it to define the condition of criticality. First we put

$$L_0 = \frac{1}{2}\alpha^2 + 2(\frac{1}{2})\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = -\alpha' M^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n, \quad (17)$$

where M^2 is the squared mass of the string. Requiring L_0 to be zero gives us an expression for the mass operator:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n. \tag{18}$$

2.4. Quantization of the string

The central subject of this paper appears when we try to quantize the string. The approach we follow in this case is simply to write the commutators of the canonical conjugate variables, which in this case goes as follows:

$$\begin{aligned} [X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] &= i\eta_{\mu\nu}\delta(\sigma-\sigma'), \\ [X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] &= 0, \\ [x^\nu, p^\mu] &= i\eta_{\nu\mu}. \end{aligned} \tag{19}$$

Remember that x^μ and p^ν appear in the solution of the string Eq. (12) and survive even if the string modes are zero. Sometimes they are wrongly referred as *Center of Mass (CM)* position and momentum of the string, but we know that CM is a misleading, relativistic concept and the reader surely understand why. The commutation relation Eq. (19) leads to

$$[\alpha^\nu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{20}$$

Notice that in analogy with the harmonic oscillator operators $(1)/(\sqrt{m})\alpha^\mu_{-m}$ behaves like a creation operator and $1/(\sqrt{m})\alpha^\mu_m$ as an annihilation operator. We shall use this feature to construct the spectrum of states of the string. But now we have an ordering problem. If $m+n \neq 0$, the operators α^μ_m of the string commute. But this is not the case in L_0, and as a result we have an ambiguity in the definition of the energy of the string:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \text{constant}. \tag{21}$$

This infinite sum is given by the commutator Eq. (20). Each time we reverse a couple of non-commuting operators, we have a contribution of the form $-(1/2)\sum_{n=1}^{\infty} n$ for each dimension of the string. But how to interpret this infinite sum? The solution is given by the work of mathematicians who realize that it is possible to give a meaning to this sum, as was anticipated in the introduction. What we are going to do is to define this sum as the generalization to -1 of the Riemann zeta function $\zeta(s=-1)$, (Ref. 2).

Let us now give a further insight into this entire subject. In the quantum case the constraints are implemented as a condition over the states $|\phi>$ generated by the amplitude oscillation operators α^μ_m appearing in Eq. (12). But for L_0 we cannot do that because of the order ambiguity that implies a constant (Eq. 21), so instead we impose

$$(L_0 - a)|\phi>= 0, \tag{22}$$

where a is the constant that appears in Eq. (21).

Let us now focus on the spectrum of the system and study the one-excitation state. This state is defined as the action of one creation operator over the vacuum $|0;k\rangle$, where k^μ is the momentum of the *CM of the string*. We need to specify a polarization vector ϵ^μ. We denote this state as $|\epsilon;k\rangle \equiv (\epsilon\cdot\alpha_{-1})|0;k'\rangle$.

Observe that

$$\begin{aligned} \langle\epsilon;k|\epsilon;k'\rangle &= \langle 0;k|\epsilon^*\cdot\alpha_1\epsilon\cdot\alpha_{-1}|0;k\prime\rangle \\ &= \epsilon^*_\mu\epsilon_\nu\langle 0;k|[\alpha^\mu_1,\alpha^\nu_{-1}]0;k'\rangle \\ &= \epsilon^*\cdot\epsilon(2\pi)^D\delta^D(k-k'). \end{aligned} \tag{23}$$

So we could have in general negative norm states or *ghosts* if the polarization vectors are timelike. The constraint over this state produces the result

$$\begin{gathered} (L_0 - a)|\epsilon;k\rangle = 0 \quad \text{which gives} \\ \alpha' k^2 + 1 = a \quad \text{or} \quad M^2 = \frac{1-a}{\alpha'}. \end{gathered} \tag{24}$$

Critical strings are defined precisely for the case $a=1$, which means that $M^2=0$. For this class of stings, the momentum k must be null, since we are dealing with a massless state. Since the polarization vector is orthogonal to momentum, we have another constraint, $\epsilon\cdot k=0$. So we remove two polarization degrees of freedom. This gives

$$a = -(1/2)(D-2)\sum_{n=1}^{\infty} n, \tag{25}$$

which implies the well known result that D=26 dimensions.

We have now reviewed the central points in the physics related to the sum $\sum_{n=0}^{\infty}, n$; we want now to study this sum in the next section from the mathematical point of view.

3. The Riemann $\zeta(s)$ in the complex plane

In order to extend the Riemann ζ function, we are going to relate it to the $\Gamma(s)$ function and in turn take advantage of the analytic continuation of $\Gamma(s)$ function to extend the ζ function to the complex plane. To that end we are now going to prove two results.

Let us start with the definition

$$\begin{gathered} \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s\in\mathbb{C}, \\ s = \sigma + it \text{ with } \sigma > 1. \end{gathered} \tag{26}$$

This function is a Dirichlet series, *i.e.* of the form

$$\sum_{n=1}^{\infty}(a_n/n^s),$$

and is convergent for $\sigma > 1$, at least. In fact, it converges uniformly in the region defined by $\sigma \geqslant 1+\delta$, $\delta > 0$. This defines $\zeta(s)$ as an analytic and regular function if $\sigma > 1$.

On the other hand we know that the Euler-Gamma function is defined as

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} ds, \tag{27}$$

where $s = \sigma + it$ with $\sigma > 1$. We want to relate it to the Riemann $\zeta(s)$ function. If in Eq. (27) we replace $y = nx$ with $n \in N$, $n \geqslant 1$, we obtain

$$\Gamma(s) n^{-s} = \int_0^\infty x^{s-1} e^{-nx} dx. \tag{28}$$

Summing both sides of this last equation from $n = 1$ to ∞, we easily obtain that

$$\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1} \sum_{n=1}^\infty e^{-nx} dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx. \tag{29}$$

Now, we shall prove another result which, combined with this one, will give us a new expression that will prove to be suitable for the desired analytic continuation we are looking for.

Theorem 1 $For\, \sigma > 1$,

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz, \tag{30}$$

where $(-z)^{s-1}$ is defined in the real axis as $e^{(s-1)\log(-z)}$ with $-\pi < Im \log(-z) < \pi$.

Proof

Let us consider the integral

$$I(s) = \int_C \frac{z^{s-1}}{e^z - 1} dz, \tag{31}$$

where the contour *C* starts in ∞, goes up to the origin, surrounds it, and then returns along the line $z = \rho e^{2\pi i}$, going up to $\infty e^{2\pi i}$ and surrounds the cut line that goes from 0 to $+\infty$, as is shown in Fig. 1.

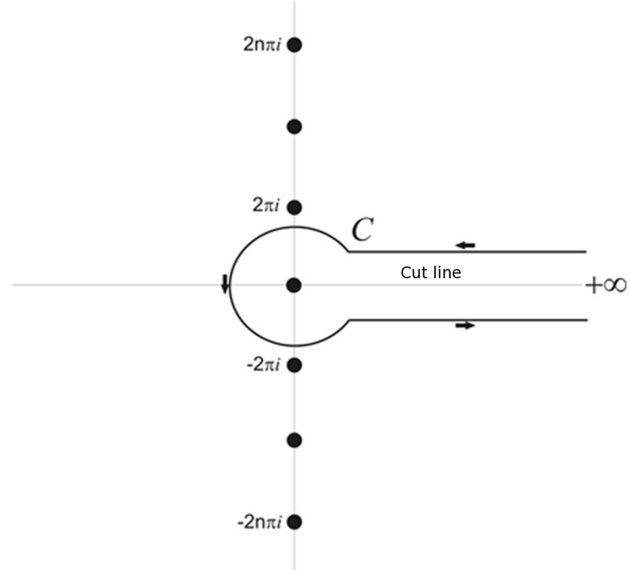

FIGURE 1. Integration contour *C*.

We now split the contour integral into three parts, one over the real axis going the opposite direction, from $+\infty$ to the origin, the second over a small circle C_0 of radius R surrounding the origin. The third part goes from the origin to infinity below the cut line. Taking the limit when $R \to 0$, we obtain

$$I(s) = \int_{C_0} \frac{z^{s-1}}{e^z - 1} dz = \int_\infty^0 \frac{x^{s-1}}{e^x - 1} dx + \int_0^\infty \frac{\left(xe^{2\pi i}\right)^{s-1}}{e^x - 1} dx$$
$$= -\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + e^{2\pi i(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Combining this last result with (29), we easily get

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz, \tag{32}$$

which is what we wanted to prove

Notice the importance of this last result. The integral in Eq. (31) is an entire function of s, whereas $\Gamma(1-s)$ is meromorphic with poles in $s = 1, 2\cdots$. But we know that $\zeta(s)$ is analytic for $\sigma > 1$, so the poles for $n = 2, 3\cdots$ should cancel out with the zeros of the integral. To proceed we also need the following result due to Ahlfors [8].

Theorem 2 *The function $\zeta(s)$ satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{33}$$

for any value of s except for $s = 1$.

Please notice the similarity of this result with the well known identity for the gamma function $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$.

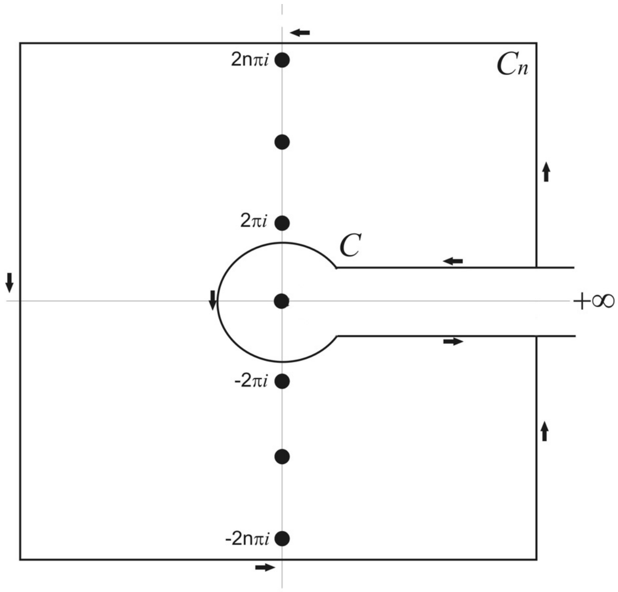

FIGURE 2. Contour $C_n - C$.

Proof

First, let us consider the integral

$$I(s) = \int_{C_n - C} \frac{z^{s-1}}{e^z - 1} dz, \tag{34}$$

where the contour C_n is a square formed by the horizontal line that passes a little bit over the point $2n\pi$ for the upper line of the square, a little bit below $-2n\pi$ for the lower side of the square. To complete the square, we just put two other vertical lines placed in such a way as to form a perfect square, as can be seen in Fig. 2.

The contour $C_n - C$ has winding number one about the points $\pm 2m\pi$, $m = 1, 2\cdots$. At these points, the function $(-z)^{s-1}/(e^z - 1)$ has simple poles with residues $(\mp 2m\pi i)^{s-1}$, so we must have

$$\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1} dz = 2i\pi \sum_{m=1}^{n} \left[(2mi\pi)^{s-1} + (-2mi\pi)^{s-1} \right] = 4\pi i \sum_{m=1}^{n} (2m\pi)^{s-1} \sin\frac{\pi s}{2}. \tag{35}$$

Let us now focus on the integral

$$\int_{C_n} \frac{z^{s-1}}{e^z - 1} dz. \tag{36}$$

On the contour C_n, this integral is clearly bounded. In general, if a function $|f(z)| \leq M$ is bounded we can put

$$\left| \int_{C_n} f(z) dz \right| \leq ML,$$

where L is the length (or perimeter) of C_n. In our case the length of C_n grows linearly with n, so if $s = \sigma + it$, then

$$\left| z^{s-1} \right| \leq |z|^{\sigma-1} e^{|t| \arg(z)} \leq |z|^{\sigma-1}.$$

But as the term $|1/(e^z - 1)|$ is bounded by 1 as $|z|$ grows, we have

$$\left| \frac{z^{s-1}}{e^z - 1} \right| \leq M n^{\sigma-1}$$

for some constant M independent of n. We finally conclude that

$$\left| \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz \right| \leq M n^{\sigma-1} n = M n^{\sigma}.$$

Now let us take the case $\sigma < 0$ and take n to infinity. Since $\sigma < 0$, the integral over C_n goes to zero when $n \to \infty$ because $n^\sigma \to 0$. So Eq. (34) gives

$$\int_{C} \frac{z^{s-1}}{e^z - 1} dz = 4\pi i e^{\pi s i} \sin\left(\frac{\pi s}{2}\right) \sum_{m=1}^{\infty} (2m\pi)^{s-1}. \tag{37}$$

Using theorem 1 we finally obtain

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \tag{38}$$

which is what we wanted to prove

The final calculation is simple. From theorem 2 we easily find

$$\zeta(1-s) = \frac{2^{-s}\pi^{1-s}\zeta(s)}{\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)} = 2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s),$$

where we use the well known relation

$$\Gamma(s)\Gamma(s-1) = \pi/\sin(\pi s).$$

And we are done. We need only to put s=$2m$, m=$1, 2, \cdots$, in the above relation to obtain that

$$\zeta(1-2m) = 2^{1-2m}\pi^{-2m}\cos(m\pi) \times \Gamma(2m)\zeta(2m) = -\frac{B_{2m}}{2m}, \tag{39}$$

where we use the well known relation for the Bernoulli numbers

$$B_n = -\frac{(-1)^{n/2} 2n!}{(2\pi)^n} \zeta(n). \tag{40}$$

We sketch a simple proof of this result in the appendix. To obtain the result we are looking for, we need only to put $m = 1$, knowing that $B_2 = 1/6$

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}. \tag{41}$$

4. Concluding remarks

Critical strings become a central subject matter in string theory for many reasons. One of them is that the *conformal anomaly,* as is called, is canceled out. As a consequence, critical strings retain their conformal properties even after the quantization procedure is performed (Ref. 12). The no-ghost theorem for $D \leqslant 26$, that is, the non-existence of negative norm states if the dimension of the space defined by the $\mu = 0, 1, \cdots$, is less than or equal to 26 dimensions is another important property of critical strings frequently quoted. Also, if Lorentz invariance is to hold, the commutator $\left[J^{i-}, J^{j-}\right]$ should vanish, where the J^{i-} are obtained from the Lorentz generators $J^{\mu\nu}$ as $J^{i-} = 1/\sqrt{2}(J^{i0} - J^{iD-1})$. This happens to be the case precisely if $a = 1$ and $D = 26$ (Ref. 3 to 7).

So we know critical strings are important, but why are they so strange? After all, we should look for no conformal anomaly, no-ghost theorem, Lorentz invariance *and* only four dimensions. This means that in Eq. (25), if we insist on living in four dimensions, then, certainly

$$\sum_{n=1}^{\infty} n = -1.$$

The point is, this result is not true. If we believe in the theory of meromorphic, entire functions and in the theory of analytic functions in the complex plane, including the residue theorem, we must conclude the strange result that strings live in a space of dimension greater than four. This fact has extremely deep and profound consequences in the theory as it unveils a new world of far reaching consequences in our understanding of nature. At least if we believe (or not!) in strings.

This issue is precisely one of the motivations for this paper, since analytic continuation of

$$\sum_{n=0}^{\infty} 1/n^s$$

for $s = -1$ implies the result -1/12. So a deep understanding of the mathematical arguments leading to the existence of extra hidden dimensions is extremely useful here. It is my experience that students feel much more comfortable when they understand well the mathematical background of a physical subject. When they have both approaches, the physical intuitive approach and the rigorous mathematical one, they feel more confident and usually return to the more intuitive physical formulation, understanding better its benefits. Fortunately, it turns out that the mathematical insight needed to understand the regularization process in this case is not very complicated nor involves a profound knowledge of advanced mathematical techniques. This point can be contrasted with the situation present in Quantum Field Theory, where the renormalization procedure is in fact much more complicated and upset the students that begin to study this field.

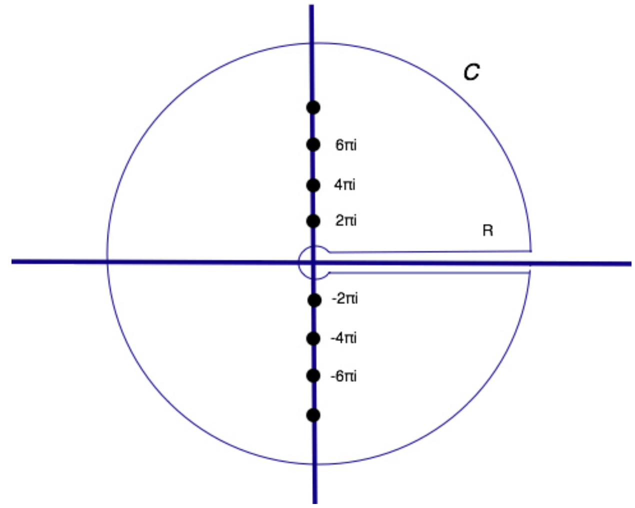

FIGURE 3. Contour $C_n - C$.

But, as it turns out, the present treatment of this point in standard string theory courses is unsatisfactory (Ref. 13). In many cases, results dealing with the analytic continuation of the Riemann $\zeta(s)$ function are simply quoted. In other, fewer cases, the reader is invited to solve some non-rigorous exercise in problems given in the final chapter. But we wanted to study a more rigorous approach. What we discover is that the mathematical basis of this problem is within reach of any advanced undergraduate physics student and the background mathematical basis deserves to be more known than the present status we have in this area.

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Appendix

A. Bernoulli numbers

Bernoulli numbers B_n are defined by the following expansion in the complex plane $\mathbb{C}$:

$$\frac{z}{e^z - 1} = \sum_{n=1}^{\infty} \frac{B_n}{n!} z^n. \tag{A.1}$$

This is a Taylor series that we can easily invert using the residue theorem

$$B_n = \frac{n!}{2\pi i} \oint_{c_0} \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \tag{A.2}$$

where the contour C_0 surrounds the origin in the positive sense, but we take care to put $|z| < 2\pi$ to avoid the poles at $\pm 2\pi i$.

For $n = 0$ we have a simple pole with residue 1. So we conclude that $B_0 = 1$. For $n = 1$ the pole becomes a second order pole. Following standard methods, for instance series expansion of the exponential followed by a binomial expansion, we find that $B_1 = -1/2$. We use this result in the main text of this paper.

But for $n \geqslant 2$ this method becomes cumbersome and we resort to a different and much more interesting approach. Let us deform the contour C_0 to obtain the new contour C as shown in Fig. 3. The new contour still encircles the origin but it also encircles in the negative direction an infinite series of singular points along the imaginary axis, at $z = \pm 2m\pi i$.

We are going now to consider the limit when $R \to \infty$. The lines that go above and below the real axis cancel out. At $z = 2\pi i m$ we have a simple pole only with residue $(2\pi i m)^{-n}$. We then have for $n \geqslant 2$ that

$$\oint_C \frac{z}{e^z - 1} \frac{dz}{z^{n+1}} = -2\pi i \sum_{m=1}^{\infty} \frac{1}{(2\pi i m)^n}. \tag{A.3}$$

Note that when n is odd, the residue from $z = 2\pi i m$ cancels out that from $z = -2\pi i m$ and $B_{2n+1} = 0$, that is $B_3, B_5, \cdots = 0$. For n even the residues add, instead, giving Eq. (40):

$$B_n = \frac{n!}{-2\pi i} 2 \sum_{m=1}^{\infty} \frac{1}{(2\pi i m)^n} = -\frac{(-1)^{n/2} 2n!}{(2\pi i)^n} \times \sum_{m=1}^{\infty} \frac{1}{(m)^n} = -\frac{(-1)^{n/2} 2n!}{(2\pi i)^n} \zeta(n). \tag{A.4}$$

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