Enhancing the process of teaching and learning physics via *dynamic problem solving strategies*: a proposal

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The large number of published articles in physics journals under the title "Comments on $\cdots$" and "Reply to $\cdots$" is indicative that the conceptual understanding of physical phenomena is very elusive and hard to grasp even to experts, but it has not stopped the development of Physics. In fact, from the history of the development of Physics one quickly becomes aware that, regardless of the state of conceptual understanding, without quantitative reasoning Physics would have not reached the state of development it has today. Correspondingly, quantitative reasoning and problem solving skills are a desirable outcome from the process of teaching and learning of physics. Thus, supported on results from published research, we will show evidence that a well structured problem solving strategy taught as a dynamical process offers a feasible way for students to learn physics quantitatively and conceptually, while helping them to reach the state of an *Adaptive Expert* highly skillful on innovation and efficiency, a desired outcome from the perspective of a *Preparation for Future Learning* approach of the process of teaching and learning Physics effectively.

Keywords: Physics problem solving; physics learning; teaching of physics; quantitative reasoning.

El gran número de artículos publicados en revistas de física bajo el título "Comentarios sobre $\cdots$" y "Réplica a $\cdots$" es indicativo de que la comprensión conceptual de los fenómenos físicos es muy escurridiza y difícil de entender incluso para los expertos, pero ello no ha detenido el desarrollo de la Física. De hecho, de la historia del desarrollo de la Física rápidamente nos damos cuenta de que, independientemente del estado de comprensión conceptual, sin el razonamiento cuantitativo la Física no hubiese alcanzado el estado de desarrollo que tiene actualmente. En consecuencia, tanto razonamiento cuantitativo como habilidades en la resolución de problemas son resultados deseables a obtener del proceso de enseñanza y aprendizaje de la Física. Así, con apoyo en resultados de investigaciones publicadas, mostraremos evidencias de que cualquier estrategia para la resolución de problemas presentada como un proceso dinámico ofrece una forma viable para que los estudiantes aprendan física tanto cuantitativa como conceptualmente, mientras que al mismo tiempo los ayuda a alcanzar el estado de un *experto adaptable* altamente calificado en la innovación y la eficiencia, un resultado deseado desde la perspectiva del enfoque del proceso de enseñanza y aprendizaje de la Física con efectividad en función de una *Preparación para el aprendizaje futuro*.

Descriptores: Resolución de problemas en física; aprendizaje de física; enseñanza de la física; razonamiento cuantitativo.

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1. Introduction

From the perspective of a *Preparation for Future Learning* approach of the process of teaching and learning physics effectively, a contemporary view encourage addressing the process so that students become *adaptive experts* [1,2], who are individuals highly efficient in applying (transferring) what they know to tackle new situations and are also extremely capable of innovation in the sense of being able to inhibit inadequate blocking "off the top of the head processes" or "to break free of well-learned routines" so that they can move to new learning episodes by finding, perhaps ingenious, ways to approach first time situations.

In this regard, it is undeniable that *Physics Education Research* (PER) and psychological research on learning and instruction have made available a good deal of teaching strategies which are helpful in reaching the aforementioned goal (a few such strategies are listed in Ref. 3 and some of them have been reviewed elsewhere [4-6]). Nevertheless, some controversial debates in relation to the effectiveness of some of these *Research-Based Instructional Strategies* (RBIS) can also be found in the literature [7-15].

Nobel Prize winner Professor Carl Wieman has also called for cautiousness when measuring RBIS teaching outcomes as one could create illusions about what students actually learn [16]. As mentioned in an article about transfer: "standard methods of investigating transfer have tended to depend on success-or-failure measures of participants' behavior on transfer tasks designed with very specific performance expectations on the part of the investigator. These methods have failed to identify the productive knowledge that students often do bring to bear on the tasks given to them" [17]. In this regard, Professor Sobel is more emphatic "Yes, in a special (possibly grant-supported) program, with smaller groups, with highly motivated instructors and students, with less content, students might do well, but that's not the real world" [9]. Or as warned by Professors Reif and Allen "Because nominal expertise does not necessarily imply good performance, one must be cautious in interpreting cognitive studies of novices and experts for which 'experts' have been chosen on the basis of nominal criteria. Data about such experts must be interpreted cautiously to avoid misleading conclusions about thought processes leading to good performance" [18].

In this panorama, of particular concern considering the intrinsically quantitative nature of physics is the fact that in physics classes students should actually be trained to apply what they have learned in their math classes and the aforementioned opinions might be a result of the fact that many of the published papers on teaching and learning physics seem to overemphasize the importance of teaching conceptual physical aspects [19-23], and to deemphasize the significance of standard mathematical reasoning [24], which are crucial for understanding physical processes, and which are not stressed, or even taught, because, rephrasing a passage from a recent editorial, they interfere with the students' emerging sense of physical insight [12]. A view which is further stressed in a physics textbook instructor manual ([25], page 1-9): "the author believes that for students struggling to grasp many new and difficult concepts, too high a level of mathematics detracts from, rather than aids, the *physics* we want them to learn. There is ample time in upper division courses for a more formal and rigorous treatment. It's counterproductive to burden students with unfamiliar and frightening mathematical baggage during their first exposure to the subject." And these kind of opinions seems to be reflected in outcomes obtained from the application of the relatively recently developed CLASS survey, which measures student's beliefs about physics and learning physics, showing a decrease of roughly 15% (out of 397) after instruction on student's beliefs about problem solving in physics and a decrease of 12% (out of 41) after instruction on student's beliefs about the connection between physics and mathematics (see respectively Tables I and V of [26]. Both courses were calculus-based Physics I. In relation to Table I (N=397), the authors of the study mention that "These are typical results for a first semester course—-regardless of whether it is a traditional lecture-based course or a course with interactive engagement in which the instructor does not attend to student's attitudes and beliefs about physics."

Moreover, controversial outcomes coming from some highly publicized RBIS [27-30] hardly help physics instructors in finding suitable advice about how to approach the teaching of physics in the most efficient way and an answer to the question of how much time should be spent on intuitive, conceptual reasoning and how much time in developing quantitative reasoning.

In view of the aforementioned facts, the aim of this paper is to begin a discussion on how the process of teaching and learning physics via *dynamic problem solving strategies* can help to tackle not only the conceptual but also the quantitative reasoning deficiencies persistently reported in the literature regarding the performance of students in introductory and upper-division physics courses [31-36,18,37-39]. It is obvious that both conceptual and quantitative reasoning are desired skills students should acquire and develop in our physics courses in order to foster in them their willingness to explore more complex scientific or engineering problems with confidence: *a preparation for future learning*.

The rest of the paper is organized as follows. Recalling events from the history of physics, in the next section, *Conceptual versus quantitative understanding*, we argue that in spite of conceptual gaps in some key physical ideas the development of physics did not stop because of the quantitative nature of physics. We also present in this section an empirical result showing how a student changes his/her wrong initial (conceptual) intuition by using quantitative analysis when solving a problem about electrical circuits. The next section discusses some of the needs for teaching students the use and application of structural problem solving strategies. The following section, before presenting the conclusions, develops the central theme in the article: promoting deep approaches to learning via *Dynamic problem solving strategies*. An illustrative example is also discussed in this section.

2. Conceptual versus quantitative understanding

From a practical point of view, the conceptual understanding of the principles of physics is a difficult and in some cases a very elusive task. This is confirmed by the large body of research dealing with "Student ideas about $\cdots$", "Student understanding of $\cdots$", "Students misconceptions about $\cdots$", "students' misunderstanding of $\cdots$" and so forth.

For instance, in a study [40] performed in a rather highly suitable and exceptionally favorable teaching environment, it was found that students answered correctly some conceptual questions on the nature of sound propagation in the same proportion before and after receiving instruction on the subject. In that study, in addition to active teaching instruction, students also watched at their own pace video-lectures on the nature of sound propagation by an experienced instructor, Professor Paul Hewitt. After the pre-instruction test, the students were told that each one of the test questions will be answered in these video-lectures. Similar results on students performance have also been reported from a study about why the seasons change. Students maintained their conceptual misconceptions about the subject even after watching a video that clearly explained the phenomena [6].

Perhaps more dramatic is the repeatedly reported case in which students respond to conceptual questions about the behavior of physical quantities the same way as students did at the beginning of the 80's [36].

Should we be surprised by these findings? Not, at all. As mentioned by Ambrose and collaborators "It is important to recognize that conceptual change occurs gradually and may not be immediately visible. Thus, students may be moving in the direction of more accurate knowledge even when it is not yet apparent in their performance" [6]. In fact, this is also observed in well trained physicists. For example, great debates about the proper understanding of the concept of physics have taken place among brilliant physicists. To be specific, in this regard one could refer to the conceptual debates in statistical and quantum physics and the electromagnetic theory [41-47]; the controversies between Lorentz and Einstein on the conceptual understanding and the meaning of the principles of special relativity [48]; and in other areas [49,50]]. What is even more relevant to PER is the fact that many of these controversies persist still today among expert physicists who invest a great deal of their time thinking about and working with these matters [51-58]. Additionally, the large number of "Comment on $\cdots$" and "Reply to $\cdots$" articles in physics journals are also a reminder of the difficulty of understanding and applying the concepts of physics.

Nevertheless, in spite of the aforementioned controversies on the conceptual understanding of physical principles, the development of physics has not stopped. One could argue that the reason for it is rooted in the fact that "nature is too subtle to be described from any single point of view. To obtain an adequate description, you have to look at things from several point of views, even though the different viewpoints are incompatible and can not be viewed simultaneously" [59]. And, as a matter of fact, physics is fortunately a combination of two basic compatible viewpoints "physical reasoning" and "quantitative reasoning".

Correspondingly, in spite of debates on the nature and significance of the concepts of physics, the intrinsic quantitative nature of physics is what has propelled the development of physics, helped by the sometimes questioned *Scientific Method* [60,61]. While the scientific method helps us to organize and test systematically every single hypothesis (enhancing our conceptual understanding) [62,63], quantitative reasoning helps us to be precise in which body of knowledge (mathematical models of the physical world) needs to be further developed: those *de accord* or that are consistent with observations and experiments [60,64,65].

To make the point clearer, one could think about the kind of progress physics would have reached had not Kepler struggled to fit the orbit of Mars to an elliptical one, stopping because of his lack of understanding (provided by Newton around 80 years later) of why the orbits of the planets were following the laws he was uncovering. Or think about the course of knowledge had Galileo given up his view of doing experiments and finding mathematical explanations for them in favor of the Catholic Inquisition's conceptual ideas about the universe. Or think about the current state of development in physics had Planck (because of lacking the respective conceptual understanding) restrained himself from introducing (in 1900) the Planck's constant to resolve the ultraviolet catastrophe. Or think about Einstein not continuing the development of his Theory of General Relativity because of the lack of conceptual understanding for not keeping in his theory the cosmological constant leading to a static universe.

But, have expert physicists today overcome the conceptual understanding undermining those developments? The answer in no. To be explicit in one case, one could see how physicists are still trying to understand quantum mechanics at its deepest conceptual level, a problem which arose more than one hundred years ago, at the beginning of the last century, with the introduction of Planck's constant. Yet, in spite of the conceptual shortcoming, the mathematical formulation of quantum mechanics and its refinements have allowed physics, regardless of the conceptual gap, to progress to levels that in today's world at many physics and engineering research centers, researchers are making conclusive observations about the nano-scale world for unanimated matter.

Thus, in each one the aforementioned cases, and of the many others that can be cited, there is no doubt that it was the quantitative analysis undertaken by the scientist involved that raised further the value of scientific knowledge, even when at the time it was hard to provide satisfactory conceptual explanations of the phenomena (such explanations came much later, after further developments of the mathematical understanding of each phenomena). Without them one would be talking today about "philosophical or scriptural proclamations" rather than scientific ones. Consequently, each one of these facts speak about the necessity of having our students of science and engineering to become properly acquainted with quantitative reasoning in their early training, even if they are lacking deep conceptual understanding. In this way they will be able to further deepen their understanding as they arrive to study upper division courses. In terms of the teaching and learning of physics, an empirical example of how quantitative reasoning can help students to accurately reason conceptually, even though the students' initial intuition might be wrong, can be seen in data from a post-test written examination given to students enrolled in a first year introductory physics course at the University of Maine (UMaine). The formulation of the problem is shown in Fig. 1. At the start, answering the question intuitively, one student wrote that the brightness of bulb A should decrease. Then the student went on to explain why it would happen that way as follows: "The brightness should decrease because the brightness of the bulb depends on the current of the circuit. So when the switch is open to find the current of the two light bulbs in series you would use the formula $I_{\text{total}} = V_{\text{total}}/R_{\text{total}}$. So let say $V = 12$ v and each bulb acts as a resistor with 2 Ω of impedance. When it's open there are two resistors in the circuits so $I = 12/(2+2) = 3$ A compared to when it's closed we simplify it to a series circuits so the R_{total} of the parallel would be 1 Ω so $1\Omega + 2\Omega = 3\Omega$ as R_{total}. $V = 12$ v so $12/3 = 4$ A so when the switch is thrown there's more current and the bulb is brighter".

Analysis of this and other students' answers of this study will be published elsewhere [68]. Here we should only notice how the use of quantitative reasoning helped the student to correct his/her initial (wrong) intuition to the correct result that after closing the switch bulb A becomes brighter. Thus, this example provides evidence that the learning with emphasis on equations and stressing the conceptual physical meaning of the respective symbols in the equation is a very feasible task. In particular, proper guidance and additional training in applying physical equations with understanding will help this student to reason analytically, using symbols, instead of resorting to numerical values (though we are not against this practice, specially in more difficult situations), which is more useful when dealing with situations on which physical intuition might fail [69] (assuming that resistance of bulb A is known, a non-intuitive question regarding the cir-

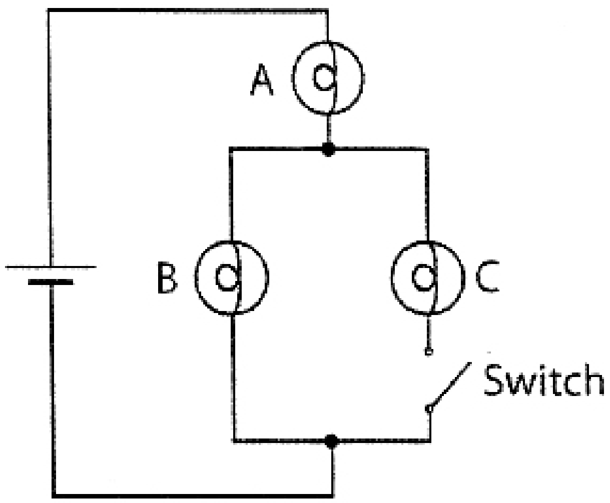

FIGURE 1. The circuit contains an ideal battery, three identical light bulbs and a switch. Initially the switch is open. After the switch closes, does the brightness of bulb A increase? Explain. This question was given as a post-test written examination to a total of 23 students at the University of Maine this year (2011). Only two students attempted to answer the question quantitatively. The others applied rather unsuccessfully a *Case Based Reasoning* approach [66], trying to answer the question recalling from memory some classification patterns devised for conceptual understanding of this type of problems [67].

cuit of Fig. 1 would be to ask about values for the resistance of bulbs B and C in order to have a maximum power in that section of the circuit when closing the switch. This is a kind of open-ended problem which has multiple solutions.)

From the cognitive point of view, the helpfulness of quantitative reasoning in conceptual understanding can be grounded in the assumption that "...the use of mathematics in physics presupposes measurements. Measurements transform difficult to relate perceptual quantities into a common numerical ontology that supports precise comparisons and relations" [70].

Thus, while "guiding students through a process of conceptual change is likely to take time, patience, and creativity" [6], students needs to becomes acquainted in applying what they have been learning quantitatively, so they can readily use that knowledge appropriately in their more advanced courses. To make this possibility a reality one needs to devote time to approaching the process of teaching and learning physics with emphasis on equations and the meaning, not only of the equation but also of each one of the symbols in the equation (for example, the meaning and consequences of $y = y_0 + mx$ when m represents a constant acceleration are different from the situation on which it (m) might represents a constant speed. In each case, the symbols, y, y_0 and x, might have different meanings too. For instance, in the case of m representing constant acceleration, x could represent either position, then y and y_0 should represent the square of a velocity, or time, then y and y_0 should represent a speed). In this way, one could minimize PER findings on students being proficient in manipulating formulas without understanding the meaning of the symbols in the equation [71]. Correspondingly, as pointed out by Professor Hewitt [72] "Isn't teaching emphasis on symbols and their meanings in an introductory [physics] course a worthwhile effort?"

3. On why a problem solving strategy is needed

Another worrying outcome from the learning and instruction of physics and science research literature is the observed lack of ability by students to apply a structured reasoning methodology that could, among other things, help them to identify the nature of a problem as well as the principles and quantitative models to solve it. An example can be found in a study [22] reporting that out of 22 students solving a set of six physics problems, 9 "Analyzes the situation based on required variables. Proceeds by choosing formulas based on the variables in a trial and error manner.", 6 "Proceeds by trying to use the variables in a random way.", 2 "Proceeds by trying to 'fit' the given variables to those examples.", and 5 "Plans and carries out solution in a systematic manner based on that analysis." Similar observations can be drawn from the analysis of interview excerpts reported in other studies [35,17,71].

This lack of a structural way of reasoning is also observed in the responses to the circuit question presented in the previous section (see Fig. 1). In relation to the same question, another student reasoned as follows: "No, Decrease. Adding more resistance drop the current. Using the #'s I used, power decreases ∴ Brightness goes Down." Writing the answer the student wrote Brightness $= P = IR^2$ (she/he wrote the following numbers next to every circuit element $V = 6$ v for the battery and 2 Ω at each bulb). The student continued with $A + B$ =4 Ω and $A + (1/B + 1/C)$ =3 Ω. Then she/he wrote $V = IR$; $V/R = I \Longrightarrow 6v/4$ Ω = 1.5 A; $V/R = I \Longrightarrow$ 6v/3 Ω = 2 A; $P = IR^2 = 1.5 \cdot (4)^2 = 24$ W and $2 \cdot (3)^2 = 18$ W.

In addition, several studies have shown how initial thought processes block students' thinking preventing them from going beyond a circular way of reasoning [73,74]. To mention an example, in analyzing the interview of a student solving a physics problem it is reported that "Dee-Dee was very unwilling to give up her qualitative ideas about force and motion, even though she has already written down the correct algebraic form of Newton's second law. Her qualitative and quantitative dynamics knowledge appear to be associated (she articulated them very close together in time), but they have not been reconciled into a consistent knowledge structure" [21].

Thus, in all of the mentioned cases we can observe in most of the students the lack of a structured and systematic reasoning strategy which could guide them to further analyze, verify, and make sense of the solution procedure they use when solving a problem. Additionally, this fact has also been reported in many studies comparing novice and expert reasoning abilities [69,18,75,39]. The most common observed behavior in students is the plugging of numbers in equations without much hesitation. As a matter of fact, in a study [76] on which students were asked to write down self-reflections on problem-solving one of them wrote "Instead of learning the material and then doing the assignments, I would just try to search for equations in the text that would solve the problem, and if I couldn't figure it out, just guess." Other student wrote "The way I approach physics problems is by looking for a formula to follow. I will start a problem, look in my notes for a formula, look on the discussion board for a formula, look on the formula sheet for a formula, and lastly, look in the book for a formula. This is probably exactly I will approach some problems in Computer Science. If I need to implement a function, I will need to find the syntax for the function and a little excerpt saying what inputs it has and what outputs it has."

Unfortunately, that behavior is encouraged in some instructional settings [77] and throughout the summary of equations at the end of each chapter on commonly used textbooks. And it is further stressed by the way illustrative examples are worked out in those textbooks [72,78,79]. Correspondingly, the problem solving methodologies commonly found in many textbooks have been heavily criticized in PER literature [80,4]. And we agree with such criticism because what is preached and mostly illustrated in commonly recommended textbooks for introductory physics is a mechanical and static problem solving methodology: find an equation, plug in the numbers, and get the answer, taking away the joy of finding and exploring the different ways of solving a problem and the making sense not only of the solution procedure but also of the obtained answer.

More disappointing, textbooks examples end when the answer to the problem is found. At most, in some few cases, there is a checking for dimensional consistency of the result, and in very rare cases one can find a discussion about the feasibility of the obtained result. But in general, there is no further exploration of the solution procedure (*i.e.* whether or not it is a logically applicable to the situation at hand) and no advice is given to students on how they can be certain that the followed solution procedure is correct (as a matter of fact, many wrong solution procedures found in textbooks have been reported in the literature without being corrected by textbook writers [81,84]). More importantly, no advise is given to students regarding the fact that some incorrect solution procedures can lead to a correct solution [79] or that some solution procedures can be applied or transfered to solve other problems in different contexts (just to mention one example, the computation of the gravitational and the electrical field of a mass and a charge distribution respectively share some similarities, but that is not mentioned in most commonly used textbooks. Additional examples on this matter are mentioned in Ref. 85).

Correspondingly, students don't get trained in developing a sense of solving problems by analogy. And they might even think that the mathematics used in physics is different from the mathematics they learn in their calculus classes [86]. Furthermore, in most commonly used textbooks, physics equations are not fully discussed to properly connect the physical or conceptual meaning of each symbol with the place the symbol has in the equation (*i.e.* why and what does it mean that the symbol is multiplying? or why and what does it mean that it appear as a negative exponential factor? What happens if the value of a symbol increases?, and so on). This lack of further analyzing the significance of the symbols in equations might explain difficulty of students reasoning in situations involving multivariable equations [32,87-89]. Moreover, the inadequacy of text-book worked-out examples to enhance students learning has been made evident in a study by Chi and collaborators [90]. In the study it was found that the textbook worked-out examples did not provide any clue for students make generalizations of the underlying domain theory, a necessary training in order to help students to develop important skills to successfully transfer and apply the acquired knowledge in more complex contexts.

Accordingly, the need for explicitly incorporating on the teaching of physics a well structured *dynamic problem solving strategy* (introduced and explained in the next section) is thus justified. As a matter of fact, in the same way as the *Scientific Method* can be used to decide among competing theories, a *dynamic problem solving strategy* is a means by which students can guide their thoughts in deciding on the plausibility or reasonability of each one of the steps taken when solving a (physics) problem. Furthermore, a *dynamic problem solving strategy* helps students to make sense of what they are learning by fitting it into what they already know or believe. Via the questioning of intermediated results and asking questions about what is being done, students can detect flawed/wrong conceptual and/or computational procedures. As students gradually make reflections on what is being done while solving a problem, they start to create or strengthen their own "mental library" of what works and what does not work. As quoted by the Nobel Laureate in Economic Sciences (1978) Herbert Simon (cited in Ref. 6) "*Learning results from what the student does and think and only from what the student does and think. The teacher can advance learning only by influencing what the student does to learn*."

Thus, introducing high school and university students to a *dynamic problem solving strategy* will enhance and strengthen their argumentative thinking skills and will also help them to organize their reasoning skills by focusing their mental effort. It additionally could help to internalize early in students the fact that,

(i) the process for solving scientific problems requires creative thought, the use of available resources (books, computers, articles, etc.), and personal interaction with peers and colleagues.

(ii) There may be no simple answer to questions that have been posed. In some cases the outcome of a calculation can be contrary to what is expected by physical intuition. In other instances an approximation that appears feasible turns out to be unjustified, or one that looks unreasonable turns out to be adequate.

(iii) There might be several competing "correct" answers based on available knowledge and the careful and judicious application of a *dynamic problem solving strategy* is necessary to pick the correct one. And

(iv) the full understanding of a problem and its solution requires both the quantitative formulation of the problem and the conceptual meaning of the symbols that appear in the equation(s) describing the problem quantitatively.

In short, the learning of a *dynamic problem solving strategy* would be the starting point to generate a culture of reasoning and of making sense in the classroom, a necessary condition in "preparation for future learning" [91,92,1,2].

4. A dynamic problem solving strategy

An important issue to be resolved in PER is the lack of an integrative theoretical framework that can make sense of the richness of the large amount of empirical results collected via the many *Research Based Instructional Strategies* (RBIS) generated in the field (for a partial listing of RBIS see [3]). In this regards, some obstacles need to be overcome. In addition to controversial outcomes mentioned in the introduction of this article concerning some RBIS, important ongoing debates alluding to fundamental issues associated with the psychology of learning physics [93,94] indicate that we are still far from reaching such a theoretical framework. Until then, inspired by the scientific method framework, we find it plausible to enhance the teaching and learning of physics via *dynamic problem solving strategies* in order to strengthen students' quantitative problem solving skills, a necessity which is further stressed by the occurrence of some disasters [95,96].

In general terms, a *dynamic problem solving strategy*, not to be confused with *modes of reasoning* (see below), could be constructed from the following steps [79]:

(1) understand and describe the problem;

(2) provide a qualitative description of the problem;

(3) plan a solution;

(4) carry out the plan;

(5) verify the internal consistency and coherence of the equations used and the applied procedures; and

(6) check and evaluate the obtained solution.

One step more or one step less, the aforementioned *problem solving strategy* looks similar, you might rightly wonder, to any other problem solving strategy commonly found in textbooks and which have been heavily criticized in PER literature [80]. In fact, as mentioned earlier we agree with those criticism. For one additional reason, once introduced, the strategy is not consistently applied in the textbook illustrative examples, much less in the student and instructor companion manuals. For another, the steps of the problem solving strategy are presented as rigid steps which need to be followed in the particular order they are written and applied without connection or interaction between them (certainly such an inflexible problem solving strategy can only be of limited value). And finally, for another additional reason, textbooks end of chapter problems are constructed and organized in such a way that students only need to find the right equation to plug-and-chug some numbers to get the answer to the problem. Correspondingly, consequences of such strategies are reflected in the findings of PER studies ([80] and references there in).

On the other hand, a study comparing a typical textbook problem solving strategy with a consistently and coherently applied one reports the overwhelming advantage of the later in relation to the former [97]. To be specific, the authors of the comparative study propose a prescriptive theoretical model of effective human problem solving according to which the problem solving process embraces three major stages:

1) generation of an initial problem description (including qualitative analysis) which is helpful in the construction of a problem solution;

2) obtaining the solution using appropriated methods;

3) evaluation and improvement of the solution. General comments on the nature and significance of each stage are given in Ref. 75.

In the referred work [97], Professors Heller and Reif discussed further details of the controlled study they performed comparing the quantitative problem solving performance of three groups: one guided by their proposed problem solving strategy, other group guided by typical textbooks problem solving directions, and a comparison group working without any external guidance. In addition to showing the remarkable superiority of the participants trained according to the authors methodology, three major lessons can be drawn from the study:

a) that participants guided by a problem solving strategy performed better than those who did not have any guidance;

b) that "completeness and explicitness of procedures for constructing initial problem description", missing from textbooks problem solving strategies, are crucial for attaining better problem solving performance; and

c) that the participants implemented any systematic problem solving procedure fairly easily once they became familiar with it, but to get to that level, the procedure needs to be applied constantly, consistently and coherently because "human subjects tend to be fallible and distractable, prone to forget steps in a procedure or to disregard available information."

The authors comment that after the resistance was overcome, some participants remarked that the steps "really work" and that the problems seemed suddenly "easy" to solve, showing in turn that students beliefs could be changed via a well designed and applied problem solving strategy, while at the same time strengthening their computational skills (similar results are reported in a study in mathematics [98]. The author mentions that "Compared with the students who had received traditional mathematics instruction, the students who had received problem-solving instruction displayed greater perseverance in solving problems, more positive attitudes about the usefulness of mathematics, and more sophisticated definitions of mathematical understanding.")

At this point one now turns to the topic of this section trying to answer the question: what, then, makes a *dynamic problem solving strategy*?

First, the implementation of the methodology is not an inflexible static process (*i.e.* there is not a specific order for implementing the considered steps and one can avoid some of the steps or even add a new one). For instance, working via inductive reasoning one could start from the answer to an unknown problem (*i.e.* an observed natural phenomena) and work backward to provides a well posed problem whose solution is the observed answer. Students can be trained in this process via well designed worked out exercises by presenting them with a careful and detailed backward analysis of the followed solution process. More importantly, verification of a solution procedure via backward analysis in addition to helping students build confidence in the obtained solution, it also guides them to organize information in their memory by making connections with prior knowledge, a condition which increases the accessibility of useful knowledge.

Thus, going back and forth in the application of a *dynamic problem solving strategy* provides the required feedback that experts apply when dealing with the solution of new situations. This flexibility helps to avoid getting trapped or stuck by top-of-the-head thoughts, preformed convictions, or intuitions that don't help in going forward [73,74]. One just continue developing and applying (mentally and/or in writing) thought processes until a solution is found, further analyzed and reconciled with intuitions. We can either change the wrong intuition (i.e. as one student did when solving the exercise of Fig. 1) or strengthen the correct one. Additional guidelines on what sort of analysis should be included can be extracted from the work of Professors Polya [99], Schoenfeld [100,101], and Reif [102]. This way of building understanding can be contrasted with the mechanical way in which textbook's illustrative examples are worked out.

Second, proper application of a *dynamic problem solving strategy* requires the consistent use of any applicable *mode of reasoning*: deductive and/or inductive reasoning; reasoning via analogy and/or via counterexamples; reasoning by *reductio ad absurdum*; and many others, including rule, case, model, and collaborative-collective modes of reasoning. These modes of reasoning are rarely mentioned in commonly used introductory physics textbooks. Nevertheless, in their mathematics courses students might have already studied some of these modes of reasoning (*i.e.* when proving by inductive reasoning the convergence of a sequence), and it is in their physics courses where they should have the opportunity to further explore such modes of reasoning from different perspectives, and became acquainted with them.

Third, a very important skill in applying a *dynamic problem solving strategy* is the ability to ask questions. Questioning, particularly at the higher cognitive levels, is an essential aspect of problem solving. By asking questions while solving a problem one becomes engaged in a process of self-explaining components of the underlying theory being applied to solve the problem and that were not explicitly exposed when learning the theory. Asking questions also help in the detection of "comprehension failures" and to take action to overcome them. As put by Chi and collaborators "Good students ask very specific questions about what they don't understand. These specific questions can potentially be resolved by engaging in self-explanations" [90].

Thus, at each step of a *dynamic problem solving strategy* one needs to stop and ask ourselves about the significance of what has been done so far, trying to find meaningful associations between the new knowledge being applied and related concepts one already might know. Examples of questions to be asked constantly include: how this knew knowledge is related to what I already know?; in which context I have seen this problem before?; in which context could I use this piece of knowledge?; how these seemingly disparate discrete pieces of knowledge be functionally and causally related?; does the principles to be applied can be used in this situation?; is this approach the right one? How can I be certain of it? Are you sure you can do that? Fortunately, the education research literature has provided a good deal of research on how one can help students develop the habit of asking questions [103-107]. But, students will not get the benefit of this process unless they are explicitly taught in using them. In this sense, being an intrinsic part of a *dynamic problem solving strategy*, teaching it will develop on students the habit of asking questions, a process throughout which students could build new useful knowledge that they can then use in further developments as they engage themselves in productive thinking and learning, not only within the teaching and learning environment but also outside it.

We finalize this section in the hope of have made it clear that the delivery of instruction following a *dynamic problem solving strategy* will create the habit on students to look at problems carefully and from multiple perspectives, choosing to be more mindful about the making of sense of each solution procedure as they engage themselves in the exploration of alternatives ways to solve a problem and to find explanations. The development of such way of thinking certainly requires time, effort, practice, self-reflection, and feedback (from peers and the instructor). This is how experts become experts, developing new intuition to cope properly with situations where there is no right answer.

4.1. Illustrative example

In general, most problem solving strategies found in the literature only mention checking the feasibility of the obtained final result. Since some wrong procedures could lead to right results, we have found it necessary to include an additional step (step 5 mentioned at the beginning of this section) in constructing a *dynamic problem solving strategy* so faulty reasoning schemes could be detected [79].

To illustrate the aforementioned thoughts, one could mention the fallacious application of the idea of *separation of variable* (borrowed from solving partial differential equations [108]) to prove in two dimensions the constant acceleration kinematic relationships $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$ and $v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$ (here $\vec{\mathbf{a}} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$, $\vec{\mathbf{v}} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$, and $\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ represent respectively the acceleration, the velocity, and the displacement of a particle, while $\vec{\mathbf{v}}_0 = v_{0x}\hat{\mathbf{x}}+v_{0y}\hat{\mathbf{y}}$, and $\vec{\mathbf{r}}_0 = x_0\hat{\mathbf{x}}+y_0\hat{\mathbf{y}}$ represents respectively initial velocity and initial position of the particle. $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are orthogonal unit vectors).

Starting from $d\vec{\mathbf{v}} = \vec{\mathbf{a}}dt$, dot product both sides of this equation by $\vec{\mathbf{v}}$ and rearrange terms in the form

$$d(v^2/2) = \vec{\mathbf{a}}\cdot\vec{\mathbf{v}}dt = \vec{\mathbf{a}}\cdot d\vec{\mathbf{r}},$$

which after integration, considering $\vec{\mathbf{a}} =$ constant, yields $v^2 = v_0^2 + 2\vec{\mathbf{a}}\cdot(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$. This equation can be written in the form

$$v_x^2 - v_{0x}^2 - 2a_x(x - x_0) = -(v_y^2 - v_{0y}^2 - 2a_y(y - y_0)). \quad (1)$$

So far, there is nothing wrong in any of the given steps (in passing, let's mention that by asking questions according to a *dynamic problem solving strategy* a student could gain conceptual understanding about the obtained relationship: when it is applicable?, why time dependence isn't explicit? Can it be applied to free fall?, can it be applied to projectile motion? if so, under which conditions?, etc.)

The faulty reasoning starts when a student, wrongly invoking the *separation of variable* technique, considers that Eq. (1) is in that form and, according to the technique, the student proceeds to write that $v_x^2 - v_{0x}^2 - 2a_x(x - x_0) = \alpha$ and $v_y^2 - v_{0y}^2 - 2a_y(y - y_0) = -\alpha$, with $\alpha =$ constant. Now, considering that at $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0$, $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0$, then $\alpha = 0$ and the proof is obtained, the student think.

Thus, by means of following typical textbook problem solving strategy, the student, without questioning any of the solution steps, will consider that the proof is correct because the obtained answer is correct, and to show that that it is so the student could point to any introductory physics textbook on which the same result is obtained by other procedures.

At this point one needs to mention that it is really hard to convince students that despite leading to the correct answer, a solution procedure could be mistakenly wrong. How, the student might wonder, can it be that a wrong solution procedure could yield the right response?. That does not make any sense!

How a *dynamic problem solving strategy* can be helpful in making explicit the faulty reasoning? The trick is on the step 5 mentioned at the beginning of this section: once a solution procedure path has been established, one needs to verify the consistency of each given step (a missing step from any textbook problem solving strategy). And this can be attained by working backward after the solution is obtained and by asking questions and each step, wondering and justifying the rightfulness of each one.

In this case, before calling for the separation of variable, the student needs to ask about whether the technique is applicable or not to the situation at hand. Since the method of separation of variable comes from a differential equation context, a first question to ask, following a *dynamic problem solving strategy*, is how to write Eq. (1) so it looks like a differential equation. This can be achieved by writing down the definition for each velocity component in the equation, namely $v_x = dx/dt$ and $v_y = dy/dt$. After doing so it turns out that the working equation is actually a nonlinear first order differential equation on which the time variable t is common to both sides of the equation, making it impossible to apply the separation of variable procedure (which requires that each side of the Eq. (1) be a function of only independent variables each to another. In this case both sides of the equation turns out to be dependent of the time t variable).

Another way to settle the issue following a *dynamic problem solving strategy* is to wonder whether a counter example can be built to rule out the use of the separation of variable in this case. It turns out that recalling basic calculus a very simple algebraic problem can be posed to see if the technique is or not applicable: solve $(2x - x^2) = -(2y - y^2)$.

Applying the proposed separation of variables, instead of an infinite set of solutions, only a finite set of solution will be found (namely $x = 0$ and $y = 0$ or $y = 2$; $x = 2$ and $y = 0$ or $y = 2$).

Nevertheless, in spite of the evidence, some students might be still reluctant to accept the wrongness of their solution procedure and can start to mention inexistent theoretical frameworks, like a *separation of coordinates* without being able to provide any reference to backup such a reasoning. In this situation, only self-reflection and careful analysis of the quantitative procedure will be the way to solve the dilemma.

Episodes like this are not difficult to find in PER literature [71], and they can be described as "reasoning to obtain the desired result", something that brings to memory what is called Einstein's biggest mistake [109,110], which is associated to the introduction by Einstein of a cosmic repulsion term in order to model, in spite of the mathematical evidence, a static universe. Could this be called Einstein's conceptual misunderstanding? or Is it simple a case in which a good thought blocks a better one? [73,74]

5. Concluding remarks

In the understanding that physics is essentially a quantitative based subject, in this article we have largely argue that the teaching of solving problems in introductory physics courses via a *dynamic problem solving strategy* can help students to develop quantitative reasoning skills (necessary in the short term), while enhancing their capabilities to develop conceptual understanding via the analysis of the equation(s) representing physical phenomena followed by the correct interpretation of the physical meaning of each symbol in the respective equation(s) [111] (for instance, as mentioned earlier, the meaning and consequences of $y = y_0 + mx$ when m represents a constant acceleration are different from the situation on which it might represents a constant speed. In each case, the other symbols, y_0 and x, might have different meanings too. For instance, in the case of m representing constant acceleration, x could represent either position, then y and y_0 should represent the square of a velocity, or time, then y and y_0 should represent a speed).

This view of enhancing the teaching and learning of introductory physics courses via *dynamic problem solving strategies* is de accord with the finding that "Good students (those who have greater success at solving problems) tend to study example-exercises in a text by explaining and providing justifications for each action. That is, their explanations refine and expand the conditions of an action, explicate the consequences of an action, provide a goal for a set of actions, relate the consequences of one action to another, and explain the meaning of a set of quantitative expressions" [90].

Our proposal is further reinforced by the fact that teaching directly or indirectly (*i.e.* via scaffolding tutoring) specific problem-solving strategies does improve students' scientific critical thinking and reasoning skills [91,112-114]. As mentioned in a study "When students used the procedural specification, they did so properly and obtained correct answers - although they did not always implemented all steps explicitly and resorted to some shortcuts" [115]. All of that is in agreement with the fact that learning to approach problems in a systematic way starts from learning the interrelationships among conceptual knowledge, mathematical skills and logical reasoning [116]. And it is in physics courses where students can strengthen their quantitative reasoning skills and even become acquainted with innovative non-standard ways of solving problems [117,118].

Thus, since the process of teaching and learning can rarely be done in a completely closed and controlled environment, a well learned *dynamic problem solving strategy* will equipped students with reasoning capabilities for them to properly address advice (easily found on the Internet and in some articles [24]) encouraging to memorize results and to ignore the mathematical analysis leading to them.

In other words, considering that "it is not possible to expect novice students to become more successful problem solvers by simply telling them the principles which govern the way experts sort physics problems" [90], a well structured problem solving strategy taught as a dynamical process offer a feasible way for students to learn physics quantitatively and conceptually, while helping them to reach the state of an *Adaptive Expert* highly skillful on innovation and efficiency: *a preparation for future learning*.

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1. D. L. Schwartz, J. D. Bransford, and D. Sears, *Efficiency and innovation in transfer*. In J. P. Mestre, editor, (Transfer of learning from a modern multidisciplinary perspective, Information Age Publishing, 2005). p. 1-52.
2. G. Hatano and K. Inagaki, *Two courses of expertise*. In H. Stevenson, H. Azuma, and K. Hakuta, editors, Child development and education in japan, (W. H. Freeman and Company, New York. Available at 1986). p. 262-272. http://hdl.handle.net/2115/25206,
3. C. Henderson and M. H. Dancy, *Phys. Rev. ST Phys. Educ. Res.* **5** 020107 (2009) 1-9.
4. J.L. Docktor and J.P. Mestre, *A synthesis of discipline-based education research in physics* (2011). http://www7.nationalacademies.org/bose/DBER_Docktor_October_Paper.pdf
5. T.A. Litzinger, L.R. Lattuca, R.G. Hadgraft, and W.C. Newstetter, *J. Eng. Educ.* **100** (2011) 123-150.
6. S. A. Ambrose, M.W. Bridges, M. DiPietro, M. C. Lovett, and Norman M. K. How learning works: Seven research-based principles for smart teaching. (San Francisco, CA: Jossey-Bass, 2010).
7. M. Sobel, *Physics for the non-scientist: A middle way. Phys. Teach.* **47** (2009) 346-349.
8. N. Lasry, N. Finkelstein, and E. Mazur, *Phys. Teach.* **47** (2009) 418-422.
9. M. Sobel, *Phys. Teach.* **47** (2009) 422-423.
10. N. Finkelstein, E. Mazur, and N. Lasry, *Phys. Teach.* **47** (2009) 484-484.
11. S. Glazek and D. Grayson, *Phys. Today* **61** (2008) 11-12.
12. D. Klein, *Am. J. Phys.* **75** (2007) 101-102.
13. L.J. Atkins, Comment on *School math books, nonsense, and the National Science Foundation*, by David Klein [12]. *Am. J. Phys.* **75** (2007) 773-775.

14. T. Millar, Comment on *School math books, nonsense, and the National Science Foundation*, by David Klein [12]. *Am. J. Phys.* **75** (2007) 775-776.
15. D. Klein, Reply to comments on *School math books, nonsense, and the National Science Foundation*, by David Klein [12]. *Am. J. Phys.* **75** (2007) 776-778.
16. C. Wieman, *APS News* **16** (2007) 8-8.
17. J.F. Wagner, *Cognition Instruct.* **24** (2006) 1-71.
18. F. Reif and S. Allen, *Cognition Instruct.* **9** (1992) 1-44.
19. C. Hoellwarth and M.J. Moelter, *Am. J. Phys*. **79** (2011) 540-545.
20. R. Mualem and B.S. Eylon, *Phys. Teach.* **45** (2007) 158-163. See also references there in.
21. M.S. Sabella and E.F. Redish, *Am. J. Phys.* **75** (2007) 1017-1029.
22. L.N.Walsh, R.G. Howard, and B. Bowe, *Phys. Rev. ST Phys. Educ. Res.* **3** (2007) 1-12.
23. C. Hoellwarth, M.J. Moelter, and R.D. Knight, *Am. J. Phys.* **73** (2005) 459-462.
24. K.S. Taber, *Phys. Educ.* **44** (2009) 336.
25. R.D. Knight, *Instructor Guide. Physics for scientists and engineers. A strategic approach* second edition. (Pearson, Addison-Wesley, 2008).
26. W.K. Adams, K.K. Perkins, N.S. Podolefsky, M. Dubson, N.D. Finkelstein, and C.E. Wieman, *Phys. Rev. ST Phys. Educ. Res.* **2** (2006) 1-14.
27. S. Ates and E. Cataloglu, *Eur. J. Phys.* **28** (2007) 1161-1171.
28. V.P. Coletta, J.A. Phillips, A. Savinainen, and J.J. Steinert. *Eur. J. Phys.* **29** (2008) L25-L27.
29. S. Ates and E. Cataloglu, *Eur. J. Phys.* **29** (2008) L29-L31.
30. L.E. Kost-Smith, S.J. Pollock, and N.D. Finkelstein, *Phys. Rev. ST Phys. Educ. Res.* **6** (2010) 1-17.
31. C.S. Wallace and S.V. Chasteen, *Phys. Rev. ST Phys. Educ. Res.* **6** (2010) 1-8.
32. J.R. Thompson, W.M. Christensen, E.B. Pollock, B.R. Bucy, and D.B. Mountcastle, *Student understanding of thermal physics concepts and the underlying mathematics in the upper division* In Proceedings of the Frontiers in Science Education Research Conference, 24-24 March 2009. (Eastern Mediterranean University, Famagusta, North Cyprus, 2009).
33. D.C. Meredith and K.A. Marrongelle, *Am. J. Phys.* **76** (2008) 570-578.
34. J.A. Marshall and D.J. Carrejo, *J. Res. Sci. Teach.* **45** (2008) 153-173.
35. A. Boudreaux, P.S. Shaffer, P.R.L. Heron, and L.C. McDermott, *Am. J. Phys.* **76** (2008) 163-170.
36. P.S. Shaffer and L.C. McDermott, *Am. J. Phys.* **73** (2005) 921-931.
37. L.C. McDermott and P.S. Shaffer, *Am. J. Phys.* **60** (1992) 994-1003.
38. J.H.P. vanWeeren, F.F.M. de Mul, M.J. Peters, H. Kramers-Pals, and H.J. Roossink, *Am. J. Phys.* **50** (1982) 725-732.
39. J. Larkin, J. McDermott, D.P. Simon, and H.A. Simon, *Science* **208** (1980) 1335-1342.
40. Z. Hrepic, D. Zollman, and N. Rebello, *Journal of Science Education and Technology* **16** (2007) 213-224.
41. T.S. Biró. *Is there a temperature?* (Springer, 2011).
42. A. Plotnitsky, *Epistemology and Probability* (Springer New York, 2010).
43. M. Braga, A. Guerra, and José Reis, *Sci. & Educ.* (2010) 1-14.
44. G. Bacciagaluppi and A. Valentini, *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference* (Cambridge University Press, 2009). Available at http://arxiv.org/abs/quant-ph/0609184v2.
45. D. Lindley, *Boltzmanns Atom: The Great Debate That Launched A Revolution In Physics* (Free Press, 2001).
46. K.L. Caneva, *The British Journal for the History of Science* **13** (1980) 121-138.
47. E.T. Whittaker, *A history of the theories of aether and electricity : from the age of Descartes to the close of the nineteenth century.* (Longmans, Green and CO. 1910). Available at http://www.archive.org/details/historyoftheorie00whitrich,
48. M. Janssen, *Physics in Perspective (PIP)* **4** (2002) 421-446.
49. M. Lange, *Am. J. Phys.* **79** (2011) 380-388.
50. M. Bunge, *Am. J. Phys.* **34** (1966) 585-596.
51. R.H. Swendsen, *Am. J. Phys.* **79** (2011) 342-348.
52. A. Hobson, by N.G. van kampen [*am. j. phys*. **76** (2008) 989-990]. *Am. J. Phys.* **77** (2009) 293-293.
53. N.G. van Kampen, *Am. J. Phys.* **76** (2008) 989-990.
54. R. Baierlein, *Am. J. Phys.* **74** (2006) 193-195.
55. F. Laloë, *Am. J. Phys.* **69** (2001) 655-701.
56. J.D. Jackson, *Eur. J. Phys.* **21** (2000) L29.
57. J. Roche, *Eur. J. Phys.* **21** (2000) L27.
58. J.D. Jackson, *Eur. J. Phys.* **20** (1999) 495.
59. F.J. Dyson, *to teach or not to teach*, Freeman J. Dyson's acceptance speech for the 1991 Oersted Medal presented by the *American Association of Physics Teachers*, 22 january 1991. *Am. J. Phys.* **59** (1991) 491-495.
60. A.F. Chalmers, *What Is This Thing Called Science? Hackett Pub Co Inc*, 3 edition, (1999).
61. C.J. Giunta, *J. Chem. Educ.* **78** (2001) 623.
62. N. Kipnis, *Sci. & Educ.* **20** (2011) 655-685.
63. H. Bassow, *J. Chem. Educ.* **68** (1991) 273.
64. R. Peierls, *More Surprises in Theoretical Physics* (Princeton University Press, 1991).
65. R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, 1979).
66. J.L. Kolodner, *Artif. Intell. Review* **6** (1992) 3-34.
67. P.S. Shaffer and L.C. McDermott, *Am. J. Phys.* **60** (1992) 1003-1013.
68. J. Hawkins, (In preparation, 2011).
69. C. Singh, *Am. J. Phys.* **70** (2002) 1103-1109.

70. D.L. Schwartz, M. Taylor, and P. Jay, *J. Cogn. Dev.* **6** (2005) 65-88.
71. B.L. Sherin, *Cognition Instruct.* **19** (2001) 479-541.
72. P.G. Hewitt, *Phys. Teach.* **49** (2011) 264-264.
73. D. Kahneman, Maps of bounded rationality: A perspective on intuitive judgment and choice. *Economic Sciences Nobel Price Lecture*, December 8, (2002) (pages 449-489). The lecture is available at http://nobelprize.org/nobel_prizes/economics/laureates/2002/kahnemann-lecture.pdf
74. M. Bilalic, P. McLeod, and F. Gobet, *Cognition* **108** (2008) 652-661.
75. F. Reif and J.I. Heller, *Educ. Psychol.* **17** (1982) 102-127.
76. C.A. Ogilvie, *Phys. Rev. ST Phys. Educ. Res.* **5** (2009) 1-14.
77. K.M. Hamed, *Phys. Teach.* **46** (2008) 290-293.
78. A.A. Kamal, *1000 Solved Problems in Classical Physics* (Springer Berlin Heidelberg, 2011).
79. S. Rojas, *Rev. Mex. Fís. E* **56** (2010) 22-28.
80. E. Kim and S. Pak, *Am. J. Phys.* **70** (2002) 759-765.
81. G. Saccomandi, *Eur. J. Phys.* **31** (2010) 657-670.
82. C.F. Bohren, *Am. J. Phys.* **77** (2009) 101-103.
83. S. Hutzler, G. Delaney, D. Weaire, and F. MacLeod, *Am. J. Phys.* **72** (2004) 1508-1516.
84. T.R. Sandin, *Am. J. Phys.* **41** (1973) 426-427.
85. S. Rojas, *A non-standard approach to introduce simple harmonic motion* (2010). eprint arXiv:1011.0687, http://arxiv.org/abs/1011.0687v1
86. S. Rojas, *Rev. Mex. Fís. E* **54** (2008) 75-80.
87. D. Kuhn, M. Pease, and C. Wirkala, *Journal of Experimental Child Psychology* **103** (2009) 268-284.
88. D. Kuhn, K. Iordanou, M. Pease, and C. Wirkala, *Cognitive Development* **23** (2008) 435-451.
89. A.A. diSessa, *Cognition Instruct.* **26** (2008) 560-566.
90. M.T.H. Chi, M. Bassok, M.W. Lewis, P. Reimann, and R. Glaser, *Cognitive Science* **13** (1989) 145-182.
91. J. Osborne, *Science* **328** (2010) 463-466.
92. D. Fortus, *Sci. Educ.* **93** (2009) 86-108.
93. J.D. Slotta, *Journal of the Learning Sciences* **20** (2011) 151-162.
94. D. Hammer, A. Gupta, and E.F. Redish, *Journal of the Learning Sciences* **20** (2011) 163-168.
95. B.J. Feldman, *Phys. Teach.* **48** (2010) 541-542.
96. S.H. Strogatz, D.M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, *Nature* **438** (2005) 43-44.
97. J.I. Heller and F. Reif, *Cognition Instruct.* **1** (1984) 177-216.
98. K.M. Higgins, *The Journal of Experimental Education* **66** (1997) 5-28.
99. G. Polya, *How to Solve it. A new aspect of mathematical method* 2nd. edition (Princeton University Press, 1973).
100. A.H. Schoenfeld, *Teaching problem-solving skills. The American Mathematical Monthly* **87** (1980) 794-805.
101. A.H. Schoenfeld *Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics* (In D. Grouws Ed., Handbook for Research on Mathematics Teaching and Learning, MacMillan, 1992). p.p. 334-370.
102. F. Reif, *Applying cognitive science to education: thinking and learning in scientific and other complex domains* (MIT Press, 2008).
103. A.C. Graesser and N.K. Person, *American Educational Research Journal 31* (1994) 104-137.
104. E. V. Zee and J. Minstrell, *The Journal of the Learning Sciences* **6** (1997) 227-269.
105. K.A. Harper, E. Etkina, and Y. Lin, *J. Res. Sci. Teach.* **40** (2003) 776-791.
106. C. Chin and J. Osborne, *J. Learn. Sci.* **19** (2010) 230-284.
107. C. Chin and J. Osborne, *J. Res. Sci. Teach.* **47** (2010) 883-908.
108. G. Arfken, *Mathematical Methods for Physicist*. 3rd. edition, (Academic Press, 1985).
109. S. Weinberg, *Phys. Today* **58** (2005) 31-35.
110. P.C.W. Davies, *Astrophys. Space Sci.* **244** (1996) 219-227.
111. E. Gaigher, J. Rogan, and M. Braun, *Int. J. Sci. Educ.* **29** (2007) 1089-1110.
112. A. Mason and C. Singh, *Am. J. Phys.* **78** (2010) 748-754.
113. M.T.H. Chi, *Topics in Cognitive Science* **1** (2009) 73-105.
114. R.S. Nickerson, *Review of Research in Education* **15** (1988) 3-57.
115. P. Labudde, F. Reif, and L. Quinn, *Int. J. Sci. Educ.* **66** (1988) 181-221.
116. P.R.L. Heron and D.E. Meltzer, *Am. J. Phys.* **73** (2005) 390-394.
117. F. Olness and R. Scalise, *Am. J. Phys.* **79** (2011) 306-312.
118. M.A. Bernstein and W.A. Friedman, *Thinking About Equations: A Practical Guide for Developing Mathematical Intuition in the Physical Sciences and Engineering* (JohnWiley & Sons, New York, 2009).