Some classical properties of the non-abelian Yang-Mills theories

J.A. Sánchez-Monroy[a] and C.J. Quimbay[a,b]
[a]Grupo de Campos y Partículas, Universidad Nacional de Colombia,
Sede Bogotá.
[b]Associate researcher of CIF,
Bogotá, Colombia.

We present some classical properties for non-abelian Yang-Mills theories that we extract directly from the Maxwell's equations of the theory. We write the equations of motion for the $SU(3)$ Yang-Mills theory using the language of Maxwell's equations in both differential and integral forms. We show that vectorial gauge fields in this theory are non-fermionic sources for non-abelian electric and magnetic fields. These vectorial gauge fields are also responsible for the existence of magnetic monopoles. We build the continuity equation and the energy-momentum tensor for the non-abelian case.

Keywords: Yang-Mills theory; Maxwell's equations; integral and differential forms; magnetic monopoles.

En este artículo se presentan algunas propiedades clásicas de las teorías de Yang-Mills no abelianas, que se extraen directamente de las ecuaciones de Maxwell de la teoría. Obtenemos las ecuaciones de movimiento para una teoría de Yang-Mills del grupo $SU(3)$ en su forma diferencial e integral, utilizando el lenguaje de las ecuaciones de Maxwell. Mostramos que los campos gauge en esta teoría son fuentes no fermiónicas para campos eléctricos y magnéticos no abelianos. Estos campos de gauge son responsables de la existencia de monopolos magnéticos. Finalmente, se construyen la ecuación de continuidad y el tensor energía-impulso para el caso no abeliano.

Descriptores: Teorías de Yang-Mills; ecuaciones de Maxwell; forma diferencia e integral; monopolos magnéticos.

PACS: 03.50.-z; 03.50.Kk

1. Introduction

In the context of the relativistic quantum theory of electromagnetism, the interaction among two electrically charged particles is mediated through the exchange of virtual photons. Photons are the quantum excitations of the electromagnetic field, which is a vectorial gauge field invariant under $U(1)$ abelian transformations. Similarly, the strong and weak interactions are described by means of non-abelian vectorial gauge fields. Field theories describing the behavior of pure vectorial gauge fields are known as Yang-Mills theories. Symmetries and properties of Yang-Mills theories are basic ingredients for the theoretical treatment of the fundamental interactions between elementary particles. The study of the classical properties of Relativistic Electrodynamics, a particular case of abelian Yang-Mills theory, is a topic very well known in the literature [1, 2]. On the other hand, the classical properties of non-abelian Yang-Mills theories is a subject less studied in the field theory literature. But it is possible to find some books which perform a treatment about this subject [3-6]. These books give a similar emphasis to the presentation of Yang-Mills equations as functions of gauge potentials, however the introduction of these equations in terms of non-abelian electric and magnetic fields is practically absent [6–9].

The main goal of this paper is to present some classical properties for non-abelian Yang-Mills theories. We can extract these properties writing the equation of motion for non-abelian Yang-Mills theories using the language of electric and magnetic fields. We write these non-abelian Maxwell's equations in both differential and integral forms as is usual for Maxwell's equations of Classical Electrodynamics. We restrict our interest to the case of the $SU(3)$ Yang-Mills theory, however the analysis is the same for any group $SU(N)$. We show that non-abelian electric and magnetic fields can be generated by vectorial gauge fields. These vectorial gauge fields are also responsible for the existence of magnetic monopoles. Finally, we build the continuity equation and the energy-momentum tensor for the non-abelian case.

2. Non-abelian Maxwell's equations

Non-abelian gauge theories have some differences respect to the abelian ones, as for instance the existence of a multiplicity of gauge fields, self-interactions, and gauge transformations that involve the gauge fields [11]. Particularly these differences are clearly observed if we contrast the $SU(3)$ Yang-Mills theory, including color charge sources, with the Relativistic Classical Electrodynamics including electric charge sources. The $SU(3)$ Yang-Mills theory is described by the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\rho}F^{\mu\rho}_a + gJ^\mu_a A^a_\mu, \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^a_{bc}A^b_\mu A^c_\nu, \tag{2}$$

is the non-abelian field strength tensor, A^a_μ the gluon fields, J^μ_a the colour charge sources, C^a_{bc} the structure constants of the Lie algebra associated to the $SU(3)$ gauge group, g the running coupling constant and $a, b, c = 1, 2, ..., 8$. The first

term of Eq. (1) describes the kinetic energy of the eight gluon fields and their respective auto-interactions. The second term describes the interactions of the gluon fields with the fermionic fields. Applying the variational method of Classical Field Theory, we obtain that the equations of motion of the $SU(3)$ Yang-Mills theory with color charge sources are

$$\partial_\mu F_b^{\mu\nu} + g C_{bc}^a A_\mu^c F_a^{\mu\nu} = g J_b^\nu = g\bar{\psi}\gamma^\nu \lambda_b \psi, \qquad (3)$$

where λ_b are the Gell-Mann matrices and ψ the fermionic fields. The equations of motion given by Eq. (3) represent a system of non-linear equations.

The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields, in Classical Electrodynamics, are defined as the components of the electromagnetic field strength tensor ($F_{\mu\nu}$), in the following form:

$$E_i := F_{0i} = -F^{0i} = -F_{i0}, \qquad (4)$$

$$B_n := -\frac{1}{2}\varepsilon_{nij} F^{ij}, \qquad (5)$$

being $n, i, j = 1, 2, 3$. Starting from the Lagrangian density of the electromagnetic field with sources it is possible to obtain the Yang-Mills equations using the Euler-Lagrange equations. From these equations it is possible to obtain the homogeneous Maxwell's equations.

In an analogous way, we consider the non-abelian Maxwell's equations for the $SU(3)$ Yang-Mills theory with color charge sources. The non-abelian field strength tensor is given by Eq. (2), where the covariant non-abelian gauge field is written as $A_\mu^a = (A^{0a}, -\vec{A}^a)$. The electric and magnetic color fields are defined respectively as

$$E_i^a := F_{i0}^a = \partial_i A_0^a - \partial_0 A_i^a + g C_{bc}^a A_i^b A_0^c, \qquad (6)$$

$$B^{aj} := -\frac{1}{2}\varepsilon^{jik} F_{ik}^a$$

$$= -\frac{1}{2}\varepsilon^{jik}\left(\partial_i A_k^a - \partial_k A_i^a + g C_{bc}^a A_i^b A_k^c\right), \qquad (7)$$

being $n, i, j = 1, 2, 3$. In vectorial notation, the electric and magnetic color fields are [7]

$$\vec{E}^a = -\vec{\nabla} A_0^a - \partial_t \vec{A}^a + g C_{bc}^a A_0^b \vec{A}^c, \qquad (8)$$

$$\vec{B}^a = \vec{\nabla} \times \vec{A}^a - \frac{1}{2} g C_{bc}^a \left(\vec{A}^b \times \vec{A}^c\right). \qquad (9)$$

In contrast with the magnetic field of Electrodynamics, the magnetic color field for the $SU(3)$ Yang-Mills theory can be written as the sum of a rotor term and a non-rotor term.

Using the definition of Eq. (6) in Eq. (3), we obtain that the first Maxwell's equation for the SU(3) Yang-Mills theory with color charge sources is given by [6–9]

$$\partial_i E_i^a + g C_{bc}^a A_{bi} E_i^c = g\rho^a, \qquad (10)$$

or in vectorial notation

$$\vec{\nabla} \cdot \vec{E}^a = g C_{bc}^a \vec{A}^b \cdot \vec{E}^c + g\rho^a, \qquad (11)$$

where we have used the fact that the color charge source can be written as $J_\mu^a = (\rho^a, -\vec{J}^a)$ and $A_\mu^a = (A^{0a}, -\vec{A}^a)$. Accordingly the magnetic color field can be written as

$$B^{aj} = -\frac{1}{2}\varepsilon^{jik} F_{ik}^a,$$

$$\varepsilon_{jpq} B^{aj} = -\frac{1}{2}\varepsilon_{jpq}\varepsilon^{jik} F_{ik}^a,$$

$$\varepsilon_{jpq} B^{aj} = -\frac{1}{2}(\delta_p^i \delta_q^k - \delta_p^k \delta_q^i) F_{ik}^a,$$

$$-\varepsilon_{jpq} B^{aj} = F_{pq}^a = F^{apq}, \qquad (12)$$

then, the second Maxwell's equation is given by

$$\partial^\mu F_{\mu j}^a = -g C_{bc}^a A^{b\mu} F_{\mu j}^c + g J_j^a,$$

$$\partial^0 E_j^a + \partial^i F_{ij}^a = -g C_{bc}^a A^{b0} E_j^c - g C_{bc}^a A^{bi} F_{ij}^c + g J_j^a,$$

$$\partial^0 E_j^a - \partial^i \varepsilon_{lij} B^{al} = -g C_{bc}^a A^{b0} E_j^c$$
$$+ g C_{bc}^a A^{bi} \varepsilon_{lij} B^{cl} + g J_j^a, \qquad (13)$$

$$\partial^0 E_j^a + \varepsilon_{jil} \partial^i B^{al} = -g C_{bc}^a A^{b0} E_j^c$$
$$+ g C_{bc}^a A^{bi} \varepsilon_{jil} B^{cl} + g J_j^a, \qquad (14)$$

which in vectorial notation can be written as [6]

$$\vec{\nabla} \times \vec{B}^a - \partial_t \vec{E}^a = g\vec{J}^a + g C_{bc}^a A_0^b \vec{E}^c - g C_{bc}^a \vec{A}^b \times \vec{B}^c. \quad (15)$$

The other two Maxwell's equations are obtained from the definitions of the electric color fields, given by Eq. (6); and the magnetic color field, given by Eq. (7). These Maxwell's equations are [6]:

$$\vec{\nabla} \cdot \vec{B}^a = -\frac{1}{2} g C_{bc}^a \nabla \cdot (\vec{A}^b \times \vec{A}^c) \qquad (16)$$

and

$$\vec{\nabla} \times \vec{E}^a + \partial_t \vec{B}^a = -\frac{1}{2} g C_{bc}^a \partial_t \left(\vec{A}^b \times \vec{A}^c\right)$$
$$+ g C_{bc}^a \left[\vec{\nabla} \times (A_0^b \vec{A}^c)\right]. \qquad (17)$$

We observe that Maxwell's equations for the $SU(3)$ Yang-Mills theory with color charge sources, which are given by Eqs. (11), (15), (16) and (17), do not only depend on $\vec{E}^a$ and $\vec{B}^a$ but also on $\vec{A}^a$ and A_0^a. It is clear that for the abelian Yang-Mills theory case, these equations do not have the dependence observed before. If we put $J_\mu^a = (\rho^a, -\vec{J}^a) = 0$ in Eqs. (11) and (15), we observe the presence of sources of electric and magnetic color fields whose origin are the gluon fields. It is possible to see that the bosonic fields are charged and simultaneously are source of magnetic field, i e. the gluonic fields have color charge. Additionally, as the divergence of $\vec{B}^a$ non vanishing then there exist color magnetic monopoles and the sources are the gluons but not the quarks. It is also possible to see that the electric and magnetic color

fields are not gauge invariant and therefore they do not have physical meaning.

So as classical electrodynamics predicts the existence of electromagnetic waves, the $SU(3)$ Yang-Mills theory predicts the existence of non-abelian waves associated to the strong interaction. These waves are the solutions of the following wave equations that can be obtained from Maxwell's equations (see appendix):

$$\Delta A_\nu^a = gC_{bc}^a A_\mu^b (\partial_\nu A_\mu^c - 2\partial_\mu A_\nu^c - gC_{mn}^c A_\mu^m A_\nu^n). \quad (18)$$

The obtained Maxwell's equations can be extended directly for any non-abelian Yang-Mills theory, taking in account that for a $SU(N)$ gauge group there are $N^2 - 1$ generators, being N the dimension group. For this reason, Maxwell's equations for a non-abelian Yang-Mills theory are given by Eqs. (11), (15), (16) and (17), taking $ab, c=1, 2, ..., (N^2-1)$. For the case in which $\vec{E}^a$ and $\vec{A}^a$ are independent fields and if there exist particular boundary conditions in the problem, the solutions of the Maxwell's equations for a non-abelian Yang-Mills theory are unique [10].

In a similar way as in the abelian Yang-Mills theory case, it is required that the non-abelian gauge fields transform as

$$A_\mu'(x) = U(x)(A_\mu - ig^{-1}\partial_\mu)U^{-1}(x), \quad (19)$$

where $U(x) = e^{-ig\lambda_a \chi^a(x)}$. The transformation of the non-abelian field strength tensor has the form

$$F_{\mu\nu}'(x) = U(x)F_{\mu\nu}U^{-1}(x). \quad (20)$$

If we now consider an infinitesimal gauge transformation given by

$$U(x) \approx I - ig\lambda_a \chi^a(x), \quad (21)$$

it is easy to prove that the non-abelian field strength tensor transforms as

$$F_{\mu\nu}'^a(x) = F_{\mu\nu}^a + gC_{bc}^a \chi^b F_{\mu\nu}^c. \quad (22)$$

This tensor is only invariant for the abelian case. Starting from Eq. (22), it is possible to find that the electric and magnetic color field can be written as

$$E_i' := F_{i0}' = E_i^a + gC_{bc}^a \chi^b E_i^c, \quad (23)$$

$$B_n' := -\frac{1}{2}\varepsilon_{nij}F'^{ij} = B_n^a + gC_{bc}^a \chi^b B_n^c, \quad (24)$$

obviously the electric and magnetic color elds are not gauge invariant. For this reason, these field do not have any physical meaning. In similar way as the scalar and vectorial potentials are auxiliary constructions in electrodynamics, the electric and magnetic color fields do not represent measurable quantities in the non-abelian Yang-Mills theory. For these theories, it is possible to identify two scalar invariant quantities, which can be written down using the physical fields in $(3 + 1)$ dimensions. These quantities are

$$F_{\mu\nu}^a F^{a\mu\nu} = 2(B^2 - E^2), \quad (25)$$

$$\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^a F_{\alpha\beta}^a = \vec{B} \cdot \vec{E}. \quad (26)$$

Now we present bellow the integral form of non-abelian Maxwell's equations. We first integrate Eqs. (11) and (16) over the volume of a three-dimensional domain V enclosed by a surface ∂V, and we apply the Gauss-Ostrogradski theorem. We obtain that the integral form for the equations (11) and (16), that are [7, 8]

$$\oint_{\partial V} \vec{E}^a \cdot d\vec{S} = gC_{bc}^a \int_V \vec{A}^b \cdot \vec{E}^c dV + \int_V g\rho^a dV, \quad (27)$$

and

$$\oint_{\partial V} \vec{B}^a \cdot d\vec{S} = -\frac{1}{2}gC_{bc}^a \oint_{\partial V} (\vec{A}^b \times \vec{A}^c) \cdot d\vec{S}. \quad (28)$$

We consider a two-dimensional surface S which is bounded by a loop L. We calculate the flux of the vector Eqs. (15) and (17) through S and apply Stokes theorem. We obtain the integral form for Eqs. (15) and (17), that are

$$\oint_L \vec{B}^a \cdot d\vec{l} - \frac{d}{dt}\int_S \vec{E}^a \cdot d\vec{S} = g\int_S \vec{J}^a \cdot d\vec{S}$$
$$+ gC_{bc}^a \int_S A_0^b \vec{E}^c \cdot d\vec{S} - gC_{bc}^a \int_S \vec{A}^b \times \vec{B}^c \cdot d\vec{S}, \quad (29)$$

and

$$\oint_L \vec{E}^a \cdot d\vec{l} + \frac{d}{dt}\int_S \vec{B}^a \cdot d\vec{S} = -\frac{1}{2}gC_{bc}^a \frac{d}{dt}$$
$$\times \int_S \left(\vec{A}^b \times \vec{A}^c\right) \cdot d\vec{S} + gC_{bc}^a \left[\oint_L (A_0^b \vec{A}^c) \cdot d\vec{l}\right]. \quad (30)$$

Equations (27), (28), (29) and (30) represent the integral form of non-abelian Maxwell's equations.

One of the most remarkable results in theoretical physics is provided by Noether's theorem, which establishes a relationship among symmetries of a given action and conserved quantities of the system described by the action [12]. This theorem is very important for classifying the general physical characteristics of the quantum field theories. cambiar por This theorem is very important for classifying the general physical characteristics of quantum field theories [11]. In electromagnetism there are only positive and negative electrical charges that label the different kinds of matter that respond to the electromagnetic field. More kinds of charge are required to label particles which respond to the nuclear forces. With more kinds of charge, there are many more possibilities for conservation than the one obtained from the sum of all positive and negative electric charges [3]. This fact can be examined by the continuity equation which is obtained from the non-abelian Maxwell's equations. Taking the divergence of the Eq. (15) and the time derivation of Eq. (11), then by summing we can obtain the following expression

$$\partial_t \rho^a + \vec{\nabla} \cdot \vec{J}^a = -C_{bc}^a[\partial_t(\vec{A}^b \cdot \vec{E}^c)$$
$$- \vec{\nabla} \cdot (A_0^b \vec{E}^c) + \vec{\nabla} \cdot (\vec{A}^b \times \vec{B}^c)]. \quad (31)$$

Finally we can write the energy-momentum tensor for the Yang-Mills fields as [3, 5, 13, 14]

$$\Theta_{\mu\nu} = \frac{1}{4} \left(F_{a\mu}^{\ \alpha} F_{a\alpha\nu} + \frac{\eta_{\mu\nu}}{4} F_{\alpha\beta}^a F_a^{\alpha\beta} \right). \qquad (32)$$

Let us now give the physical meaning of the various components of $\Theta_{\mu\nu}$. The component Θ_{00} given by [7–9]

$$\Theta_{00} = \frac{1}{8} \left(\vec{E}^a \cdot \vec{E}_a + \vec{B}^a \cdot \vec{B}_a \right), \qquad (33)$$

is interpreted as the energy density. The component Θ_{i0} given by

$$\Theta_{i0} = \frac{1}{4} \left(\vec{E}^a \times \vec{B}_a \right), \qquad (34)$$

is the momentum density for the non-abelian field. Finally, the component Θ_{ii} given by

$$\Theta_{ij} = \frac{1}{4} \left(E_i^a E_{aj} + B_i^a B_{aj} \right.$$
$$\left. + \frac{\delta_{ij}}{2} \left(\vec{E}^a \cdot \vec{E}_a + \vec{B}^a \cdot \vec{B}_a \right) \right), \qquad (35)$$

is the jth component of the momentum flow in the non-abelian field through a unit surface perpendicular to the x_i-axis. We observe that the energy-momentum tensor in the non-abelian case has the same interpretation as in electrodynamics [1, 3, 5].

3. Conclusions

In this paper we have obtained the equations of motion for the $SU(3)$ Yang-Mills theory including color charge sources in an analogue way as the Maxwell's equations are obtained for electrodynamics including electric charge sources. These non-abelian Maxwell's equations have been obtained for the $SU(3)$ Yang-Mills theory, but they are directly extendable for a $SU(N)$ Yang-Mills theory. We have found that Maxwell's equations do not only depend on $\vec{E}^a$ and $\vec{B}^a$, but also on $\vec{A}^a$ and A_0^a. From the divergences of $\vec{E}^a$ and $\vec{B}^a$, it is possible to conclude that there exist sources of electric and magnetic color fields which are not fermions. It is possible to see that the bosonic field is charged and is alson simultaneously is source of magnetic field, *i.e.* the gluonic fields have color charge. Additionally, as the divergence of $\vec{B}^a$ non vanishing then there exist color magnetic monopoles and the sources are the gluons but not the quarks. As happens with the gauge potential in Electrodynamics, we have also found that the electric and magnetic color field are not gauge invariant. For this reason, these field do not have any physical meaning.

In addition, we have presented the integral formulation of non-abelian Maxwell's equations by using the Gauss-Ostrogradski and Stokes theorems. For these, we have first built the continuity equation and then we have introduced the energy-momentum tensor as a function of electric and magnetic color fields. The energy-momentum tensor has the same interpretation as in electrodynamics.

Acknowledgments

Appendix:

Wave equations

Using the $SU(3)$-Yang-Mills equations Eq. (3) and the definition of the non-Abelian gauge field tensor Eq. (2), it is possible to obtain:

$$\partial^\mu F_{\mu\nu}^a = -g C_{bc}^a A^{b\mu} F_{\mu\nu}^c,$$
$$\partial^\mu (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g C_{bc}^a A_\mu^b A_\nu^c) = -g C_{bc}^a A^{b\mu}$$
$$\times (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g C_{mn}^c A_\mu^m A_\nu^n),$$
$$\Delta A_\nu^a - \partial_\nu \partial^\mu A_\mu^a + g C_{bc}^a \partial^\mu (A_\mu^b) A_\nu^c = -g C_{bc}^a A^{b\mu}$$
$$\times (2\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g C_{mn}^c A_\mu^m A_\nu^n). \qquad (36)$$

Using the gauge of Lorentz (fixing the gauge), we obtain a wave equation given by

$$\Delta A_\nu^a = g C_{bc}^a A_\mu^b (\partial_\nu A_\mu^c - 2\partial_\mu A_\nu^c - g C_{mn}^c A_\mu^m A_\nu^n). \qquad (37)$$

1. J.D. Jackson. *Classical Electrodynamics* 2nd ed (Addison-Wesley, Reading, Ma, 1980).

2. W. Panofsky and M. Philips, *Classical Electricity and Magnetism* (Cambridge, MA: Addison-Wesley, 1955).

3. M. Burgess, *Classical covariant fields* (Cambridge University Press, 2003).

4. V. Rubakov, *Classical Theory of Gauge Fields* (Princeton University Press, 2002).

5. B. Kosyakov, *Introduction to the Classical Theory of Particles and Fields* (Springer, 2007).

6. T. Frankel, *The Geometry of Physics* (Cambridge University Press, 2001).

7. R. Jackiw, *Phys. Rev. D* **21** (1980) 426.

8. R. Jackiw, *Rev. Mod. Phys.* **52** (1980) 661.

9. L. D. Faddeev, *Theor. Math. Phys.* **1** (1969) 1; *Teor. Mat. Fiz.* **1** (1969) 3.

10. J. Sniatycki and G. Schwarz, *Commun Math Phys.* **159** (1994) 593.

11. D.L. Karatas and k.L. Kowalski, *Am. J. Phys.* **58** (1990) 123.

12. E. Noether, *Invariante variations probleme*, (Nachr. Ges. Wiss. Göttingen, 1918). P. 235.

13. M. Montesinos and E. Flores, *Rev. Mex. Fís, A* **52** (2006) 29.

14. L. Faddev and A. Slavnov, *Gauge fields: introduction to quantum theory* (Benjamin-Cummings Publishing Co., 1981).