Quantum echoes in classical and semiclassical statistical treatments

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Some quantal systems require only a small part of the full quantum theory for their analysis in classical terms. In such understanding we discuss some recent literature on semiclassical treatments and add some results of our own. This analysis allows one to see that some important quantum features of the harmonic oscillator, a system of great didactic value, can indeed be already encountered at the classical or semiclassical statistical levels.

Keywords: Information theory; phase space; semiclassical information; delocalization; Fisher measure.

Algunos sistemas cuánticos requieren sólo una pequeña parte de la teoría cuántica completa para su análisis en términos clásicos. Para una mejor comprensión discutimos algunos tratamientos semiclásicos de la literatura reciente y añadimos algunos resultados propios. Este análisis permite ver que algunas importantes características cuánticas del oscilador armónico, tan importante para la didáctica de conceptos físico, de hecho ya pueden ser encontradas en los niveles estadísticos clásicos o semiclásicos.

Descriptores: Teoría de la información; espacio de fases; información semiclásica; delocalización; medida de Fisher.

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1. Introduction

It has been pointed out long ago that some quantal systems require only a small part of the full quantum theory for their analysis in classical terms [1]. Some exciting contemporary ideas in similar vein are those of [2]. Here we wish to present a didactic discussion concerning these issues in what we hope is an original manner, suitable for students that have had one/two semester(s) of instruction in boths quantum mechanics and statistical mechanics.

With this notion in mind, and with reference to some recent work [3–9] we will try to show, after inspection, reflection, and re-elaboration, that some typical quantal peculiarities can be explained, to a rather surprising extent, by recourse to just classical or semiclassical considerations. We have in mind here such "purely-quantum" concepts as those of decoherence factor, Mandel parameter, and escorts distributions, a well-known tool of contemporary statistical mechanics' research [see, for instance, [10, 11] and references therein]. We will encounter quantum echoes regarding such notions, outside the Schrödinger or Heisenberg representations. Our main research tools will be escort distributions, intertwined with information-quantifiers, of which the semiclassical Wehrl's entropy and Fisher's information measure are singled-put.

The harmonic oscillator (HO) constitutes the focus of our attention. This is, of course, much more than a mere example, since in addition to the extensively used Glauber states in molecular physics and chemistry [12, 13], nowadays the HO is of particular interest for the dynamics of bosonic or fermionic atoms contained in magnetic traps [14, 15], as well as for any system that exhibits an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids. For starters we briefly review below the notions underlying this communication.

1.1. Escort distributions

Given a probability distribution (PD) $f(x)$, there exists an infinite family of associated PDs $f_q(x)$ given by

$$f_q(x) = \frac{f^q(x)}{\int \mathrm{d}x\, f^q(x)}, \tag{1}$$

with q a real parameter, that have proved to be quite useful in the investigation of nonlinear dynamical systems, as they often are better able to discern some of the system's features than the original distribution [10, 16]. It should be emphasized that both types of distributions, f_q and f accrue similar status in contemporary statistical physics' research [10], as they notoriously occur in the formulation of several recent formulations of statistical mechanics [10, 11, 16].

Here we will take advantage of the $q-$degree of freedom to look for effects not visible at $q = 1$ that hopefully emerge at other $q-$values. Additionally, it will be seen that physical considerations constrain the $q-$choice.

1.2. Decoherence

Decoherence is that interesting process whereby the quantum mechanical state of any macroscopic system becomes rapidly correlated with that of its environment in such a manner that no measurement on the system alone (without a simultaneous measurement of the complete state of the environment) can exhibit any interference between two quantum states of the system. Decoherence is a rather exciting phenomenon and a subject of widespread attention [17]. However, it is difficult to provide a quantitative definition of it. All pertinent attempts always depend on the relevant experimental configuration and on the authors' taste [18]. An important related quantity is the square of the density matrix, in whose terms one can define a decoherence parameter $\mathcal{D}$ [19], ranging between 0 (pure states) and one. It is defined as

$$\mathcal{D} = 1 - \frac{\mathrm{Tr}(\hat{\rho}^2)}{(\mathrm{Tr}\,\hat{\rho})^2}. \tag{2}$$

This is a clearly non-negative quantity. The quantity $\mathrm{Tr}(\hat{\rho}^2)$ is often called the purity of $\hat{\rho}$, equal to unity for pure states.

1.3. Mandel parameter and Fano factor

A convenient noise-indicator of a non-classical field is the so-called Mandel parameter which is defined by [20]

$$\mathcal{Q} = \frac{(\Delta\hat{N})^2}{\langle\hat{N}\rangle} - 1 \equiv \mathcal{F} - 1, \tag{3}$$

which is closely related to the normalized variance (also called the quantum Fano factor $\mathcal{F}$ [21]) $\mathcal{F} = (\Delta\hat{N})^2/\langle\hat{N}\rangle$ of the photon distribution. For $\mathcal{F} < 1$ ($\mathcal{Q} \leq 0$), emitted light is referred to as sub-Poissonian since it has photo-count noise smaller than that of coherent (ideal laser) light with the same intensity ($\mathcal{F} = 1;\ \mathcal{Q} = 0$), whereas for $\mathcal{F} > 1$, ($\mathcal{Q} > 0$) the light is called super-Poissonian, exhibiting photo-count noise higher than the coherent-light noise. Of course, one wishes to minimize the Fano factor.

2. Basic tools

We introduce next the basic tools needed for our endeavor.

2.1. Phase-space, coherent states, and Husimi distributions

In phase-space, exact quantum solutions are given by Wigner distributions [22–24]. The paradigmatic semiclassical concept to be appealed to is that of Husimi probability distribution, $\mu(x,p)$, built upon using coherent states [5,25,26]. The pertinent definition reads

$$\mu(x,p) = \langle z|\hat{\rho}|z\rangle, \tag{4}$$

a "semi-classical" phase-space distribution function associated to the density matrix $\hat{\rho}$ of the system [12,26]. Coherent states are eigenstates of the annihilation operator $\hat{a}$, *i.e.*, satisfy $\hat{a}|z\rangle = z|z\rangle$. The distribution $\mu(x,p)$ is normalized in the fashion

$$\int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\,\mu(x,p) = 1. \tag{5}$$

Indeed, $\mu(x,p)$ is a Wigner-distribution D_W smeared over an $\hbar$ sized region of phase space [22]. The smearing renders $\mu(x,p)$ a positive function, even if D_W does not have such a character. The semi-classical Husimi probability distribution refers to a special type of probability: that for simultaneous but approximate location of position and momentum in phase-space [22].

The usual treatment of equilibrium in statistical mechanics makes use of the Gibbs's canonical distribution, whose associated, "thermal" density matrix is given by

$$\hat{\rho} = Z^{-1} e^{-\beta\hat{H}}, \tag{6}$$

with $Z = \mathrm{Tr}(e^{-\beta\hat{H}})$ the partition function, $\beta = 1/k_B T$ the inverse temperature T, and k_B the Boltzmann constant.

2.2. Information quantifiers in phase-space

The operative semiclassical entropic measure is here Wehrls's entropy W, a useful measure of localization in phase-space [27]. Its definition reads

$$W = -\int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\,\mu(x,p)\,\ln\mu(x,p). \tag{7}$$

The uncertainty principle manifests itself through the inequality

$$1 \leq W, \tag{8}$$

which was first conjectured by Wehrl [27] and later proved by Lieb [28]. In order to conveniently write down an expression for W consider an arbitrary Hamiltonian $\hat{H}$ of eigen-energies E_n and eigenstates $|n\rangle$ (n stands for a collection of all the pertinent quantum numbers required to label the states). One can always write [22]

$$\mu(x,p) = \frac{1}{Z}\sum_n e^{-\beta E_n}|\langle z|n\rangle|^2. \tag{9}$$

A useful route to W starts then with Eq. (9) and continues with Eq. (7). In the special case of the harmonic oscillator the coherent states are of the form [26]

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle, \tag{10}$$

where $|n\rangle$ are a complete orthonormal set of eigenstates and whose spectrum of energy is $E_n = (n{+}1/2)\hbar\omega$, $n = 0, 1, \ldots$ In this situation we have the useful analytic expressions obtained in Ref. 22

$$\mu(z) = (1 - e^{-\beta\hbar\omega})\,e^{-(1-e^{-\beta\hbar\omega})|z|^2}, \tag{11}$$

$$W_{HO} = 1 - \ln(1 - e^{-\beta\hbar\omega}). \tag{12}$$

When $T \to 0$, the entropy takes its minimum value W_{HO}=1, expressing purely quantum fluctuations. On the other hand when $T \to \infty$, the entropy tends to the value $-\ln(\beta\hbar\omega)$ which expresses purely thermal fluctuations. Fisher's information measure I is the local counterpart of the global Wehrl quantifier. It is an indicator of how much information is contained in a probability distribution function (PDF) [29]. In phase-space, the local quantifier adopts the appearance [6]

$$I = \frac{1}{4}\int \frac{\mathrm{d}^2 z}{\pi}\,\mu(z)\,\left(\frac{\partial\ln\mu(z)}{\partial|z|}\right)^2, \tag{13}$$

so that inserting the $\mu-$expression into the above expression we obtain for the HO the analytic form

$$I_{HO} = 1 - e^{-\beta\hbar\omega}, \tag{14}$$

so that $0 \leq I_{HO} \leq 1$.

2.2.1. A first observation

Introducing (14) into the Wehrl expression we find

$$W_{HO} = 1 - \ln\left(I_{HO}\right), \tag{15}$$

which together with the Lieb inequality seems to be telling us that too much information might be incompatible with the uncertainty principle. Closer inspection shows, however, that the above expression is valid for any values of either β or ω. We will return to this point later on, in connection with escort distributions.

2.3. Escort Husimi distributions

Things can indeed be improved in the above described scenario by recourse to this concept of escort distribution, introducing it in conjunction with semiclassical Husimi distributions. Thereby one might try to gather "improved" *semiclassical information* from escort Husimi distributions ($q-$HDs) $\gamma_q(x,p)$:

$$\gamma_q(x,p) = \frac{\mu(x,p)^q}{\int \frac{\mathrm{d}^2 z}{\pi}\, \mu(x,p)^q}, \tag{16}$$

where $\mathrm{d}^2 z/\pi = \mathrm{d}x\mathrm{d}p/2\pi\hbar$ and whose HO-analytic form can be obtained from Ref. 7, *i.e.*,

$$\gamma_q(z) = q(1 - e^{-\beta\hbar\omega})e^{-q(1-e^{-\beta\hbar\omega})|z|^2}. \tag{17}$$

As for the associated escort-Fisher measure $I_{sc}^{(q)}$ one easily gets

$$I_{sc}^{(q)} = \frac{1}{4}\int \frac{\mathrm{d}^2 z}{\pi}\,\gamma_q(z)\left(\frac{\partial \ln \gamma_q(z)}{\partial |z|}\right)^2, \tag{18}$$

that using (17) leads to

$$I_{sc}^{(q)} = q(1 - e^{-\beta\hbar\omega}) = q\, I_{HO}, \tag{19}$$

entailing that $0 < I_{sc}^{(q)} \leq q$.

2.4. Coherent states and Mandel parameter

For a coherent state (a pure quantum state) the Mandel parameter vanishes, *i.e.*, $\mathcal{Q} = 0$ and $\mathcal{F} = 1$. A field in a coherent state is considered to be the closest possible quantum-state to a classical field, since it saturates the Heisenberg uncertainty relation and has the same uncertainty in each quadrature component. It should be clear that both $\mathcal{Q}$ and $\mathcal{F}$ function as indicators on non-classicality. Indeed, for a thermal state one has $\mathcal{Q} > 0$ and $\mathcal{F} > 1$, corresponding to a photon distribution broader than the Poissonian. For $\mathcal{Q} < 0$, ($\mathcal{F} < 1$) the photon distribution becomes narrower than that of a Poisson-PDF and the associated state is non-classical. The most elementary examples of non-classical states are number states. Since they are eigenstates of the photon number operator $\hat{N}$, the fluctuations in $\hat{N}$ vanish and the Mandel parameter reads $\mathcal{Q} = -1$ ($\mathcal{F} = 0$) [23]. Taking into account that the number operator is connected with the harmonic oscillator Hamiltonian $\hat{H}$ via $\hat{N} = \hat{H}/\hbar\omega - 1/2$, we can rewrite the HO-Mandel parameter in this fashion

$$\mathcal{Q} = \mathcal{F} - 1 = \frac{(\Delta\hat{H})^2}{\hbar\omega\langle\hat{H}\rangle - \hbar^2\omega^2/2} - 1, \tag{20}$$

where we have used that $\hat{H} = \hbar\omega|z|^2$ [8]. Of course, classically the hamiltonian phase-space function is

$$\mathcal{H}(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{21}$$

3. Decoherence parameter

We shall calculate the decoherence (2) in three different versions: quantum, classical, and semiclassical. In the two last instances, one replaces $\hat{\rho}$ by an ordinary, normalized PDF f and the trace operation by integration over phase space, *i.e.*,

$$\mathcal{D} = 1 - \frac{\int \frac{\mathrm{d}x\mathrm{d}p}{h}\, f^2}{(\int \frac{\mathrm{d}x\mathrm{d}p}{h}\, f)^2}. \tag{22}$$

Classically, $\mathcal{D}$ is not guaranteed to be of a nonnegative character. *As a first new result of this communication we will see now that interesting physical results ensue if we nonetheless demand nonnegativity*.

3.1. Quantal HO-version

We begin with the orthodox quantum recipe. All our calculations are performed in phase-space. For technical details consult, for instance, Ref. 5. The quantum HO- density operator is $\hat{\rho} = e^{-\beta\hat{H}}/Z$, $\hat{H}$ the HO-Hamiltonian, and $Z = e^{-\beta\hbar\omega/2}/(1 - e^{-\beta\hbar\omega})$ the partition function of this system, so that one straightforwardly finds

$$\mathcal{D}_{\text{quant}} = \frac{2}{1 + e^{\beta\hbar\omega}}. \tag{23}$$

It is easy to see that for $\beta \to \infty$ one has $\mathcal{D}_{\text{quant}} = 0$ while for $\beta \to 0$ one has $\mathcal{D}_{\text{quant}} = 1$, as expected. As stated above, intriguing things may happen if we try to replace $\hat{\rho}$ by a classical PDF and the trace operation by integration over phase-space.

3.2. Classical HO-version

Classically (or semiclassically), the delocalization factor can be gotten by using probability distributions instead of density matrices [5]. For the HO one has

$$\mathcal{D}_{\text{class}} = 1 - \frac{1}{Z^2_{\text{class}}} \int \frac{\mathrm{d}x\mathrm{d}p}{h} e^{-2\beta\hbar\omega|z|^2}, \qquad (24)$$

where $Z_{\text{class}} = 1/(\beta\hbar\omega)$ is the classical partition function for the HO. The pertinent computation yields

$$\mathcal{D}_{\text{class}} = 1 - \frac{\beta\hbar\omega}{2}. \qquad (25)$$

Interestingly enough, $\mathcal{D}_{\text{class}} \to 1$ as $T \to \infty$, as in the quantum instance.

3.2.1. First quantum echo

When dealing with Gaussian distributions one finds $\mathcal{D}_{\text{class}} \geq 0$ [Cf. Ref. 2] only in special cases. For $f = Ae^{-a|z|^2}$ one readily finds

$$\mathcal{D}_{class} = 1 - \frac{a}{2}. \qquad (26)$$

Thus, $\mathcal{D}_{\text{class}} \geq 0$ implies $a \leq 2$. In our case, $a = \beta\hbar\omega$ and the requirement turns out to be that the "thermal" energy $k_B T$, *i.e.*, the average classical energy per degree of freedom $\langle e \rangle$, is such that

$$\langle e \rangle_{\text{min}} \geq \frac{\hbar\omega}{2}. \qquad (27)$$

This entails a rather surprising result, ***a minimum possible mean energy per degree of freedom*** $\langle e \rangle_{\text{min}}$. For energies smaller of this value the quantity (22) becomes negative. Thus, we encounter a quantum-flavored result at the classical level. One might be tempted to suggest that the vacuum energy $\hbar\omega/2$ has a statistical origin. Why? Because a minimum possible HO-energy arises just by demanding that the pertinent distribution f verify

$$\left(\int \frac{\mathrm{d}x\mathrm{d}p}{h} f \right)^2 \geq \int \frac{\mathrm{d}x\mathrm{d}p}{h} f^2. \qquad (28)$$

3.3. Semiclassical HO-version

In a semiclassical version, this parameter takes the form

$$\mathcal{D}_{sc} = 1 - \int \frac{\mathrm{d}^2 z}{\pi} \mu(z)^2, \qquad (29)$$

whose analytic expression is

$$\mathcal{D}_{sc} = \frac{1 + e^{-\beta\hbar\omega}}{2}. \qquad (30)$$

One ascertains then that for $T \to \infty$ we have, as expected, $\mathcal{D}_{\text{sc}} = 1$. On the other hand, at $T = 0$ we get $\mathcal{D}_{\text{sc}} = 1/2$.

3.3.1. Second echo

The above result can be interpreted ([6] via the relationship between the decoherence factor and the so-called participation ratio $\mathcal{R}$, that "counts" the number of pure states associated to a density matrix). We find here that just two pure states would "enter" the semiclassical PDF at $T = 0$, if it could be regarded as being of a quantal character, since

$$\mathcal{D} = 1 - \frac{1}{\mathcal{R}}. \qquad (31)$$

3.4. Escort semiclassical HO-version

For more interesting results we turn now our attention to escort distributions in the hope that making $q \neq 1$ may help us to elucidate more details of our problem. The ensuing semiclassical version becomes

$$\mathcal{D}^{(q)}_{sc} = 1 - \int \frac{\mathrm{d}^2 z}{\pi} \gamma_q^2, \qquad (32)$$

i.e.,

$$\mathcal{D}^{(q)}_{sc} = 1 - \frac{q}{2}(1 - e^{-\beta\hbar\omega}) = 1 - \frac{q}{2}\gamma; \quad 0 \leq \gamma \leq 1. \qquad (33)$$

Non-negativity implies $(q/2)\gamma \leq 1$. One can satisfy this relationship and still retain ample liberty to find acceptable triplets of values $\mathcal{D}^{(q)}_{\text{sc}} = x,\, q,\, \beta$.

Additionally, from (33) we find, calling $x = \mathcal{D}^{(q)}_{\text{sc}}$

$$q = \frac{2(1-x)}{1 - e^{-\beta\hbar\omega}}. \qquad (34)$$

Now, in this case the Wehrl entropy and Fisher measure turn out to be, respectively, [6]

$$\begin{aligned} W_q &= 1 - \ln\left[q(1 - e^{-\beta\hbar\omega})\right] \\ I_q &= q(1 - e^{-\beta\hbar\omega}), \end{aligned} \qquad (35)$$

so that the Lieb inequality becomes in this instance

$$\begin{aligned} -\ln\left[q(1 - e^{-\beta\hbar\omega})\right] \geq 0, \quad i.e., \\ -\ln q\gamma \geq 0 \;\Rightarrow\; q\gamma \leq 1, \end{aligned} \qquad (36)$$

which does pose some further constraints on q, namely,

$$q(1 - e^{-\beta\hbar\omega}) = 2(1-x) \leq 1, \qquad (37)$$

that is

$$\mathcal{D}^{(q)}_{sc} \geq 1/2; \;\; \mathcal{R}^{(q)}_{sc} \geq 2. \qquad (38)$$

3.4.1. Third echo

The meaning of the above result is quite interesting. Mathematically, q (and thus I_q) can be larger than what is allowed by (37), since in such vein one only needs asking that $W_q \geq 0$, entailing $q \leq e/\gamma$, instead of $q \leq 1/\gamma$. However, for

$$1/\gamma \leq q \leq e/\gamma, \qquad (39)$$

Lieb's inequality is violated, which is tantamount to asserting that the uncertainty principle is ignored. Thus, we see here that *"too much" information violates Heisenberg's principle* in a semi classical setting.

3.5. Classical escort version

The escort classical HO-phase-space probability distribution reads [9]

$$P_q(x,p) = \frac{e^{-q\beta\hbar\omega|z|^2}}{\int \frac{\mathrm{d}x\mathrm{d}p}{h}\, e^{-q\beta\hbar\omega|z|^2}}, \tag{40}$$

so that, after integration one finds

$$P_q(x,p) = q\beta\hbar\omega\, e^{-q\beta\hbar\omega|z|^2}. \tag{41}$$

Thus, a simple computation for

$$\mathcal{D}^{(q)}_{\text{class}} = 1 - \int (\mathrm{d}x\mathrm{d}p/h) P_q(x,p)^2$$

yields a result that entails a mere re-scaling of the inverse-temperature β by a factor q

$$\mathcal{D}^{(q)}_{\text{class}} = 1 - \frac{q\beta\hbar\omega}{2}. \tag{42}$$

This entails a shifting of the minimum allowable energy.

3.5.1. Fourth echo

Here $\mathcal{D}^{(q)}_{\text{class}} \geq 0$ entails $q \leq k_B T/(\hbar\omega/2)$, so that we obtain a physical restriction on the value of q:

$$q \leq \frac{\langle H\rangle_{class}}{E_0}, \tag{43}$$

where $E_0 = \hbar\omega/2$ is the zero-point energy.

3.6. Quantal escort version

Interestingly enough, the same $\beta-$rescaling occurs in the quantum instance. In this version we have $\hat\rho_q = \hat\rho^q / \operatorname{Tr}\hat\rho^q = e^{-q\beta\hat H}(1-e^{-q\beta\hbar\omega})e^{q\beta\hbar\omega/2}$. Thus, the decoherence factor is defined as $\mathcal{D}^{(q)}_{quant} = 1 - \operatorname{Tr}\hat\rho_q^2$, and we have the analytic expression

$$\mathcal{D}^{(q)}_{\text{quant}} = \frac{2}{1+e^{q\beta\hbar\omega}}. \tag{44}$$

We see that $\mathcal{D}^{(q)}_{\texttt{quant}} \geq 0$ implies $q \geq 0$, still another physical restriction on the $q-$value.

4. Diverging HO-Fano factors

It was found in Ref. 3 that the semiclassical q-Husimi-HO treatment reveals the appearance of "poles", *i.e.*, divergences of the Fano factor for specific $q-$values. We delve further into this issue below.

4.1. Quantal Fano factor

If we take the mean value $\langle \hat H\rangle = \operatorname{Tr}(\hat\rho \hat H)$ we have for the quantal Fano factor the expression

$$\mathcal{F}_{\text{quant}} = \frac{1}{1-e^{-\beta\hbar\omega}}. \tag{45}$$

For our present objectives we note that this quantity "diverges" *only* for $T=\infty$.

4.2. Classical Fano factor

In the classical instance some further considerations become necessary. The HO's classical partition function was given above by $Z_{\text{class}} = 1/\beta\hbar\omega$ [30]. Accordingly,

$$\langle \mathcal{H}\rangle = \frac{\hbar\omega}{Z_{\text{class}}} \int (\mathrm{d}x\mathrm{d}p/h)\, |z|^2\, e^{-\beta\hbar\omega|z|^2} = \frac{1}{\beta}, \tag{46}$$

$$\langle \mathcal{H}^2\rangle = \frac{\hbar^2\omega^2}{Z_{\text{class}}} \int (\mathrm{d}x\mathrm{d}p/h)\, |z|^4\, e^{-\beta\hbar\omega|z|^2} = \frac{2}{\beta^2}, \tag{47}$$

which entails $(\Delta\mathcal{H})^2 = 1/\beta^2$. As a consequence, we have

$$\mathcal{F}_{\text{class}} = \frac{1}{\frac{\hbar\omega}{k_B T} - \frac{\hbar^2\omega^2}{2k_B^2 T^2}}. \tag{48}$$

At low T, $k_B T \ll \hbar\omega T$ and $\mathcal{F}_{\text{class}} = 0$. $\mathcal{F}_{\text{class}}$ diverges at high temperatures. Indeed, it does so at $k_B T = \hbar\omega/2$, when the thermal energy equals the HO-ground state energy.

4.2.1. Fifth echo

This is a quite interesting result. The classical treatment somehow "knows" that this is a strange energy value, meaningless (but unattainable) in the classical world, and reacts with a "pole". In any case, classical considerations do lead to the vacuum HO-energy (again!).

4.3. Semiclassical Fano factor

The semiclassical version $\mathcal{F}_{sc}$ of Fano factor evaluated with Husimi's distribution was found in Refs. 3 and 9

$$\mathcal{F}_{sc} = \frac{(\Delta_\mu N)^2}{\langle \hat N\rangle_\mu}, \tag{49}$$

where $\langle \ldots\rangle_\mu$ denotes the semiclassical mean value of any general observable and the subindex μ indicates that we have taken the Husimi distribution (11) as the weight function. It is then easy to see that $\mathcal{F}_{sc}$ reads

$$\mathcal{F}_{sc} = \frac{2}{(1-e^{-\beta\hbar\omega})(2-(1-e^{-\beta\hbar\omega}))}. \tag{50}$$

No divergences ensue in this instance. However, they will appear if we appeal to escort distributions.

5. Escort Fano factors

5.1. Semiclassical escort Fano factor for the HO

The "escort"-expression for the Fano factor is [3]

$$\mathcal{F}_{sc}^{(q)} = \frac{2}{q(1-e^{-\beta\hbar\omega})(2-q(1-e^{-\beta\hbar\omega}))}. \quad (51)$$

We note that when q tends to unity we have $\mathcal{F}_{\text{sc}}^{(1)} \equiv \mathcal{F}_{\text{sc}}$. We see now the Fano-divergences may occur whenever

$$\frac{2}{q} = G(\beta) = 1 - e^{-\beta\hbar\omega}. \quad (52)$$

Since $0 \leq \exp(-\beta\hbar\omega) \leq 1$

$$0 \leq G(\beta) \leq 1, \quad (53)$$

and

$$2 \leq q \leq \infty. \quad (54)$$

Additionally, the inverse temperature at which the divergence of the Fano factor takes place is given by

$$\beta_{F\,\text{diverg}}(q) = \frac{-\ln(1-2/q)}{\hbar\omega}, \quad (55)$$

a value that obviously ranges in $[0,\infty]$. We conclude that the "classical pole" can be "moved" to any temperature whatsoever by a judicious choice of q, which allows one then to mimic at will the "pole"-behavior in either the classical or the quantum (at $T=\infty$) instances.

5.1.1. Second observation

The escort distribution can mimic, after judicious $q-$selection and for specific physical facets, either quantum or classic behavior.

5.2. Escort-classical Fano factor

The escort-classical HO-phase-space probability distribution found in (41) that reads $P_q(x,p) = q\beta\hbar\omega\, e^{-q\beta\hbar\omega|z|^2}$, and using $\langle f\rangle = \int(\mathrm{d}x\mathrm{d}p/h)\, f(x,p)P_q(x,p)$ one obtains $\langle H\rangle = 1/(q\beta)$, $\langle H^2\rangle = 2/(q^2\beta^2)$, and $(\Delta H)^2 = 1/(q^2\beta^2)$. Consequently, the q-escort classical Fano factor is

$$\mathcal{F}_{\text{class}}^{(q)} = \frac{1}{q\frac{\hbar\omega}{k_BT} - q^2\frac{\hbar^2\omega^2}{2k_B^2T^2}}. \quad (56)$$

The limit $q \to 1$ leads to $\mathcal{F}_{\text{class}}^{(1)} \equiv \mathcal{F}_{\text{class}}$. The Fano "pole" becomes located at $q = 2k_BT/(\hbar\omega)$. Also, here we have

$$\beta_{F\,\text{diverg}}(q) = \frac{2}{q\hbar\omega}, \quad (57)$$

and can be chosen at will.

5.3. Quantal escort-Fano factor

Here we have

$$\mathcal{F}_{\text{quant}}^{(q)} = \frac{1}{1-e^{-q\beta\hbar\omega}}, \quad (58)$$

i.e., we find again a $q\beta-$scaling and nothing interesting happens. $e^{-q\beta\hbar\omega} = 1$ when either $q=0$ or $T\to\infty$.

6. Conclusions

We have focused attention here on two purely quantal concepts: the decoherence parameter $\mathcal{D}$ and the divergence of the Fano factor for specific q or β values. These two notions have been treated at three levels: 1) quantum, 2) classical, and 3) semiclassical. In all instances this was done both for $q=1$ and $q\neq 1$.

We have heard quantum echoes at the classical level and discovered that by changing q we can force the semiclassical results to accommodate either quantum or classical properties.

In related matters concerning stochastic electrodynamics, the illuminating work of T.H. Boyer and L de la Peña et al. (among others) has to be mentioned [1, 31, 32], what we here call echoes emerge there as well. It is safe then to assert then that the classical-quantum links deserve further scrutiny.

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