Fourier description of lock-in

J. A. Dávila Pintle
Benemérita Universidad Autónoma de Puebla, Facultad de Ciencias de la Electrónica,
Puebla, 72570
e-mail: jpintle@ece.buap.mx

Received 3 May 2012; accepted 16 November 2012

In this study, a new interpretation about the operation of a traditional lock-in and dual lock-in is presented from the viewpoint of Fourier analysis. Once the mathematical principles under which these devices operate are understood, we could take full advantage of the magnitude and phase of the Fourier coefficients to measure the physical variables, as shown in the final example of this study. Also, a comparison between signal-to-noise ratio (SNR) of a square reference lock-in and a pure sinusoid lock-in is also presented.

Keywords: Lock-in; description; measurement.

PACS: 07.50Qx; 06.20DK; 07,07Hj.

1. Introduction

It is often the case that undergraduate and graduate students in electronics or experimental physics obtain poor results in their experiments due to the presence of large amounts of noise in their measurements, caused by the relatively large bandwidth of the instruments commonly used in an undergraduate laboratory, such as the oscilloscope and the multimeter, which makes them use lock-in. However, when students begin to use a traditional lock-in (non- traditional lock-in techniques [1,2] are not considered in this work), it is difficult for many of them to understand its operation principles, and this causes, in some circumstances, inappropriate use or under-utilization. There are numerous studies on the use of lock-in, but there are only a few discuss the analysis of its operation [3-5], which, in most cases, is difficult to understand (additional references are summarized in Ref. 4). For this reason, this study focused on providing a simple interpretation of the lock-in based on the basic analytical arguments.

2. Mathematical analysis

Before starting our analysis, it must be pointed out that a lock-in can only be used for measuring periodic signals; if this condition cannot be achieved, then it is not applicable. Once this indispensable condition is satisfied, let us suppose that we want to measure a voltage or current $f(t)$, which, due to its periodicity, can be represented in a Fourier series as follows:

$$f(t) = a_0 + \sum_{m=1}^{\infty} \left\{ a_m \cos(m\omega_0 t) + b_m \sin(m\omega_0 t) \right\} \quad (1)$$

where a_m and b_m are the m-th Fourier coefficients, ω_0 is the fundamental angular frequency of $f(t)$, and T is its period. From Eq. (1), if $f(t)$ is multiplied by a constant α, all Fourier coefficients are multiplied by the same constant, and thus, any of them allows us to measure α, *i.e.*, any physical parameter that affects the amplitude of $f(t)$ can be measured in this way.

Let us derive an expression for the coefficients a_m and b_m. To calculate a_m, the signal $f(t)$ is multiplied by a reference signal, which corresponds to the function associated with the coefficient a_m; in this case, it is $\cos(m\omega_0 t + \phi_0)$, where ϕ_0 is the phase difference that may exist with $f(t)$ (for simplicity, it will be assumed that $\phi_0 = 0$; the general case will be discussed later) and the product is integrated over a period as follows:

$$\int_0^T f(t)\cos(m\omega_o t)\,dt = a_0 \int_0^T \cos(m\omega_o t)\,dt + \sum_l \left\{ \int_0^T a_l \sin(l\omega_o t)\cos(m\omega_o t)\,dt + \int_0^T b_l \cos(l\omega_o t)\sin(m\omega_o t)\,dt \right\}$$

Owing to the orthogonality of the sine and cosine [6] functions, all terms on the right-hand side of the previous equation are zero, except the term of equal frequency ($l = m$) in the expression of the desired coefficient,

$$a_m = 2\left(\frac{1}{T}\int_0^T f(t)\cos(m\omega_o t)\,dt\right) \quad (2)$$

The b_m coefficient is calculated in a similar way, except that here, $f(t)$ is multiplied by the quadrature signal $\sin(m\omega_o t)$, thus obtaining the following equation:

$$b_m = 2\left(\frac{1}{T}\int_0^T f(t)\sin(m\omega_o t)\,dt\right) \quad (3)$$

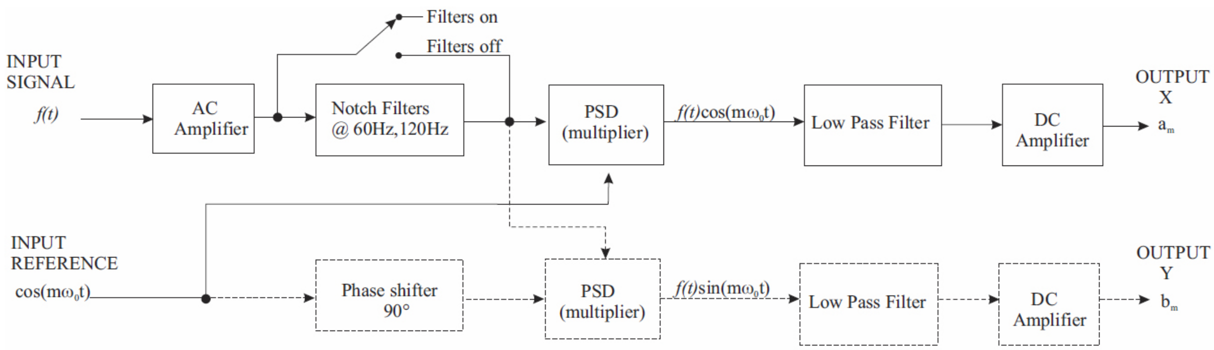

FIGURE 1. General block diagram of a lock-in (dual lock-in includes dashed blocks).

3. Implementation of the analytical method

To demonstrate that a lock-in is an apparatus that physically implements Eq. (2), let us analyze Fig. 1, which corresponds to the general block diagram of a lock-in [5] (in Fig. 1, dual lock and common elements incorporated to a lock in Ref. 15 are also included). The phase detector (PSD) multiplies the input signal $f(t)$ and reference input $\cos(m\omega_0 t)$ [16], hence, it remains to corroborate that the low-pass filter (LPF) carries out the averaging operation, subsequently providing the coefficient a_m, and we proceed as follows.

Taking into account the fact that the LPF used in a lock-in is usually a first-order one, let us consider the simple RC network shown in Fig. 2, which is a representative of all first-order LPFs.

According to the voltage law of Kirchhoff, the input $(v_i)-$output (v_o) relationship of the RC network is:

$$\tau\frac{dv_o}{dt} + v_o = v_i \tag{4}$$

where $\tau = RC$ is the constant time of the network (R is the resistance and C is the capacitance).

For periodic voltages with frequencies greater than unit,

$$\left|\tau\frac{dv_o}{dt}\right| > |v_o| \tag{5}$$

therefore, Eq. (4) can be approximated to

$$\tau\frac{dv_o}{dt} \approx v_i$$

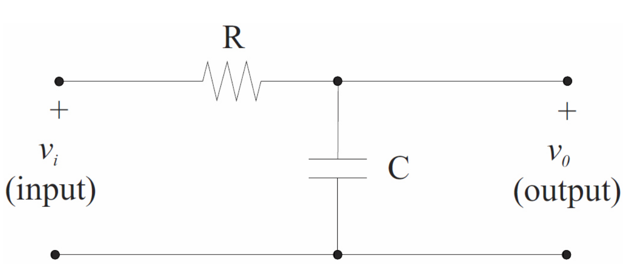

FIGURE 2. First−order low−pass filter.

as τ is increased, the previous approximation trends to equality in such a way that the exit of the LPF corresponds to Eq. (6) in the limit $\tau \to \infty$ and the observation time in which the measurement is performed ($tobs$) trends to infinity [17].

$$v_o = \frac{1}{\tau}\int_0^{tobs} v_i(t)dt. \tag{6}$$

Here, v_0 represents the exact temporal average [18] and corresponds to a_m or b_m according to the case. To conclude this section, we state that a lock-in amplifier implements Eq. (2) [dual lock-in implements Eqs. 2 and 3 named X and Y outputs respectively] divided by $\sqrt{2}$ when calibrated to deliver the root mean square (rms), and therefore, is a Fourier coefficient meter.

4. Noise reduction

In the process of measuring, there are always undesirable signals $e_n(t)$ (noise) that contaminate the signal to be measured, introducing an element of uncertainty in the measurements. A lock-in processes such noisy signals according to Eq. (2), as follows:

$$\frac{2}{T}\int_0^T (f(t) + e_n(t))\cos(m\omega_0 t)\,dt = a_m + \frac{2}{T}\int_0^T e_n(t)\cos(m\omega_0 t)\,dt. \tag{7}$$

The new term of on the right-hand side of Eq. (7) represents the noise at the output of a lock-in (the same goes for Y output of a dual lock-in). The value of this integral is virtually zero because every strange signal that does not have the frequency and phase of reference will be orthogonal to it, and therefore, eliminated. This is the great advantage of this device, although in practice, there are contributions from adjacent components to $m\omega_0$ of $e_n(t)$, which, at the input of LPF, do not meet the condition given in Eq. (5). For example, if $e_n(t)$ represents a pass-band source of noise with constant power density spectrum K, then the power of noise (n^2) at the output of a lock-in with a first-order LPF is [7,8]:

$$n^2 = \frac{K}{2\pi}\left(\frac{\pi}{2}\omega_c\right) \tag{8}$$

The expression in brackets in Eq. (8) is the bandwidth noise [10] and $\omega_c = 1/\tau$ is the cutoff frequency of the LPF; hence, according to Eq. (9), it is now easy to understand why τ has to be chosen as large as possible. However, we must not forget that this implies that the observation time also increases, and therefore, a compromise between output power noise and observation time must be done. It must be remarked that the assumption of noise sources of constant power density is not a restriction, because ω_c is usually less than a few Hz, whereas the power density for most noise sources, *e.g.*, shot noise, is approximately constant up to frequencies of 80 MHz [7].

$$n^2 = \frac{K}{4\tau} \tag{9}$$

5. The phase

Before carrying out an experimental demonstration of our hypothesis, let us reconsider the phase difference ϕ_0 that may exist between the signal to be measured and the reference signal. To simplify the notation, let us make the following definitions: The cosine and sine functions are represented as vectors in a Hilbert space [6]

$$\mathbf{i}_m \equiv \cos m\omega_0 t \tag{10}$$

$$\mathbf{j}_m \equiv \sin m\omega_0 t \tag{11}$$

The scalar product of periodic signals $f(t)$ and $g(t)$ with period T can be denoted as

$$\mathbf{f}\cdot\mathbf{g} = \frac{1}{T}\int_0^T f(t)g(t)dt \tag{12}$$

Thus, with the exception $\mathbf{i}_0\cdot\mathbf{i}_0 = 1$, the orthogonality of sine and cosine functions can be written as

$$\left.\begin{array}{l} \mathbf{i}_l\cdot\mathbf{i}_m = \frac{1}{2}\delta_{lm} \\ \mathbf{j}_l\cdot\mathbf{j}_m = \frac{1}{2}\delta_{lm} \\ \mathbf{i}_l\cdot\mathbf{j}_m = 0 \;\; \forall \;\; m,l \end{array}\right\} \tag{13}$$

where δ_{lm} is the Kronecker delta. Under this notation Eqs. (1) $-$ (3) can be rewritten as

$$\mathbf{f} = a_0\mathbf{i}_0 + \sum_{m=1}^{\infty}(a_m\mathbf{i}_m + b_m\mathbf{j}_m) = a_0\mathbf{i}_0 + \sum_{m=1}^{\infty}\mathbf{s}_m \tag{14}$$

$$a_m = 2\,(\mathbf{f}\cdot\mathbf{i}_m) \tag{15}$$

$$b_m = 2\,(\mathbf{f}\cdot\mathbf{j}_m) \tag{16}$$

where the Fourier Vector Component (FVC) can be defined as

$$\mathbf{s}_m = a_m\mathbf{i}_m + b_m\mathbf{j}_m \tag{17}$$

When $\phi_0 = 0$, (making the analogy with the Cartesian components of a vector), a dual lock-in measures the X and Y components of the FVC divided by $\sqrt{2}$ when calibrated to deliver the rms value (similar to the lock-in model SR530 of Stanford Research Systems), whereas a lock-in only measures the X component.

Let us now investigate the case $\phi_0 \neq 0$, where the reference with a phase shift ϕ_0 is expressed as follows:

$$\cos(m\omega_0 t + \phi_0) = \cos\phi_0\mathbf{i}_m - \sin\phi_0\mathbf{j}_m \tag{18}$$

and the quadrature is

$$\sin(m\omega_0 t + \phi_0) = \sin\phi_0\mathbf{i}_m + \cos\phi_0\mathbf{j}_m \tag{19}$$

The output X is the rms dot product between the signal and the reference given by Eqs. (14) and (18), respectively, therefore,

$$X = \frac{1}{\sqrt{2}}(\cos\phi_0 a_m - \sin\phi_0 b_m) \tag{20}$$

and the quadrature Y output is the rms dot product of Eqs. (14) and (19) [19]

$$Y = \frac{1}{\sqrt{2}}(\sin\phi_0 a_m + \cos\phi_0 b_m) \tag{21}$$

Equation (22) summarizes Eqs. (20) and (21) in matrix form, and represents the general expression of the outputs of a dual lock-in as follows:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{bmatrix}\begin{bmatrix} a_m \\ b_m \end{bmatrix} \tag{22}$$

Thus, we conclude, that $\phi_0 \neq 0$ makes a dual lock-in to measure the rms components X and Y of the anticlockwise$-$rotated FVC (a lock-in, of course, only measures the X component).

6. Signal-to-noise ratio

The quality of a signal in the presence of noise is measured through the signal-to-noise ratio (SNR), defined as the ratio of the output power signal P_s to the output power noise P_n,

$$SNR = \frac{P_s}{P_n} \tag{23}$$

The early lock-in or the most simple and inexpensive ones uses a square wave as the reference signal (we will name it as sq-lock-in). Let us calculate its signal-to-noise ratio (SNR_{sq}) and compare it with the SNR of a lock-in with pure sinusoid reference signal (SNR_s).

We will assume that the square reference signal (r_{sq}) raises from 0 to A in the origin of time ($t = 0$), and let us define its duty cycle as $\delta_r = t_h/T$ where T is its period and t_h is its time of high state, which is 0.5, because for this value, r_{sq} has the simplest Fourier representation and the Fourier coefficients are the maximum (this is why a reference signal with these characteristics is used).

$$\mathbf{r}_{sq} = \frac{A}{2}\mathbf{i}_0 + \sum_{l=1}^{\infty}B_l\mathbf{j}_l \tag{24}$$

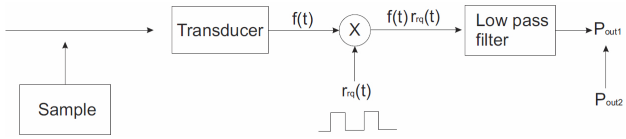

FIGURE 3. Process of measurement a physical variable.

6.1. SNR_{sq} of a square reference lock-in (sq-lock-in)

6.1.1. Signal power

Before calculating the signal power (P_s), let us analyze Fig. 3 which represents the process of measurement of a physical variable [9]. As shown in the figure, we first perform a measurement without the sample, obtaining an output signal at the sq-lock-in whose power is P_{out1}, and subsequently with the sample, the output signal power changes to P_{out2}. Thus,

$$P_s = P_{\text{out2}} - P_{\text{out1}} = \Delta P_{\text{out}} \tag{25}$$

We now proceed to calculate P_s. Let $f(t)$ be the signal delivered by the transducer illustrated in Fig. 3. The power (P_p) of the product of $f(t)$ and $r_{sq}(t)$ is:

$$P_p = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(f(t) r_{sq}(t)\right)^2 dt \tag{26}$$

Taking into account the characteristics of $r_{sq}(t)$, its power is $P_{sq} = A^2/2$ and Eq. (26) can be expressed as

$$P_p = 2P_{sq} \left(\lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} f^2(t) dt \right) \tag{27}$$

The expression between the parenthesis of Eq. (27) represents only a fraction of the total power (P_f) of $f(t)$ given by

$$P_f = \lim_{T\to\infty} \frac{1}{T} \int_0^T f^2(t) dt \tag{28}$$

Hence $P_p \leq 2\ P_{sq}\ P_f$, the equality is only met if $f(t) = k r_{sq}(t)$ where k is a constant of proportionality. Therefore, this is the best signal we can use to measure with a sq-lock-in. From Fig. 3, we can observe that the signal delivered by the sq-lock-in is the zero-frequency component of the product of $f(t)$ and $r_{sq}(t)$ (because the cutoff frequency of the LPF is usually very small). As the component at zero frequency of a square signal contributes half of its total power, we conclude, that for the best signal the output power is precisely

$$P_{\text{out}} = P_{sq} P_f \tag{29}$$

According to Eq. (25) the maximum power of the signal achieved with a sq-lock-in is

$$P_s = P_{sq} \Delta P_f \tag{30}$$

6.1.2. Noise power

For sources of noise with noise spectral density constant and considering the noise at each component of the reference as pass-band noise [7], the noise power at the output of the filter can be calculated as follows:

$$\mathbf{n} = n_o \mathbf{i}_0 + \sum_{l=1}^{\infty} (n_{cl} \mathbf{i}_l + n_{ql} \mathbf{j}_l) \tag{31}$$

Using Eqs. (24) and (31) at the output of the sq-lock-in, the noise can be given as

$$n_{\text{out}} = \mathbf{n} \cdot \mathbf{r}_{sq} = \frac{1}{2} \sum_{l=1}^{\infty} n_{ql} B_l + \frac{1}{2} A n_0 \tag{32}$$

Therefore, the output power noise ($P_{n_{\text{out}}}$) considering that $n_{ql}^2 = n^2$ for every l (n^2 is given in Eq. (9)), is the summation of every component of Eq. (32):

$$P_{n_{\text{out}}} = \left(\frac{1}{4} \sum_{l=1}^{\infty} B_l^2 + \frac{1}{4} A^2 \right) n^2 \tag{33}$$

the expression between the parenthesis of Eq. (33) is $(3/4)P_{sq}$, and for stationary noise sources and non-memory measurements, then, the signal noise is $P_n = 2P_{n_{\text{out}}}$; therefore, the maximum SNR for a sq-lock-in is

$$SNR_{sq} \leq \frac{4}{6} \frac{\Delta P_f}{n^2} \tag{34}$$

6.2. SNRs of a sinusoid reference lock-in (lock-in)

If the physical variable that we want to measure only affects the amplitude of $f(t)$, which occurs commonly, (see Fig. 3), we can use the Perseval theorem to show that the changes in the signal power [Eq. (28)] are equal to those of the power contained in every Fourier coefficient, *i.e.*

$$\Delta P_f = \frac{1}{2} \Delta b_l^2 \tag{35}$$

Therefore, the signal power for the optimum case of sq-lock-in is equal to that measured with the sinusoid lock-in, with reference being in phase with $f(t)$. We already calculated the noise output power n^2 in Eq. (9); thus, for this lock-in, $P_n = 2n^2$ for the same reasons mentioned earlier. Therefore, we conclude that

$$SNR_{sq} \leq (0.667) SNR_s \tag{36}$$

7. Experimental set up

In this section, we will experimentally demonstrate that a lock-in is a Fourier coefficient meter. To carry out this task, we will use an SR530 dual lock-in of Stanford Research Systems to perform the following: First, the magnitude of the first 40 Fourier components of a periodic positive square signal (named SIG) of amplitude A_s = 0.5 V and duty cycle $\delta_s = 0.1$, applied at the input of the SR530, will be measured and compared with the theoretical values. Second, δ_s will be varied to observe its effect on the first FVC [Eq. (17)] and the results will be compared with those of the theory of Fourier predicts, namely [13].

$$a_m = 2\frac{A_s}{m\pi}\sin\left(m\pi\delta_s\right) \tag{37}$$

$$b_m = 0 \tag{38}$$

$$\phi_m = -m\pi\delta_s + \theta_0 \tag{39}$$

The experimental set up is shown in Fig. 4, and was implemented with a signal generator Telulex model GS-100, an SR-530 and a self-designed correlating circuit [20]. It is important to note that in measurements with a lock-in, the reference signal (named REF) and SIG must be correlated to have the same frequency and phase difference, and a common way to meet this requirement is to generate REF from SIG; however, here, the opposite will be done-the signal generator provides REF, which is a periodic positive square signal TTL. SIG is generated from REF through two mono-stable timers (see frame at the bottom of Fig. 4), the first delivers a positive square signal of 100 Hz frequency, regardless of the frequency of REF, which is present at its input, and the second is used to adjust the duty cycle δ_s of the signal of the first mono-stable. In this way, SIG is generated as a positive square wave of constant frequency of 100 Hz whose magnitude in high state is $A_s = 0.5$ V and duty cycle is $\delta_s = 0.1$.

For the first part of the experiment, the frequency of REF ($f_{\rm REF}$) is modified according to the following equation: $f_{\rm REF} = m \times 100$ Hz, where $m = 1, 2, ...40$. For the second part $f_{\rm REF} = 100$ Hz and δ_s is varied from 0.05 to 0.95 in increments of 0.05.

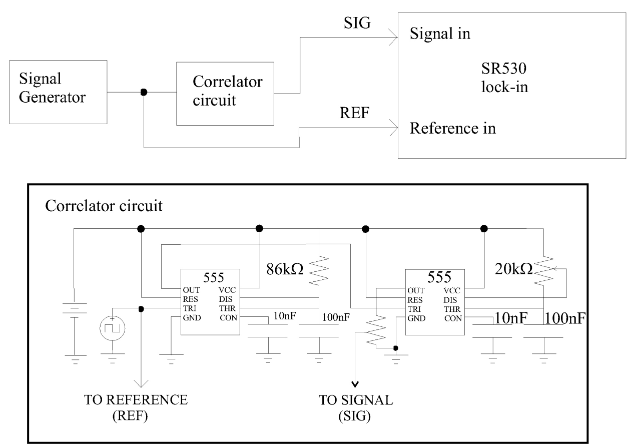

FIGURE 4. Experimental set up for measuring the Fourier coefficients of SIG.

8. Results

In Fig. 5, a comparison between the theoretical values of the magnitude of Fourier coefficients m = 1, 2,... 40 of SIG calculated using Eq. (40) is presented, with respect to the values obtained experimentally by the arrangement described in the previous section, and where two important aspects have to be taken into account: a) To obtain the amplitudes from the rms values that SR530 delivers, the experimental data were multiplied by $\sqrt{2}$, and b) all SR530 Notch filters were disabled. If the filters are not disabled, then the coefficients whose frequencies are within the bands of rejection of those elements are attenuated, because its transmittance is less than 1 within these bands; in our case, this effect was clearly observed on the first coefficient of SIG located at 100Hz, which had 0.83 times greater magnitude (see inset in Fig. 5) than that obtained with the filters disabled.

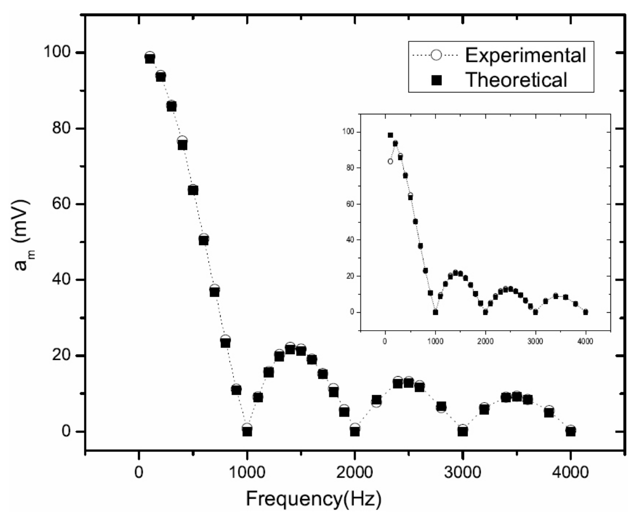

FIGURE 5. Theoretical and experimental amplitude of the Fourier coefficients (a_m) of SIG.

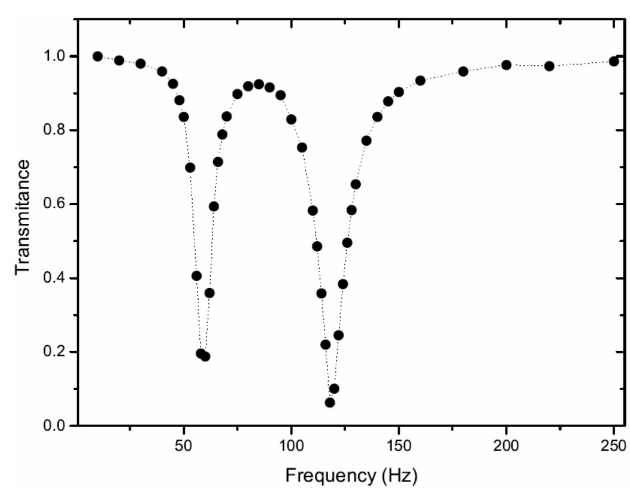

FIGURE 6. Transmittance of the Notch filters of the lock-in SR530.

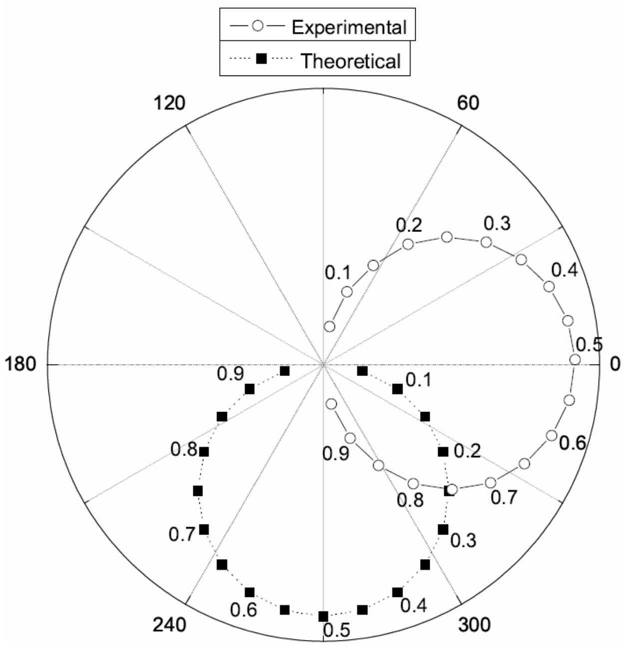

FIGURE 7. Experimental and theoretical polar plot of (a_1,ϕ_1) as function of duty cycle δ_s.

For a better understanding of the effect of the filters, we present the filters transmittance versus frequency (Fig. 6), and as can be corroborated, at 100 Hz, the transmittance was 0.83. Figure 6 was obtained by measuring the normalized amplitude of a_1 when the frequency of SIG was varied from 10 to 250 Hz, having all filters enabled.

In Fig. 7, the theoretical and experimental values of the magnitude and phase of Fourier coefficients (a_1, ϕ_1) are plotted in polar form, which were obtained by changing the duty cycle of SIG (δ_s). We can see an excellent concordance between these values, with the exception of an unimportant 91° shift corresponding to $\theta_0 = 91^\circ$ in Eq. (39). Finally, we conclude that a lock-in is a Fourier coefficient meter and the coefficient is selected by the frequency of the reference applied to its input.

9. Using a lock-in

The following example illustrates the use of a lock-in. Currently, we know that it is common to use optical fibers to transmit pulses of light representing binary information. Such pulses, as they travel in a fiber, suffer attenuation and broadening due to dispersion. To measure such attenuation and broadening with a lock-in, we can send periodic pulses ($p(t)$) at the input of the fiber; $p(t)$ can also be used to generate the reference signal (REF). The attenuated amplitude of the signal at the output of the fiber (SIG) can be considered as $p(t)$ multiplied by a factor $\alpha < 1$; therefore, the attenuation α and the broadening of the pulses can be measured through the magnitude and phase of any of the Fourier coefficients of SIG, respectively, as we did in the experimental section. Also, in reference [13], a good experiment showning the importance of the phase and amplitude has been presented.

Acknowledgments

The author thanks Dr. Veronica Cerdan Rámirez and Dr. Edmundo Reynoso for their valuable recommendations. This work has been sponsored by CONACyT, grant 51757 and VIEP with grant DAPJ-ING-12.

1. B. K. Spears and N. K. Tufillaro, *Am. J. Phys.* **76** (2008) 213-217.
2. M. Osvaldo Sonnaillon and F. Jose Bonnetto, *Am. J. Phys.* **76** (2008) 213-217.
3. M.L. Meade, *Lock-in amplifiers: principles and applications* 1st ed. (Peter Peregrinus,Short Run Press Ltd., England, 1983). pp.15.
4. Y. Kraftmakher, *Am. J. Phys.* **74** (2006) 207-210.
5. J.H. Scofield, *Am. J. Phys.* **62** (1994) 129-133.
6. K.T. Tang, *Mathematical Methods for Engineers and Scientists 3* 1st ed. (Springer-Berlag, Berlin Heidelberg, NewYork, 2007). pp.5,121.
7. B.P. Lathi, *Introducción a la teoría y sistemas de comunicación.* 14th ed. (Limusa,John Wiley and sons. Inc., México, 1995). pp.261.
8. Andreas Mandelis, *Am. J. Phys.* **65** (19943309-3323.
9. Saeed V. Vaseghi, *Advanced Digital Signal Processing and Noise Reduction* 3th ed. (John Wiley and sons, Inc., New York, 2006). pp.1.
10. C.D. Motchenbacher and J.A. Connelly, *Low noise electronic system design* 1st ed. (John Wiley and sons, Inc., New York, 1993). pp.13.
11. K. Sowka, M. Weel, S.Cauchi, L.Cockins and A. Kumarakrishnan, *Can. J. Phys.* **83** (2005) 907-918.
12. Hwei P. Hsu, *Análisis de Fourier.* 1st ed. (Adison Wesley Longman, México, 1995). pp.44.
13. William H. Hayt, Jr, *Análisis de circuitos en ingenieria.* 4th ed. (Mc. Graw Hill., México, 1989). pp.542.
14. Wuqiang Yang, *Am. J. Phys.* **78** (2010) 909-915.
15. The function of the AC and DC amplifiers is simply to amplify signals, while the function of the Notch filters are to reduce 60 and 120 Hz from the electrical network. Nevertheless we shall see that filters have, in some cases repercussions on the final result.
16. It is common to use a PLL in the cannel of reference to generate $\cos(m\omega_0 t)$ from an arbitrary shape signal with frequency $m\omega_0$
17. A practical rule is $tobs$ must be greater than 5τ.

18. It can be demonstrated that any n-order LPF under appropriated conditions carries out n times the average of the average.
19. As a dual lock-in generates the quadrature reference [Eq. (19)] from Eq. (18), it is also dephased.
20. This simple circuit may be used to synchronize the lock-in to third harmonic instead of using analog multipliers and signal filters for applications of laser stabilization using third−derivative absorption [11].