Brownian motion, diffusion, entropy and econophysics

J. D. A. Islas-García[a], A. R. Villagómez-Manrique[a], M. del Castillo-Mussot[a,b], and P. G. Soriano-Hernández[a]
[a]*Instituto de Física, Universidad Nacional Autónoma de México,*
04510 Mexico City, México.
j_da_ig@ciencias.unam.mx; alan.villa@ciencias.unam.mx; marcelodlcstll@yahoo.com; pigamma45@gmail.com
[b]*Unidad Profesional Interdisciplinaria en Ingeniería y Tecnologías Avanzadas,*
Instituto Politécnico Nacional, 07340 Mexico City, Mexico.

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To model wealth distributions there exist models based on the Boltzmann-Gibbs distribution (BGD), which is obtained by simulating binary economic interactions or exchanges that are similar to particle collisions in physics with conserved energy (or money in econophysics). Also, BGD can be reproduced by numerical simulations of diffusion for many particles which experience energy fluctuations. This latter case is analogous to non-interacting pollen particles performing Brownian motion. In order to decrease inequality, we also modify the energy-conserved diffusion by taxing the richest agent. In all cases, we calculate the corresponding Gini inequality index and the time evolution of the entropy to show the stability of the statistical distributions.

Keywords: Diffusion; entropy; econophysics; Gini index.

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1. Introduction

Econophysics is a new discipline that uses the ideas, concepts and methods of physics to contribute to the understanding of economic and financial phenomena [1]. Although the methodology used in this new field of research has generated controversy [2,3], its results have consolidated it as separate and autonomous discipline [4]. For example, the inequality in social systems exhibits a few statistical regularities, as in the case of income and wealth distributions over a wide range of societies and time periods [5,6]. Likewise, Pareto [7] found that in some European countries the wealth distribution follows a power-law tail for the richest sections of society. In general, the upper end (the richest) of the income and wealth distributions is believed to be described by a power-law, as Pareto [7] argued over 100 years ago. More recently, Silva and Yakovenko [8] found that the data analysis of income distribution in the USA reveals coexistence of two social classes; the large lower class is characterized by the exponential Boltzmann Gibbs distribution (BGD), and the very small upper class exhibits the power-law Pareto distribution with characteristic fat tails. A similar result was found in Mexico [9] as a case of less stable emergent economies, especially in times of economic crisis.

At present, the simple model of random exchanges of wealth between agents, which generates an exponential distribution [10], and the model in which agents have a saving propensity, resulting in gamma-like distributions [11], are some of the models that reliably reproduce, in a simple way, the main empirical features of the wealth distributions of real economic systems [12].

Therefore, this paper analyzes different models that do not conserve the total wealth of the system. In this way, we seek to offer another perspective of the phenomenon of wealth distribution which involves small fluctuations of the total energy of the system. Now let us mention some important concepts from physics and economics.

1.1. Boltzmann probability distribution and entropy

In physics, if the initial number of particles and the total system energy are both conserved, then the Boltzmann-Gibbs distribution (BGD) is the most probable distribution corresponding to the maximum entropy value [13]. In general, for any normalized probability distribution, the system entropy is given by ([13], 151):

$$S = -\sum_{i}^{n} p_i Ln(p_i) \tag{1}$$

where p_i is the probability at the *i-th* level of energy, and for a normalized discrete distribution *f(p)*, then the entropy equation reads:

$$S = -\sum_{i}^{n} f(p_i) Ln(f(p_i)). \tag{2}$$

1.2. Brownian motion

In 1827, the Scottish biologist R. Brown was the first to observe the random motion of pollen particles suspended in water. Reason why, still without a theory that could explain that phenomenon, the random movement of particles suspended in a fluid was baptized as ”Brownian motion”. Subsequently,in another seemingly unrelated field of knowledge, the French mathematician Louis Bachelier developed in his PhD thesis a mathematical model for the movement of stock prices in the financial markets [14]; the new theory laid the foundations that would allow to understand the unusual movement of suspended particles, but was until 1905 that A. Einstein published an article explaining how the movement that Brown observed was result of individual collisions of pollen particles and water molecules. Then, we perform simulations similar to Brownian motion, where the system particles or agents modify their amount of energy by small random variations.

1.3. The Gini inequality index

In the study of income and wealth distributions, the Gini index G is a measure of inequality given by:

$$G = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N}|x_i - x_j|}{2NM} \tag{3}$$

Here x_i and x_j are the wealth values of agents i and j, respectively; M is the total system wealth and N is the number of economic agents in the system. G, by definition, is a number between 0 and 1; where 0 corresponds to the case of perfect equality, where all agents have the same wealth and the second corresponds to the case in which one agent has all the system wealth [10]. It is known that for the case of exponential distribution, the Gini index has a value of 0.5 [15].

2. Model 1: economic collisions

There is an analogy between statistical physics and economics in which energy is analogous to money and collisions between particles are similar to economic interactions between agents [10]. Then random binary interactions are proposed, in which the total energy or wealth remains constant in each interaction. Thus, for two randomly chosen agents *i* and *j*, the economic exchange obeys the following expressions:

$$m'_j = k(m_j + m_i);$$
$$m'_i = (1-k)(m_j + m_i) \tag{4}$$

where k is a uniformly distributed random number in $[0, 1]$, m_i and m_j are the initial energy (or wealth), respectively,

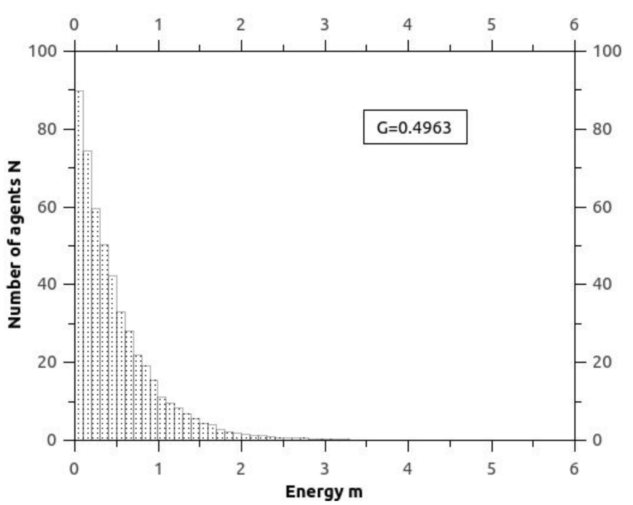

FIGURE 1. Approximation of the exponential distribution with Gini index close to 0.5, as a result of simulations of the economic collisions model.

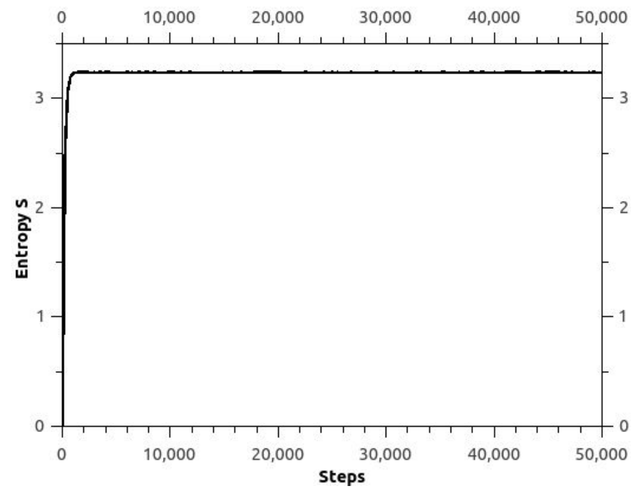

FIGURE 2. Entropy of the wealth distribution obtained with economic collisions model, as a function of the number of steps.

while m'_i and m'_j are the final energy after the interaction. With these equations, since the energy and number of agents are conserved in each step, then the BGD is obtained from any initial conditions. In Fig. 1 we show the final wealth distribution of 500 agents interacting 5.0×10^4 times, with initial energy of 0.5 unit per agent, averaged over 100 realizations.

This entropy function shows that the distribution of wealth obtained by the economic collisions model reaches a stable state with a maximum value of 3.2. For much larger systems ($N \gg 500$) we obtained values of the Gini index closer to 0.5, which is characteristic of the exponential distribution, but with $N = 500$ is sufficient for our purposes. Shaikh [16] and Ragab [17] argue that labor income in USA follows approximately an exponential BGD, while property income follows Pareto distribution.

3. Model 2: diffusion model

In the case of the economic collisions model the energy is exactly conserved in each binary interaction as shown in Eq. 4. Now, if we focus our attention on the system formed by the pollen particles, then each pollen particle can receive or concede energy from interacting with much smaller fluid molecules. In other words, each pollen particles exhibits a random motion but there is no direct interaction among pollen particles. That is, each pollen particle receives or gives a small random value of energy δ, called energy fluctuation. By making use of a symmetric distribution, with respect to zero, m'_i is given by:

$$m'_i = m_i + \delta;$$
$$\delta \in [-\epsilon, \epsilon]. \tag{5}$$

Since we want to reproduce the exponential distribution (see Fig. 1) with this diffusion model, we have to restrict the values of m'_i to be positive. Therefore, we have to propose a

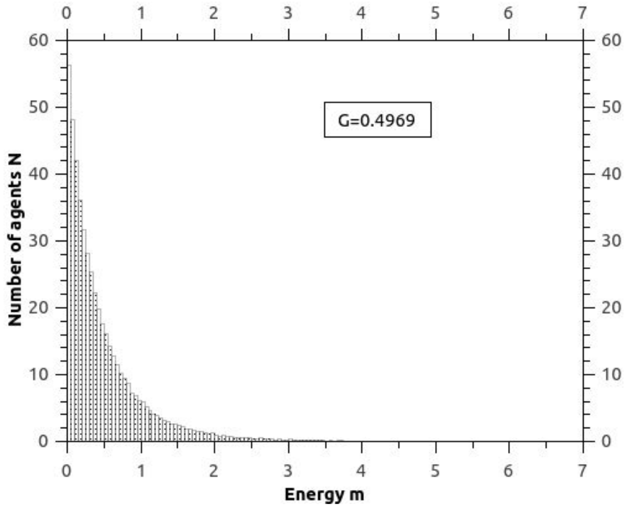

FIGURE 3. Average distribution of energy resulting from the simulations of the diffusion model whith 1.5×10^5 steps, $\delta \in [-0.1, 0.1]$.

mechanism when $m_i' < 0$ because δ is negative and has an absolute value greater than m_i. If in a given step this happens, then we stop this command and another agent m_j is randomly chosen to decrease its value by subtracting the same amount $|\delta|$. But if it again happens that in one determined step this new agent cannot absorb this energy "loss", then another agent has to be selected, and so on, until an agent can sustain the loss. Therefore, this mechanism generates an indirect interaction between the agent whose wealth would be negative and the agent who finally receives the loss, because it is similar to an exchange in which an amount $|\delta|$ of money pass from the wealthier agent to the poorer and the total system wealth is conserved. Finally, notice that in order to not affect the overall conservation of energy, this cyclic procedure does not change the value of δ in the process.

Each simulation or realization is different because agents and fluctuations are chosen randomly. We start with a uniform distribution with 0.5 units of energy per agent, so after 100 independent realizations, with 500 agents and 1.5×10^5 values of random fluctuations δ in each simulation, the average result is shown in Fig. 3. Since the number of agents is very small, then the histogram curve is not so smooth, using for each histogram 100 bins configuration.

This distribution has a Gini index close to 0.5 (0.4969), so it is similar to an exponential distribution. Figure 4 shows the corresponding semi-log graph of the distribution with an adjustment value $R = 0.9709$, confirming that the distribution is nearly exponential.

3.1. Approximated conservation of energy and entropy evolution

Unlike the case of binary collisions presented in model 1, the total energy of the pollen particles changes in the diffusion process presented in this section. However, the total energy in this model is almost conserved because the distribution of the fluctuations δ is symmetric. In other words, energy is finally conserved within a small margin of error because after n steps the total energy E_t is modified according to the following expression:

$$E_t(n) = N * e + \sum_{i}^{n} \delta_i;$$
$$\delta_i \in [-\epsilon, \epsilon] \quad \text{and} \quad i \in [1, \ldots, n] \tag{6}$$

where N is the number of agents, e is the initial energy per agent, n is the number of steps and δ_i is the modification corresponding to the *i-th* step. The normalized total fluctuation is:

$$F_{\text{total}} = \frac{\sum_i^n \delta_i}{\sum_i^n |\delta_i|}; \tag{7}$$

In the limit when n tends to infinity, F_{total} converges to zero, which assures that for a considerably large number of steps, the final energy is very close to the initial energy. We obtained a value of $F_{total} = 1.26 \times 10^{-3}$ averaged over 100

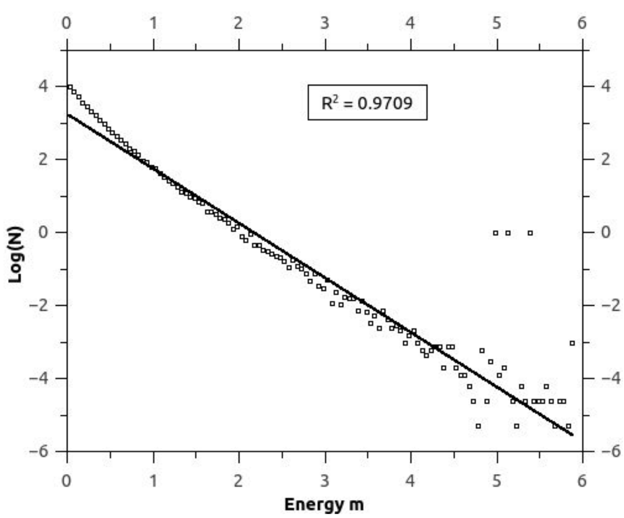

FIGURE 4.Semi-log graph of the average energy distribution corresponding to the diffusion model.

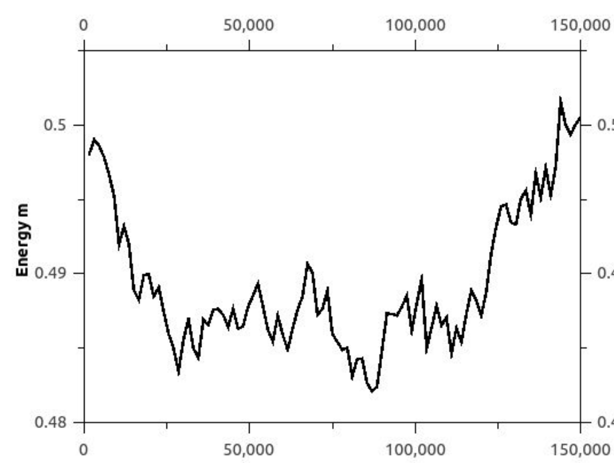

FIGURE 5. Average system energy with respect to the number of steps obtained from the diffusion model.

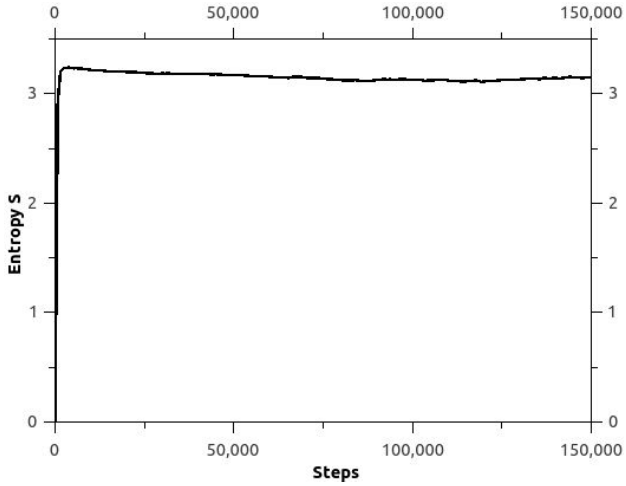

FIGURE 6. Entropy evolution, as function of the number of steps, resulting from the simulations of the diffusion model.

simulations, thus reinforcing the hypothesis of the almost perfect final conservation of energy. Figure 5 shows that the sum of all the fluctuations, and Fig. 5 shows that the system entropy reaches a maximum value and then remains constant showing stability.

In next sections we simulate the heating (and cooling) the fluid in which the particles are immersed by slowly adding (or subtracting) energy.

4. Model 3: adding energy

To show the effect of energy conservation through the diffusion model, we can consider asymmetric intervals for δ. For the cases where energy is added, that is, $\delta \in [-\epsilon+\gamma, \epsilon]$ or $\delta \in [-\epsilon, \epsilon+\gamma]$, where $0 < \gamma < \epsilon$ is the asymmetry factor, we obtain that their distribution does not preserve the exponential form, and the Gini index is different to 0.5 (G=0.3114), as shown in Fig. 7.

Figure 8 shows that the system entropy keeps indefinitely increasing, which means that the steady state is not reached.

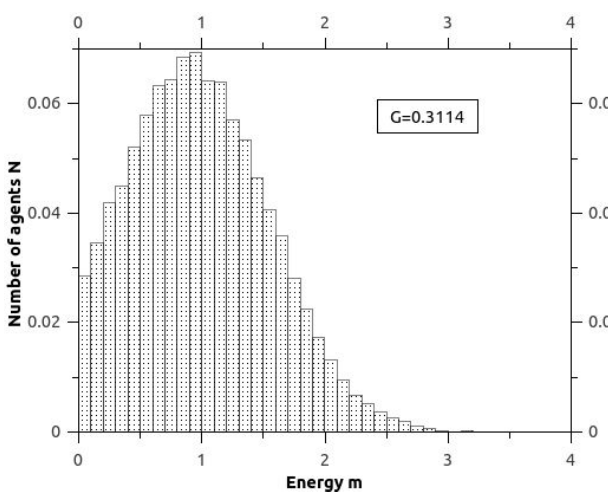

FIGURE 7. Asymmetric cases for the addition of energy model. $\epsilon = 0.1$, $\gamma = 0.01$ and 5.0×10^5 steps.

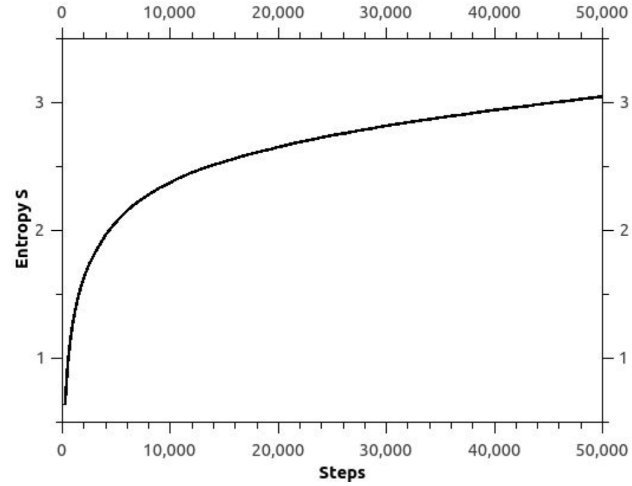

FIGURE 8. Entropy for the case of addition of energy. $\epsilon = 0.1$, $\gamma = 0.01$ and 5.0×10^5 steps.

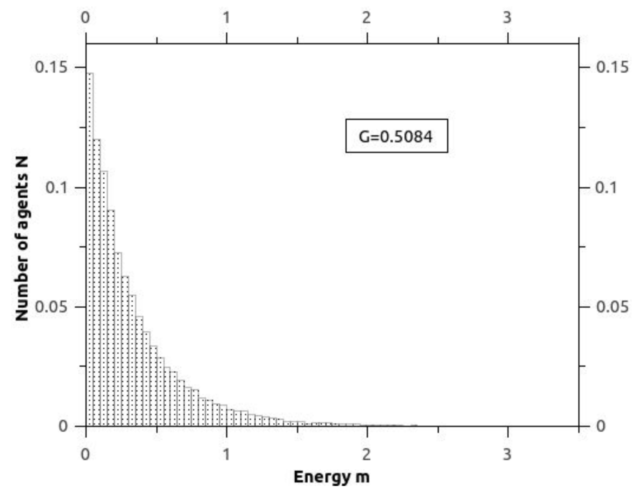

FIGURE 9. Asymmetric case for the subtraction of energy model. $\epsilon = 0.1$, $\gamma = 0.01$ and 1.5×10^5 steps.

5. Model 4: subtracting energy

For the cases when energy is subtracted ($\delta \in [-\epsilon-\gamma, \epsilon]$ or $\delta \in [-\epsilon, \epsilon-\gamma]$), we obtain that the system energy slowly goes to zero and keeps the exponential form with a Gini index close to 0.5 (G=0.5084), as can be shown in Fig. 9 and 10 shows the corresponding entropy. Notice that the change of the entropy is lager in model 3 than in model 4.

6. Model 5: richest agent always helps poorer agents

We modify the diffusion model for the case when m'_j is negative. Instead of reassigning the fluctuation to random agents, now the energy fluctuation is subtracted from the agent with

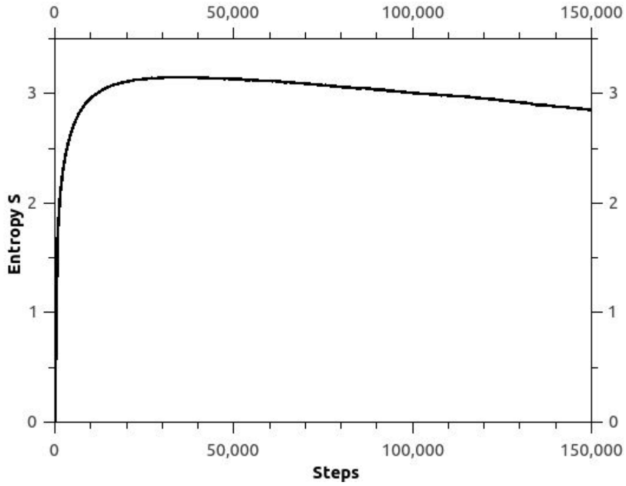

FIGURE 10. Entropy for the case of subtraction of energy. $\epsilon = 0.1$, $\gamma = 0.01$ and 1.5×10^5 steps.

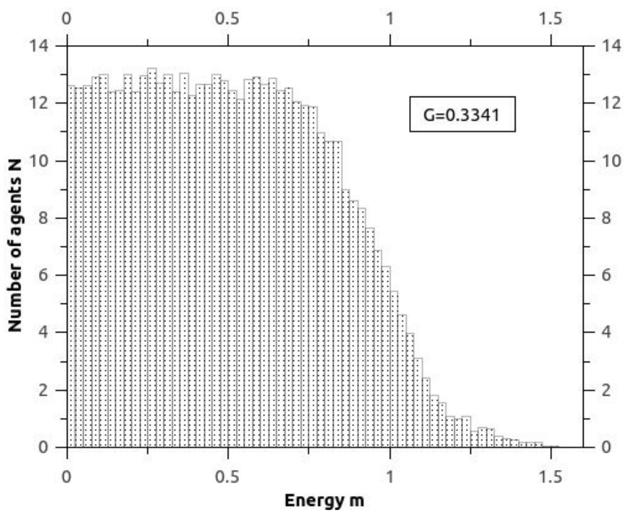

FIGURE 11. Average wealth distribution resulting from the model 5. The symmetric fluctuations have a $e = 0.1$.

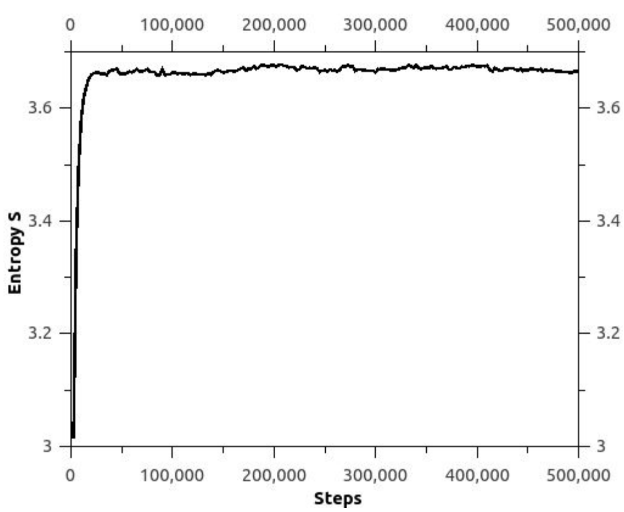

FIGURE 12. Entropy of the wealth distribution in which a rich agent always helps poorer agents.

the greatest energy, so in this model we try to decrease inequalities. With this new condition, we simulate a system with 100 agents which have an initial energy of 0.5 units each and whose energy is modified 5.0×10^5 times. The result averaged over 100 realizations is shown in Fig. 11.

The distribution obtained from model 5 is not similar to that of the diffusion model (see Fig. 3) and has a G value of 0.3341. We obtained a very compact energy or wealth distribution, and the distribution is stable since the entropy function reaches a maximum value which remains constant as shown in Fig. 12, with higher values of entropy and larger entropy fluctuation than in all previous models.

It is important to mention that any model involving economic taxes (similar to the well-known energy selection process known as the Maxwell demon) is not easy to reproduce in a real physical system, like a particle gas.

7. Discussion and conclusions

We simulated many different processes: binary collisions, diffusion under symmetric and asymmetric energy fluctuations, and a model in which we avoid negative energy values by taxing the richest agent. We used the collision model as a reference model to point out that in a system with fixed number of agents and total wealth, the final wealth distribution is the BGD, which is an exponential, and its entropy shows us the stability of the system. For the analysis of our models, the Gini index was used to quantitatively compare the similarity of the distribution obtained with the exponential Boltzmann-Gibbs distribution whose Gini index has a known value of 0.5. Additionally, the Gini index provides information on the degree of inequality in wealth distribution. So that, G was a relevant parameter for the study of our distributions.

The main difference between our models and the widely known *traditional* models [10,11] lies in the condition of wealth conservation. In contrast with the all-step strict energy conservation in the binary collision model, the interactions in the diffusion models do not conserve energy; this is because, at each step, an agent undergoes a small random change (or fluctuation) in his wealth (or energy) that is not the result of an exchange with another agent.

The diffusion model is similar to that of pollen particles suspended in a fluid; however, the total energy of the pollen particles (or agents) is approximately conserved when δ is distributed in a symmetric interval around zero, *i.e.*, when the expectation value of δ is zero. On the other hand, to simulate the heating or cooling of the environment, we employed asymmetric δ distribution. If the expectation value of such distribution is positive, the system gains energy, the entropy grows very fast, and the distribution obtained is not exponential-like, as shown in Figs. 7 and 9. If this expectation value is negative, then the system loses energy and the obtained distribution is exponential-like, but it is not stable, as shown by the entropy evolution in Figs. 8 and 10.

Notice that, in all our models, if an agent's wealth would become negative (because the fluctuation δ is negative and larger than agent energy), we seek for "richer" agents until we found one with an energy larger than δ. Then, instead of randomly selecting an agent to reassign the fluctuation, we use a very simple model in which we select the richest agent in that step to help the poorer agent. So, the mechanism used for this purpose affects the shape of the distribution obtained, as can be seen when comparing Fig. 3, which shows an exponential distribution, with Fig. 11. in which a more compact distribution of wealth is observed. On the other hand, an increase in the maximum value reached by the entropy is observed, which is a consequence of the special considerations of the model. It is important to emphasize that, in this reassignment process, an indirect interaction is created because the fluctuation is exchanged from one agent to another. Furthermore, this exchange does keep constant the system energy and, therefore, the mechanism is similar to an economic collision.

In summary, the exponential distribution is very robust. It is obtained even if the total system energy or wealth is not conserved as in the case of subtracting energy. Nevertheless, if we reduce the freedom of the interactions by affecting a specific group of elements or by limiting the exchange of money, then the distribution can change substantially. Some restrictions on the freedom in economic exchanges produce more drastic modifications on the final distributions than others, but the exponential distribution seems to be pretty robust to changes due to the fundaments of the theory of probability.

We hope that this work may stimulate further research in diffusion processes, econophysics systems and related phenomena.

Acknowledgments

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1. C. Schinckus, *Physica A: Statistical Mechanics and its Applications*, **389** (2010) 814-3821.
2. M. Gallegati, S. Keen, T. Lux, P. Ormerod, *Physica A: Statistical Mechanics and its Applications*, **370** (2010) 1-6.
3. J. L. McCauley, *Physica A: Statistical Mechanics and its Applications*, **371** (2010) 601-609.
4. E. J. de Area Leão Pereira, M. Fernandes da Silva, H.B. Pereira, *Physica A: Statistical Mechanics and its Applications*, **473** (2010) 251-261.
5. A. Chaterjee, S. Yarlagadda, B.K. Chaterjee, *Econophysics of Wealth Distributions: Econophys-Kolkata I*. (Springer-Verlag, 2005).
6. V. M. Yakovenko, J.B. Rosser Jr, *Rev. Modern Phys.,* **81** (2009) 1703.
7. V. Pareto, *Cours d'Economie Politique*. (Rouge, Lausanne, 1896).
8. A.C. Silva, V.M. Yakovenko, *Europhys. Lett.* **69** (2005) 304-310).
9. P. Soriano-Hernández, M. del Castillo-Mussot, O. Córdoba-Rodríguez, R. Mansilla-Corona, *Physica A: Statistical Mechanics and its Applications*, **465** (2017) 403-413.
10. V. M. Yakovenko, A. A. Dragulescu, *European Physical Journal B*, **17** (2000) 723-729
11. A. Chakraborti, B. K. Chakrabarti, *European Physical Journal, B*, **17** (2000) 167-170
12. B. K. Chakrabarti *et al.*, *Econophysics of Income and Wealth Distributions.* (New York, Cambridge University Press, 2013) pp 3-17.
13. W. Greiner, L. Neise, H. Stocker, *Thermodynamics and statistical mechanics.* (Trans: D. Rischke, Springer, corrected second printing, 1997), Cap. 5, p. 151.
14. L. Bachelier, *Théorie de la spéculation*. (PhD Thesis, Soborne Paris, 1900)
15. Drăgulescu, A. & Yakovenko, V. *Eur. Phys. J. B* **20** (2001) 585. doi:10.1007/PL00011112
16. A. Shaikh, *Capitalism: competition, conflict, crises*. (Oxford , New York, Oxford University Press, 2016) pp. 751-755.
17. A. Ragab, *Three essays on the incomes of the vast majority* (Unpublished Doctoral Dissertation). The New School (2014).