

On a tautochrone-related family of paths

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An alternative approach to the properties of the tautochrone and brachistochrone curves is used to introduce a family of curves complying with relations where the time of descent is proportional to a fractional power of the height difference. These curves are classified according to their symmetries. Further properties of these curves are studied.

Keywords: Analytical mechanics; Huygens's isochrone curve; Abel's mechanical problem.

Utilizamos un tratamiento alternativo de las curvas tautocrona y braquistocrona para introducir una familia de curvas que cumplen con relaciones en las que el tiempo de descenso es directamente proporcional a la altura descendida, elevada a un valor fraccionario. Las mencionadas curvas son clasificadas de acuerdo con sus simetrías. Se estudian otras propiedades de dichas curvas.

Descriptor: Mecánica analítica; curva isocrona de Huygens; problema mecánico de Abel.

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1. Introduction

In 1658 Christiaan Huygens made public his discovery of the tautochrone, that is: the path which a point-like particle must follow so that the period of its motion comes out to be exactly independent of its amplitude. This is in contrast with Galileo's pendulum, where the period is independent only in the small amplitude approximation. A thought-provoking, insightful, description of Huygens's work can be found in Ref. 1. For a discussion in simpler terms see [2] and [3].

Some forty years after Huygens's work, Johann Bernoulli found the brachistochrone, *i.e.* the path of quickest descent from a fixed starting point to a fixed end point. As is well known, this was later to become one of the seminal problems in variational calculus. A sketch of Bernoulli's deduction is included in Ref. 4, while [5] contains a brief account of the discovery, as well as pointers for the deduction of the curve through variational calculus.

Both curves are related, as Bernoulli himself noticed. Huygens's tautochrone is the complete cycle of an inverted cycloid, while a brachistochrone is a segment of an inverted cycloid, with the initial point at an apex.

As can be deduced from the tautochrone property (*i.e.* that for this path the period is independent of the amplitude), the time T it takes for a point-like particle to descend along an inverted cycloid to a minimum of the path is independent of Δy , the height difference from the starting point to the minimum:

$$T \propto (\Delta y)^0 \quad (1)$$

and depends only on the parameters that define the given cy-

cloid. This is in stark contrast with the inclined plane, where the time of descent is proportional to the square root of the height difference:

$$T \propto (\Delta y)^{1/2} \quad (2)$$

It is then natural to ask for the possible values of an exponent β such that one can construct a path that complies with

$$T \propto (\Delta y)^\beta \quad (3)$$

The rest of this article deals with the solution of such question, and with its consequences. In this endeavour the brachistochrone will appear more than once. In order to address the question we have just posed, use will be made of a generalization of Abel's elegant alternative deduction of the tautochrone [4], which, as Abel himself commented, can be cast in the language of fractional integro-differential operators [6]. Albeit a long and noble history in mathematics (dating back to Leibniz), [7] fractional analysis has only begun to be used by physicist in the last thirty years or so. We thus hope that this article will serve, if nothing more, to call attention on this exciting subject (some examples of the applications of fractional analysis in physics may be found in Refs. 6 to 11).

Some recent generalizations of the tautochrone include: the tautochrone with friction [12], the relativistic tautochrone [13], the tautochrone in rotating frames of Ref. 14 and the tautochrone under an arbitrary potential [15]. Finally, for other applications of the calculus of variations in classical mechanics and other branches of physics one may recommend [16] and [17].

The rest of this article is organized as follows: in Sec. 2 we review the properties of the tautochrone and brachistochrone curves under a new light. In Sec. 3 we introduce a family of curves complying with relation (3) for $-1/2 < \beta \leq 1/2$, and classify them according to their symmetries. In Sec. 4 we study further (probably counterintuitive) properties of these paths. Sec. 5 is reserved for conclusions.

2. Tautochrone vs. brachistochrone

Mathematicians carefully distinguish between curves (with a given parametrization) and paths (which are the images of curves.) We are interested in obtaining paths along which a point-like particle would descend, but we will do so by finding curves which have those paths as images. A given curve describes a unique path, but there are an infinite number of curves that correspond to a given path. In the present context, this distinction is quite unimportant: we will make clear each time that the same path gets two different parametrizations.

Huygens’s tautochrone can be parametrized as follows:

$$\begin{aligned} x(\theta) &= \frac{a}{2}(\theta - \sin \theta) + x_L - \frac{a\pi}{2} \\ y(\theta) &= \frac{a}{2}(1 + \cos \theta) + y_L \end{aligned} \quad 0 \leq \theta \leq 2\pi \quad (4)$$

the x -axis lying along the horizontal, with the positive y -axis in the upward direction.

As already said, this describes a family of inverted cycloids, and the presence of parameters y_L and x_L allows us to translate the tautochrone’s lowest lying point, (x_L, y_L) , anywhere we want to (one can take a tautochrone from Antwerp to Paris and it will remain a tautochrone).

Now, parameter $a > 0$ allows us to choose the maximum difference of height for a given tautochrone, that is: the difference of height $y(0) - y(\pi) = a$ from apex $(x(0), y(0))$ to the lowest lying point $(x(\pi), y(\pi)) = (x_L, y_L)$. But once this maximum difference of height is fixed, the total horizontal length of the tautochrone is also fixed, its value given by

$$x(2\pi) - x(0) = a\pi.$$

Let us remark: in a tautochrone the total difference of height and the total horizontal length are not independent.

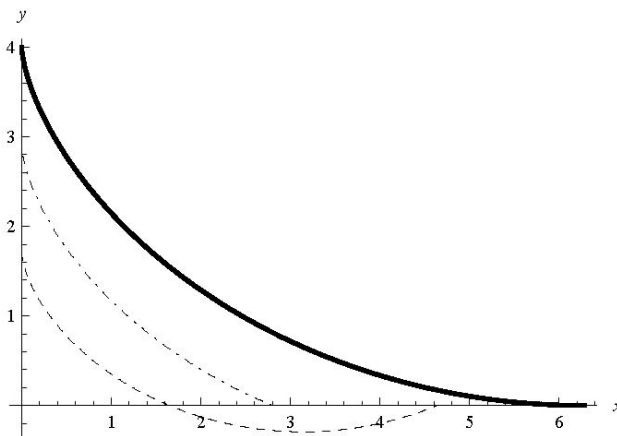


FIGURE 1. Some examples of brachistochrones. The thick curve is also a half tautochrone.

On the other side, in a brachistochrone the total horizontal length and the maximum difference of height *must* be independent as there is a brachistochrone connecting any two chosen points \mathbf{x}_i and \mathbf{x}_f .

The brachistochrone curve between (x_i, y_i) and (x_f, y_f) can be parametrized as follows:

$$\begin{aligned} x(\theta) &= \frac{a}{2}(\theta - \sin \theta) + x_i \\ y(\theta) &= \frac{a}{2}(1 + \cos \theta) + y_i - a \end{aligned} \quad 0 \leq \theta \leq \theta_f \quad (5)$$

where a and $0 < \theta_f \leq 2\pi$ are such that

$$x_f = x(\theta_f) = \frac{a}{2}(\theta_f - \sin \theta_f) + x_i$$

and

$$y_f = y(\theta_f) = \frac{a}{2}(\cos \theta_f + 1) + y_i - a.$$

Thus, an extra parameter is needed to describe the family of brachistochrones.

Equation (4) indicates us that we can divide the tautochrone into a left half, $(x_+(\theta), y_+(\theta))$, and its reflection through the vertical axis $x = x_f$, $(x_-(\theta), y_-(\theta))$, giving rise to a new parametrization:

$$\begin{aligned} x_{\pm}(\theta) &= \pm \frac{a}{2}(\theta - \sin \theta) + x_L \mp \frac{a\pi}{2} \\ y_{\pm}(\theta) &= \frac{a}{2}(1 + \cos \theta) + y_L \end{aligned} \quad 0 \leq \theta \leq \pi \quad (6)$$

Each one of these halves is a brachistochrone (its fixed points being an apex and the lowest lying point) of a very special kind: the ratio of the maximum height difference to the total horizontal length has a fixed value:

$$(y_{\pm}(0) - y_{\pm}(\pi))/|x_{\pm}(0) - x_{\pm}(\pi)| = \frac{2}{\pi}, \quad (7)$$

independent of any parameter (the lowest lying point is common to both curves, which represents no problem.)

If, on the other hand, one takes a brachistochrone (5) and joins it with its reflection through the $x = x_L$ axis, one sim-

ply does not get a tautochrone in the general case.

Perhaps a better way of understanding these curves is by noticing that both the tautochrone and the brachistochrone properties are invariant under scale transformations of the type $(x, y) \rightarrow (x', y') = (\kappa x, \kappa y)$ for $\kappa > 0$.

Indeed, in the frictionless motion of a particle along any given path σ , the conservation of the total mechanical energy dictates that T , the time it takes the particle to travel from the starting point $\mathbf{x}_i = (x_i, y_i) \in \sigma$ to the end point $\mathbf{x}_f = (x_f, y_f) \in \sigma$, is given by

$$T_\sigma = \frac{1}{\sqrt{2g}} \int_\sigma \frac{ds}{\sqrt{y}} \tag{8}$$

(where $ds = \sqrt{(dx)^2 + (dy)^2}$ is the differential of arch-length along σ) granted that the particle starts its motion with zero initial velocity and moves under the influence of a gravitational field $\mathbf{g} = (0, -g)$. Consider now a scale transformation $\mathbf{x} \rightarrow \mathbf{x}' = \kappa \mathbf{x}$ $\kappa > 0$, that sends \mathbf{x}_i to $\mathbf{x}'_i = (x'_i, y'_i)$ and sends (x_f, y_f) to $\mathbf{x}'_f = (x'_f, y'_f)$. There is a one-to-one mapping between each curve connecting \mathbf{x}_i with \mathbf{x}_f and each curve connecting \mathbf{x}'_i with \mathbf{x}'_f , given by

$$y(x) \rightarrow y'(x') = \kappa y(x) = \kappa y(\kappa^{-1}x')$$

(in symbolic terms $\sigma \rightarrow \sigma' = \kappa\sigma$). According with (8) T'_σ , the time it will take a particle to travel from \mathbf{x}'_i to \mathbf{x}'_f along σ' , is related to T_σ through:

$$T'_\sigma = \kappa^{1/2} T_\sigma$$

so that if a particular curve σ minimizes T , the corresponding curve $\sigma' = \kappa\sigma$ will minimize T' , that is:

$$\delta T'_\sigma = \kappa^{1/2} \delta T_\sigma = 0.$$

A similar result is easily proven for the tautochrone. Thus, if you build a scale model of a brachistochrone (tautochrone), the result will also be a brachistochrone (tautochrone). In the next section we will generalize this results for a wider family of curves.

Summing up, we have pointed out the differences between tautochrones and brachistochrones, while making clear that the tautochrone and brachistochrone properties are both preserved by scale transformations. From Eqs. (4) and (5) it is easily proven that (as would be expected) both properties are also preserved by translations and reflections through any vertical axis. If this seems a bit trivial, consider that neither of this properties is preserved by a general rotation in the $x - y$ plane nor by a general compression or expansion of the curve (*i.e.* by transformations of the type $(x, y) \rightarrow (x', y') = (\kappa x, y)$). In other words: a tilted tautochrone (brachistochrone) or a compressed tautochrone (brachistochrone) is no longer a tautochrone (brachistochrone). In Sec. 3 we will generalize this to a wider family of curves, but before going on there are a couple of facts regarding the tautochrone that we need to point out.

First, the period of motion in a given tautochrone is easy to calculate [2], and the result is:

$$4T = 4\pi \sqrt{\frac{a}{g}}, \tag{9}$$

we will frequently make use of this result.

Finally, consider, for a fixed value $H > 0$, and a fixed point (x_L, y_L) , the uni-parametric family of curves $\{\mathfrak{S}_{+, \phi}\}_{0 \leq \phi < \pi}$ given by:

$$\begin{aligned} x_{+, \phi}(\theta) &= \frac{H}{1 + \cos \phi} (\theta - \sin \theta) + x_L - \frac{\pi H}{1 + \cos \phi} \\ y_{+, \phi}(\theta) &= \frac{H}{1 + \cos \phi} (1 + \cos \theta) + y_L \end{aligned} \quad \phi \leq \theta \leq \pi \tag{10}$$

Notice that:

- all this curves start at a same height

$$y_i = y_{+, \phi}(\phi) = H + y_L$$

and they all end at point $(x_L, y_L) = (x_\phi(\pi), y_\phi(\pi))$.

- The curve $\mathfrak{S}_{+, \phi}$ is a segment of a left half tautochrone with apex at

$$(x_{M, \phi}, y_{M, \phi}) := (x_L - \frac{\pi H}{1 + \cos \phi}, \frac{2H}{1 + \cos \phi} + y_L)$$

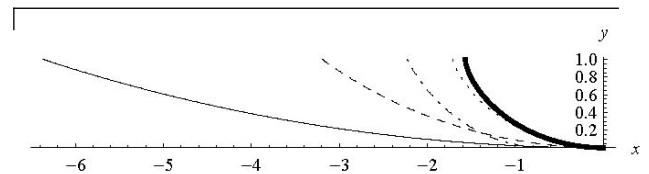


FIGURE 2. Some $\mathfrak{S}_{+, \phi}$ paths for $H=1$. The thick curve is the half tautochrone $\mathfrak{S}_{+, \phi=0}$. Dotted: $\phi=\pi/5$, dot-dashed: $\phi=2\pi/5$, dashed: $\phi=3\pi/5$, and thin: $\phi=4\pi/5$.

In so many words $\{\mathfrak{S}_{+, \phi}\}_{0 \leq \phi < \pi}$ is just the family of all left half tautochrone segments with a fixed total height difference H , ending at a fixed lowest lying point (x_L, y_L) . (But they are not segments of the same tautochrone!)

We note in passing that, with the exception of $\mathfrak{S}_{+, \phi=0}$,

none of this segments is a brachistochrone. Indeed, for each point $(x_{+, \phi}(\theta), y_{+, \phi}(\theta))$ lying in a given segment $\mathfrak{S}_{+, \phi}$ we can easily find, from (5), the brachistochrone that connects it with (x_L, y_L) ; and this is not, in general case, $\mathfrak{S}_{+, \phi}$ itself.

More to the point: each one of the $\mathfrak{S}_{+, \phi}$ curves, being a segment of a tautochrone, has what we will call *the independence of height property*. That is, if a particle moving along $\mathfrak{S}_{+, \phi}$ has zero initial velocity at some point $(x', y') \in \mathfrak{S}_{+, \phi}$, then it will reach (x_L, y_L) in a time T_ϕ , given by:

$$T_\phi = \pi \sqrt{\frac{H}{g(1 + \cos \phi)}}$$

independently of (x', y') , according with (9). The family of paths just described can be extended to include the reflections of the $\mathfrak{S}_{+, \phi}$, giving us what we will call the $\{\mathfrak{S}_{\pm, \phi}\}_{0 < \phi \leq \pi}$ family. These are, up to a translation, the only paths with the independence of height property with a given total height difference H (the unicity of the solution of Abel's equation, which we will see later, warrants this). As T_ϕ is the same for $\mathfrak{S}_{-, \phi}$ as it is for $\mathfrak{S}_{+, \phi}$ we can then state that:

Of all the paths with the height independence property, with a total height difference H , there is one (up to translations and reflections) with minimum T , given by:

$$T_{\phi=0} = \pi \sqrt{\frac{H}{2g}} \tag{11}$$

this path is $\mathfrak{S}_{+, \phi=0}$.

Granted, $\mathfrak{S}_{+, \phi=0}$ happens to be a brachistochrone, but the property just stated is logically independent of the brachistochrone property because:

- The family $\{\mathfrak{S}_{\pm, \phi}\}_{0 \leq \phi < \pi}$ is not a good sample of all the curves connecting two given points: we have not presented an alternative proof of the brachistochrone.
- On the other side, we will find other families of curves with an element that minimizes the time of descent *for that given family*; and this element cannot be, in the general case, a brachistochrone.

If we consider now the invariance under scale transformations, we will see that the half tautochrone is really unique (up to an arbitrary combination of translations, reflections and scale transformations.) We will generalize this result in the following sections.

3. A family of curves

Consider a classical, non-relativistic, point-like particle of mass m moving on the x - y -plane with potential energy

$$U(x, y) = mgy$$

for some $g \in \mathbb{R}^+$.

Suppose, further more, that the particle is constrained to move, without friction, along a curve σ univocally described by the function

$$x = x_\sigma(y) \quad \text{for all } y \in [y_L, y_M] \subseteq \mathbb{R}$$

where function $x_\sigma(y)$ is continuously differentiable in (y_L, y_M) , and such that

$$\begin{aligned} \lim_{y \rightarrow y_L^+} x_\sigma(y) &= x_\sigma(y_L), \\ \lim_{y \rightarrow y_M^-} x_\sigma(y) &= x_\sigma(y_M), \end{aligned}$$

and that the limit

$$\lim_{y \rightarrow y_M^-} \frac{dx_\sigma}{dy}(y)$$

exists.

If the particle has zero initial velocity at an initial ‘‘height’’ $y_i \in (y_L, y_M]$, then the conservation of the total mechanical energy of the system dictates that the particle will reach height y_L in a time $T_\sigma(y_i) > 0$ given by

$$T_\sigma(y_i) = - \int_{y_L}^{y_i} \frac{\frac{ds_\sigma}{dy}}{\sqrt{2g}(y_i - y)^{\frac{1}{2}}} dy \tag{12}$$

where

$$ds_\sigma(y) = - \sqrt{1 + \left(\frac{dx_\sigma}{dy}\right)^2} dy$$

is the differential of the arch length traveled by the particle from height y_i to height y while moving on σ .

Equation (12) can be written in the slightly more concise form:

$$T_\sigma(y_i) = - \sqrt{\frac{\pi}{2g}} \left(I_{y_L}^{\frac{1}{2}} \frac{ds_\sigma}{dy} \right)(y_i) \tag{13}$$

with the use of the Riemann-Liouville (right hand) fractional integral, defined as [6-7,18]:

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t - \tau)^{1-\alpha}} d\tau \tag{14}$$

for $t > a$ and $\alpha > 0$. Here, Γ stands for the gamma function (the necessary information about the properties of the gamma function may be found in Ref. 19).

We now examine the implications of imposing on curve σ a condition of the type

$$T_\sigma(y_i) = K(y_i - y_L)^\beta \tag{15}$$

for all $y \in (y_L, y_M]$ for a and fixed $\beta \in \mathbb{R}$ and a real constant $K > 0$ (it is implicit that σ depends on the values of β and K , and that the admissible values of K may be restricted by the value of β).

Let us note in passing that for $\beta = 0$ (15) essentially coincides with Abel's equation for the tautochrone:

$$\frac{d}{dy} \left(I_{y_L}^{\frac{1}{2}} \frac{ds_\sigma}{dy} \right) (y) = 0 \tag{16}$$

while $\beta = 1/2$ corresponds to the trivial case of the frictionless inclined plane.

To be more precise, for $\beta = 1/2$ and a given final position (x_L, y_L) , the solutions constitute a uni-parametric family of straight lines $\{l_\theta\}_{0 < \theta < \pi}$ given by

$$x = x_\theta(y) = (y - y_L) \cot \theta + x_L \quad 0 < \theta < \pi \tag{17}$$

for in this case $ds_\theta = -(\sin \theta)^{-1}$; and by plugging this last expression directly in (12) we get, after a straightforward calculation:

$$T(y_i) = \sqrt{\frac{2}{g} \frac{(y_i - y_L)^{1/2}}{\sin \theta}} = \sqrt{\frac{2S}{g \sin \theta}}$$

where S is the distance traveled by the particle along the inclined planed, in accordance with the well known result.

Turning to the general case, we resort to the well known fact [6] that Abel's (general) integral equation:

$$f(t) = (I_0^\alpha \phi)(t) \quad (t > 0, 0 < \alpha < 1)$$

has as solution

$$\phi(t) = \frac{d}{dt} (I_0^{1-\alpha} f)(t),$$

so that for every curve σ complying with (15) we can write

$$\frac{ds_\sigma}{dy}(y) = -\frac{\sqrt{2g}}{\pi} K \frac{d}{dy} \int_0^{y-y_L} \frac{\nu^\beta}{(y-y_L-\nu)^{1/2}} d\nu \tag{18}$$

The integral on the r.h.s. of (18) is divergent for $\beta \leq -1$; and for $\beta > -1$ one can apply the convolution theorem to get

$$\frac{ds_\sigma}{dy}(y) = -\sqrt{\frac{2g}{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+3/2)} K \frac{d}{dy} (y-y_L)^{\beta+1/2} \tag{19}$$

so that s_σ will be finite at y_L only if $\beta \geq -1/2$, and the case $\beta = -1/2$ has no geometrical (let alone physical) interpretation, as it would imply that the arch length traveled from (x_i, y_i) to any given point on the curve would be same, irrespective of the end point. Thus, physically acceptable bounded curves exist only for $\beta > -1/2$. With this caveat in mind, from (19) we may conclude

$$\begin{aligned} x_\sigma(y) &= x_L \pm \int_y^{y_L} \sqrt{\left(\frac{ds_\sigma}{dy'}\right)^2 - 1} dy' \\ &= x_L \pm \int_y^{y_L} \sqrt{\left(\frac{y' - y_L}{h}\right)^{2\beta-1} - 1} dy', \quad x_L \in \mathbb{R}, \tag{20} \end{aligned}$$

where, by definition:

$$h := \left(\sqrt{\frac{2g}{\pi}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)} K \right)^{\frac{2}{1-2\beta}} \tag{21}$$

In order to retain the physical interpretation of Eq. (20) the integrand on the r.h.s. must remain real-valued for values of y arbitrarily close to y_L , and so the expression is only valid for $\beta \leq 1/2$. Moreover, for any acceptable value of β (i.e. $-1/2 < \beta \leq 1/2$) h must take values in $[H, \infty)$ where H stands for

$$H := y_M - y_L \tag{22}$$

Now, in expression (20) not all the symmetries present in our system are explicit. With the use of some algebra, we can rewrite it as:

$$x_\sigma(y) = x_L \pm h \int_0^{\frac{y-y_L}{h}} \sqrt{\eta^{2\beta-1} - 1} d\eta \tag{23}$$

Thus, a curve σ is identified by four different parameters: β and x_L, y_L, \pm and h . From now on we shall write $\sigma_{\beta, x_L, y_L, \pm, h}$ in order to identify a specific curve. The freedom to chose x_L and y_L means that property (3) is invariant under translations. In an analogous manner, the freedom to chose the sign \pm is the result of invariance under reflections through a vertical axis. Finally, h is related with the invariance of the property under scale transformations. Indeed, if one multiplies x_L, y_L, y and h -all by a same factor κ - then $x_\sigma(y)$ is also multiplied by this factor. Thus, symbolically:

$$\sigma_{\beta, \kappa x_L, \kappa y_L, \kappa h} = \kappa \sigma_{\beta, x_L, y_L, h}$$

and the property is preserved under scale transformations.

Thus, the following **lemma** has been established:
Lemma 1 *There does not exist a continuously differentiable bounded curve in the plane, in which a classical non-relativistic point-like particle could move under the influence of a spatially homogeneous time independent gravitational field $\mathbf{g}(\mathbf{x}, t) = \mathbf{g}$, starting with zero initial velocity at position \mathbf{x}_i , complying with a condition*

$$\Delta t \propto (\Delta y)^\beta$$

for values of $\beta > 1/2$, or $\beta \leq -1/2$ where

$$\Delta y := -\mathbf{g} \cdot (\mathbf{x}_i - \mathbf{x}_L)$$

for a fixed position \mathbf{x}_L , Δt being the lapse of time required for the particle to move from \mathbf{x}_i to \mathbf{x}_L along the curve.

For $-1/2 < \beta \leq 1/2$ the solutions exist, as shown in (23), and are unique up to an arbitrary combination of reflections, translations and scale transformations.

This lemma can easily be extended for charged particles in the presence of spatially homogeneous electrostatic fields, but the charge-mass quotient would then appear in our equations.

Let us note, in passing, that arbitrary combinations of translations, reflections and scale transformations constitute a group under composition. This is the group that leaves the tautochrone property unaltered.

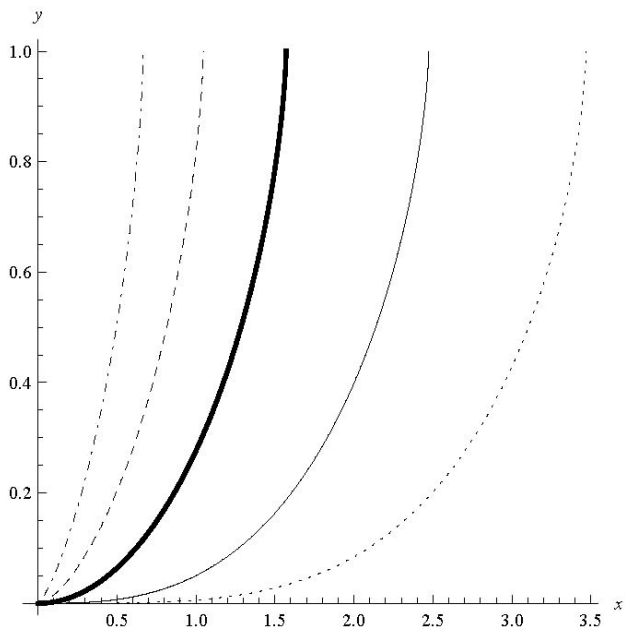


FIGURE 3. Some $\sigma^{\beta H}$ paths with $H=1$. The thick path is a half tautochrone ($\beta=0$). Dot-dashed: $\beta=0.3$, dashed: $\beta=0.15$, thin: $\beta=-0.15$, and dotted: $\beta=-0.24$.

4. Further properties

According with definition (21) and the condition established for the admissible values of h , Eq. (22), for a given β the physically admissible values of K are restricted by

$$K \geq K_{\beta}^* \tag{24}$$

where K_{β}^* stands for

$$K_{\beta}^* := \sqrt{\frac{\pi}{2g} \frac{\Gamma(\beta + 1/2)}{\Gamma(\beta + 1)}} H^{1/2-\beta}$$

Thus, for a fixed β and a fixed total height difference $H > 0$ there is a minimum total time of descent, $T_{\beta H}^*$, compatible with condition (3), this time is given by:

$$T_{\beta H}^* = H^{1/2} \sqrt{\frac{\pi}{2g} \frac{\Gamma(\beta + 1/2)}{\Gamma(\beta + 1)}} \tag{25}$$

and it is achieved by a curve (lets call it $\sigma^{\beta H}$) given by the parametric equation:

$$x_{\beta H}(y) = H \int_0^{y/H} \sqrt{\eta^{2\beta-1} - 1} \, d\eta \quad y \in (0, H]. \tag{26}$$

This curve is unique up to translations and reflections.

In other words, if a point-like particle starts its movement at position $(x_{\beta H}(H), H) \in \sigma_{\beta H}$ then it will take a time $T_{\beta H}^*$, as given in (25), for the particle to reach the final position $(0, 0)$, but if it starts at any other point of the path, with height $y \in (0, H)$,

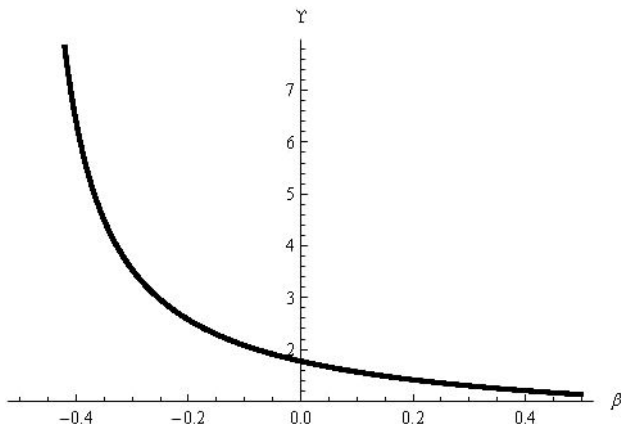


FIGURE 4. The function $\Upsilon(\beta) := \Gamma(\beta + 1/2)/\Gamma(\beta + 1)$ in the region $-1/2 < \beta \leq 1/2$.

then the particle will reach the origin in a lapse of time $T_{\beta H}(y)$, given by:

$$T_{\beta H}(y) = \left(H^{1/2-\beta} \sqrt{\frac{\pi}{2g} \frac{\Gamma(\beta + 1/2)}{\Gamma(\beta + 1)}} \right) y^{\beta} \tag{27}$$

and this is (up to translations and reflections) the swiftest path with a total height difference H and compatible with condition (3).

It is then natural to ask: of all $\sigma^{\beta H}$ paths which one is the swiftest for a fixed total height difference H ? *i.e.*, which one will make a particle descend a height H in a minimum time?

Function $\Gamma(\beta + 1/2)/\Gamma(\beta + 1)$ is monotonically decreasing in the region of interest (as can be seen in Fig. 4 so that the minimum is achieved at $\beta = 1/2$, which corresponds to a segment of length H of a vertical straight line (quite obvious), and as β approaches the value $-1/2$, $T_{\beta H}^*$ diverges:

$$\lim_{\beta \rightarrow -\frac{1}{2}^-} T_{\beta H}^* = \infty.$$

Let us note that there is a fixed ratio between the total height difference H and the horizontal length $x_{\beta,H}(H)$ in a $\sigma^{\beta H}$ curve:

$$R(\beta) := \frac{x_{\beta H}(H)}{H} = \int_0^1 \sqrt{\eta^{2\beta-1} - 1} \, d\eta \tag{28}$$

and this ratio diverges as β approaches the value $-1/2$:

$$\lim_{\beta \rightarrow -\frac{1}{2}^+} R(\beta) = \infty.$$

And so, we have established a second lemma:

Lemma 2 *Each element in a family of paths $\{\sigma^{\beta H}\}_{-1/2 < \beta < 1/2}$, defined in (26) has a fixed horizontal length to total height difference ratio, independent of H , given in (28). This ratio is a monotonically decreasing function of β . A particle starting from rest at height H would descend along a given path $\sigma^{\beta H}$ in a time $T_{\beta H}^*$, given by (25), which is, for fixed H , also a monotonically decreasing function of β . Thus, every $\sigma^{\beta H}$ with $0 < \beta \leq 1/2$ is shorter*

and swifter than the half tautochrone, and every $\sigma^{\beta H}$ with $-1/2 < \beta < 0$ is longer and slower than the tautochrone.

The path of quickest descent for a given height is a segment of a vertical straight line. As β approaches $-1/2$ with fixed height, the total arch-length of the path grows without limit, as does the total time of descent.

This result can not be held against the brachistochrone: first, because we are dealing with paths connecting two given heights; whereas brachistochrones connect two given fixed points. In second place, the vertical straight line segment can be considered as a kind of collapsed brachistochrone.

5. Conclusions and outlook

In the framework of classical mechanics, we have generalized the tautochrone path, generating a family of curves in which the time of descent is proportional to a fractional power of the height difference. The material just presented may be used to complement and enrich any university course exposition of classical mechanics and/or analytical mechanics at an intermediate or senior level, as it:

- Shows us that there are still interesting things to learn in classical mechanics at a fairly elementary level.
- Illustrates the use of mathematical tools such as the Laplace transform and the gamma function in physical problems.

- Highlights the rôle of symmetries, even in well-trodden branches of physics.

There are some interesting consequences and questions that we have left aside:

- These curves are independent of the mass of the particle: m is absent in every expression, starting from (8) because gravitational mass and inertial mass were canceled out even before writing this equation. This is just a consequence of the equivalence principle.
- The eerily ubiquitous constant $\Gamma(1/2) = \sqrt{\pi}$, present in all our calculations, just begs the question: are there possible generalizations of our curves in the frame of strong gravitational fields?
- In one-dimensional physics, the harmonic oscillator potential is the isochronous potential, *i.e.*, this is the only potential for which the frequency is strictly independent of the amplitude. Can this potential be derived from the Euler-Lagrange equations of the tautochrone curve? Can this potential be generalized for some or all of our curves?

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