# Numerical evaluation of Bessel function integrals for functions with exponential dependence

J. L. Luna[a], H. H. Corzo[a,b], and R. P. Sagar[a]
[a]*Departamento de Química, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco No.186, Col.Vicentina, Iztapalapa 09340 México D.F., México*
[b]*Department of Chemistry and Biochemistry, Auburn University, Auburn, AL 36849-5312, USA*

Received 22 April 2013; accepted 6 August 2013

A numerical method for the calculation of Bessel function integrals is proposed for trial functions with exponential type behavior and evaluated for functions with and without explicit exponential dependence. This method utilizes the integral representation of the Bessel function to recast the problem as a double integral; one of which is calculated with Gauss-Chebyshev quadrature while the other uses a parameter-dependent Gauss-Laguerre quadrature in the complex plane. Accurate results can be obtained with relatively small orders of quadratures for the studied classes of functions.

*Keywords:* Bessel function integrals; Gaussian quadrature; Hankel transform; Gauss-Laguerre; Gauss-Chebyshev

PACS: 02.30.Uu; 02.60.Jh

## 1. Introduction

Bessel function integrals involving a function $f(x)$,

$$I(\rho,\nu)=\int_0^\infty f(x)J_\nu(\rho x)\mathrm{d}x, \tag{1}$$

occur in many areas of science and engineering. Their numerical evaluation continues to be of interest especially for functions which decay slowly since care must be taken for larger values of the argument, $\rho$, as the integrand begins to oscillate rapidly [1–13]. The particular case when $f(x)$ possesses exponential behavior arises in many contexts.

One avenue of numerical approximation lies in recasting this integral as a double integral using the integral representation of the $\nu$-order Bessel function of the first kind, [9]

$$J_\nu(x)=\frac{x^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}2^{\nu-1}}\int_0^1(1-t^2)^{\nu-1/2} \times\cos(xt)\mathrm{d}t,\quad Re\ \nu>\frac{1}{2}. \tag{2}$$

Substituting this integral representation into Eq. (1) and changing the order of integration yields

$$I(\rho,\nu)=\frac{\rho^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}2^{\nu-1}}\int_0^1(1-t^2)^{\nu-1/2}F(t)\mathrm{d}t \tag{3}$$

where

$$F(t)=\int_0^\infty x^\nu f(x)\cos(\rho xt)\mathrm{d}x. \tag{4}$$

We take a fresh approach for the calculation of the second integral or cosine transform. It is based on the introduction of a parameter, $\alpha$, and transformation into the complex plane and has been previously employed to calculate sine transforms of functions with explicit exponential dependence [14]. The introduction of the parameter is used to model the exponential behavior of the function. We emphasize that the applicability of this method for the transform of functions without explicit exponential dependence has not been explored (as we will do here).

Using Euler's identity, $F(t)$ may be re-written as

$$F(t)=Re\left[\int_0^\infty x^\nu e^{\alpha x}f(x)e^{-(\alpha-\imath\rho t)x}\mathrm{d}x\right]. \tag{5}$$

Transforming into the complex plane by setting $z=(\alpha-\imath\rho t)x$ in Eq. (5), yields

$$F(t)=Re\left[\frac{1}{(\alpha-\imath\rho t)^{\nu+1}} \times\int_0^\infty z^\nu e^{\frac{\alpha z}{\alpha-\imath\rho t}}f\left(\frac{z}{\alpha-\imath\rho t}\right)e^{-z}\mathrm{d}z\right]. \tag{6}$$

The factor $e^{-z}$ is the weight function for Gauss-Laguerre quadrature. Thus Eq. (6) can be approximated by

$$F(t)\approx Re\left[\frac{1}{(\alpha-\imath\rho t)^{\nu+1}} \times\sum_{j=1}^N z_j^\nu e^{\frac{\alpha z_j}{\alpha-\imath\rho t}}f\big(\frac{z_j}{\alpha-\imath\rho t}\big)w_j\right] \tag{7}$$

where $z_j$ and $w_j$ are the abscissae and weights of the $N$-order Gauss-Laguerre quadrature. This methodology is expected to perform well when the function $f(x)$ possesses an exponential-type behavior [14].

The integrand of the outer integral over [0,1] is an even function of $t$ which allows us to rewrite the integral over [-1,1] and multiply by a factor of one half. This integral is now tailor made for the use of Gauss-Chebyshev quadratures for $\nu = 0, 1$ and Gauss-Jacobi quadratures for $\nu \geq 2$ since the factor $(1-t^2)^{\nu-\frac{1}{2}}$ appearing in the integrand is the corresponding weight function of these quadratures.

Thus,

$$I(\rho,\nu) = \frac{\rho^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}2^\nu} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} F(t)\mathrm{d}t \qquad (8)$$

$$I(\rho,\nu) \approx \frac{\rho^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}2^\nu} \sum_{k=1}^{M} F(t_k)v_k \qquad (9)$$

where the $t_k$ and $v_k$ are the abscissae and weights of the $M$-order Gauss-Chebyshev (Jacobi) quadrature.

The purpose of this work is the numerical evaluation of zero- and first-order Bessel function integrals by the proposed methodology for trial functions with explicit exponential behavior. We will also examine if the methodology can be extended to other classes of functions without explicit exponential behavior and which decay slower for larger $x$. A study of the convergence properties of the methodology with the order of the quadratures is also presented along with an analysis of the dependence of the method on the parameter.

## 2. Results and discussion

### 2.1. Zero-order Bessel function

Tables I and II give the six trial functions which were chosen for study. Among these are included functions with and without explicit exponential dependence. The $f_1$ gaussian-type function is a common function which arises in many applications. The $f_6$ function has been previously studied with regard to Bessel function integrals [1].

The results are grouped into two tables and are reported as calculated values and the logarithm of the absolute value of the relative error (E). Analytical expressions for the integrals were obtained with the Mathematica 8.0.4.0 package [15].

The results in Table I are obtained from a $(N,M)$=(15,15) quadrature order combination for the $f_1$ function while $(N,M)$=(25,50) was used for the other functions, for different values of the argument $\rho$. The Gauss-Chebyshev quadrature of the first kind for $\nu$=0 has an analytical expression with $t_k = \cos((2k-1)\pi/(2M))$ and $v_k = \pi/M$.

The efficacy of the method depends on an appropriate selection of the parameter, $\alpha$, and this information is included in the tables. It's particular selection, whether constant or $\rho$-dependent, was done on a trial and error basis seeking to obtain good results. Thus reported values are not with the optimum $\alpha$ for each point but rather $\alpha$'s which yield good results for the arguments $\rho$ which are reported. Analysis of the results with $\alpha$ will be presented later on.

Calculated values are reported to eight figures while calculations for the relative error were carried out with all figures using double precision calculations. The Gauss-Laguerre quadratures used in this work have a precision of fifteen figures.

Table I shows that the gaussian function, $f_1$, is the best performing function (for smaller $\rho$) especially when one considers that a lower order quadrature [(15,15)] is used in this case as compared to the other functions. The accuracy is ob-

TABLE I. Results for $\nu = 0$ using (N,M)=(15,15) order quadratures for $f_1$ and (N,M)=(25,50) for the other functions with $a$=5 in all functions. The first entry is the numerical approximation while the second is obtained from the analytical result.

| $\rho$ | $f_1(x) = e^{-ax^2}$, $\alpha = 20$ | | $f_2(x) = \frac{1-e^{-ax}}{x}$, $\alpha = 2\rho$ | | $f_3(x) = \frac{1}{\sqrt{x^2+a^2}}$, $\alpha = 2\rho$ | |
|---|---|---|---|---|---|---|
| | Calculation | Log[E] | Calculation | Log[E] | Calculation | Log[E] |
| 0.5 | 0.39386722 | -11.7 | 2.99822295 | -9.8 | 0.42571191 | -8.9 |
| | 0.39386722 | | 2.99822295 | | 0.42571191 | |
| 1.0 | 0.38660764 | -13.9 | 2.31243834 | -9.7 | 0.20511342 | -8.6 |
| | 0.38660764 | | 2.31243834 | | 0.20511342 | |
| 2.0 | 0.35951379 | -11.5 | 1.64723115 | -9.5 | 0.10054505 | -8.3 |
| | 0.35951379 | | 1.64723115 | | 0.10054505 | |
| 5.0 | 0.23336989 | -11.2 | 0.88137359 | -9.3 | $4.00323576 \cdot 10^{-2}$ | -8.0 |
| | 0.23336989 | | 0.88137359 | | $4.00323580 \cdot 10^{-2}$ | |
| 10.0 | 0.10702824 | -10.9 | 0.48121182 | -9.0 | $2.00040106 \cdot 10^{-2}$ | -7.9 |
| | 0.10702824 | | 0.48121183 | | $2.00040109 \cdot 10^{-2}$ | |
| 20.0 | $5.06682915 \cdot 10^{-2}$ | -4.1 | 0.24746646 | -8.7 | $1.00005002 \cdot 10^{-2}$ | -7.8 |
| | $5.06645355 \cdot 10^{-2}$ | | 0.24746646 | | $1.00005003 \cdot 10^{-2}$ | |
| 50.0 | $2.66600457 \cdot 10^{-2}$ | -0.5 | $9.98340784 \cdot 10^{-2}$ | -8.3 | $4.00003194 \cdot 10^{-3}$ | -7.8 |
| | $2.00403662 \cdot 10^{-2}$ | | $9.98340789 \cdot 10^{-2}$ | | $4.00003200 \cdot 10^{-3}$ | |

TABLE II. Results for $\nu = 0$ using (N,M)=(25,50) order quadratures for the $f_4$ and $f_5$ functions and (25,100) for the $f_6$ function, with $a$=5 in $f_4$ and $f_5$. Asterisked values correspond to $a = 1$ (**) and $a = 2$ (*). The first entry is the numerical approximation while the second is obtained from the analytical result.

| $\rho$ | $f_4(x) = \frac{x}{\sqrt{x^2+a^2}}, \alpha = 2\rho$ | | $f_5(x) = \frac{1}{x^2+a^2}, \alpha = 2\rho$ | | $f_6(x) = \frac{x}{(x^2+1)^{\frac{3}{2}}}, \alpha = 2\rho$ | |
|---|---|---|---|---|---|---|
| | Calculation | Log[E] | Calculation | Log[E] | Calculation | Log[E] |
| 0.5 | 0.164169968 | -6.7 | $8.75333951 \cdot 10^{-2}$ | -10.1 | 0.60653553 | -5.1 |
| | 0.164169997 | | $8.75333951 \cdot 10^{-2}$ | | 0.60653066 | |
| 1.0 | $6.73793238 \cdot 10^{-3}$ | -5.7 | $4.20831092 \cdot 10^{-2}$ | -9.5 | 0.36787930 | -6.4 |
| | $6.73794700 \cdot 10^{-3}$ | | $4.20831092 \cdot 10^{-2}$ | | 0.36787944 | |
| 2.0 | $2.26929113 \cdot 10^{-5}$ | -3.5 | $2.02252881 \cdot 10^{-2}$ | -8.9 | 0.13533528 | -9.4 |
| | $2.26999649 \cdot 10^{-5}$ | | $2.02252881 \cdot 10^{-2}$ | | 0.13533528 | |
| 5.0 | $1.34758648 \cdot 10^{-3}$ ** | -5.7 | $8.01299232 \cdot 10^{-3}$ | -8.2 | $6.73794693 \cdot 10^{-3}$ | -8.0 |
| | $1.34758940 \cdot 10^{-3}$ ** | | $8.01299237 \cdot 10^{-3}$ | | $6.73794700 \cdot 10^{-3}$ | |
| 10.0 | $4.53858226 \cdot 10^{-6}$ ** | -3.5 | $4.00160577 \cdot 10^{-3}$ | -7.9 | $4.53998007 \cdot 10^{-5}$ | -5.5 |
| | $4.53999298 \cdot 10^{-6}$ ** | | $4.00160582 \cdot 10^{-3}$ | | $4.53999298 \cdot 10^{-5}$ | |
| 20.0 | $-5.2 \cdot 10^{-10}$ ** | - | $2.00020015 \cdot 10^{-3}$ | -7.9 | $1.86490125 \cdot 10^{-9}$ | -1.0 |
| | $1.0 \cdot 10^{-10}$ ** | | $2.00020018 \cdot 10^{-3}$ | | $2.06115362 \cdot 10^{-9}$ | |
| 50.0 | $-1.4 \cdot 10^{-10}$ ** | - | $5.00050038 \cdot 10^{-3}$ * | -7.9 | $-1.2 \cdot 10^{-10}$ | - |
| | $3.9 \cdot 10^{-24}$ ** | | $5.00050045 \cdot 10^{-3}$ * | | $1.9 \cdot 10^{-22}$ | |

TABLE III. Results for $\nu = 0$ using (N,M)=(60,100) order quadratures with $a$=5. The first entry is the numerical approximation while the second is obtained from the analytical result. Asterisked values correspond to $a = 1$ (**) and $a = 2$ (*).

| $\rho$ | $f_4(x) = \frac{x}{\sqrt{x^2+a^2}}, \alpha = 2\rho$ | | $f_5(x) = \frac{1}{x^2+a^2}, \alpha = 2\rho$ | | $f_6(x) = \frac{x}{(x^2+1)^{\frac{3}{2}}}, \alpha = 2\rho$ | |
|---|---|---|---|---|---|---|
| | Calculation | Log[E] | Calculation | Log[E] | Calculation | Log[E] |
| 0.5 | 0.16417000 | -9.4 | $8.75333951 \cdot 10^{-2}$ | -14.0 | 0.60653067 | -7.7 |
| | 0.16417000 | | $8.75333951 \cdot 10^{-2}$ | | 0.60653066 | |
| 1.0 | $6.73794697 \cdot 10^{-3}$ | -8.3 | $4.20831092 \cdot 10^{-2}$ | -13.4 | 0.36787944 | -10.3 |
| | $6.73794700 \cdot 10^{-3}$ | | $4.20831092 \cdot 10^{-2}$ | | 0.36787944 | |
| 2.0 | $2.26999492 \cdot 10^{-5}$ | -6.2 | $2.02252881 \cdot 10^{-2}$ | -12.8 | 0.13533528 | -13.6 |
| | $2.26999649 \cdot 10^{-5}$ | | $2.02252881 \cdot 10^{-2}$ | | 0.13533528 | |
| 5.0 | $1.34758939 \cdot 10^{-3}$ ** | -8.3 | $8.01299237 \cdot 10^{-3}$ | -12.0 | $6.73794700 \cdot 10^{-3}$ | -11.9 |
| | $1.34758940 \cdot 10^{-3}$ ** | | $8.01299237 \cdot 10^{-3}$ | | $6.73794700 \cdot 10^{-3}$ | |
| 10.0 | $4.53998985 \cdot 10^{-6}$ ** | -6.2 | $4.00160582 \cdot 10^{-3}$ | -11.5 | $4.53999298 \cdot 10^{-5}$ | -9.5 |
| | $4.53999298 \cdot 10^{-6}$ ** | | $4.00160582 \cdot 10^{-3}$ | | $4.53999298 \cdot 10^{-5}$ | |
| 20.0 | $1.01503989 \cdot 10^{-10}$ ** | -1.8 | $2.00020018 \cdot 10^{-3}$ | -11.0 | $2.06112328 \cdot 10^{-9}$ | -4.8 |
| | $1.03057681 \cdot 10^{-10}$ ** | | $2.00020018 \cdot 10^{-3}$ | | $2.06115362 \cdot 10^{-9}$ | |
| 50.0 | $-5.9 \cdot 10^{-13}$ ** | - | $5.00050045 \cdot 10^{-3}$ * | -11.0 | $-6.6 \cdot 10^{-14}$ | - |
| | $3.9 \cdot 10^{-24}$ ** | | $5.00050045 \cdot 10^{-3}$ * | | $1.9 \cdot 10^{-22}$ | |

served to decrease as $\rho$ increases and yields poor results for $\rho = 50$. On the other hand, we numerically tested that greater accuracy can be obtained for larger values of $\rho$ by increasing the order of the quadrature. For example, the (25,50) order quadrature yielded a virtually exact result for $\rho = 20$ while for $\rho = 50$, the Log[E] value is -8.5.

Comparing the results for $f_2$ and $f_3$, one can gauge how the method performs for functions with and without exponential dependence. One observes good results even up to larger values of $\rho$ for both types of functions. This suggests that the method performs well even for functions without explicit exponential dependence.

Table II contains more results for slower decaying functions without exponential dependence. Reasonable results for the $f_4$ function are attainable for values of $\rho$ up to ten. Note that the larger values of $\rho$ for the $f_4$ function are with $a = 1$.

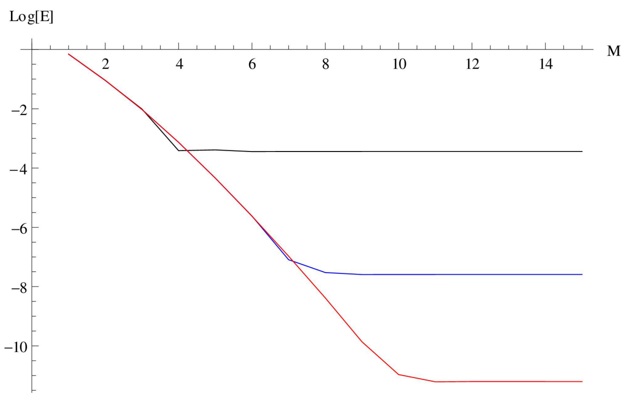


FIGURE 1. Logarithm of the relative error versus order of the Gauss-Chebyshev quadrature (M) for the $f_1$ function with $a = 5$ at argument $\rho = 5$ using $\alpha = 20$. Different curves correspond to different orders of Gauss-Laguerre quadratures; black (N=5); blue (N=10); red (N=15).

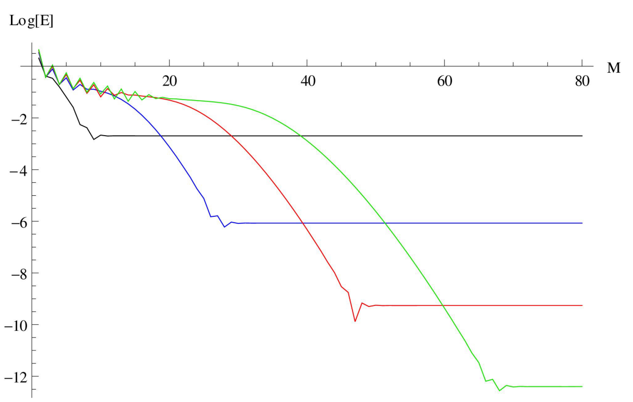


FIGURE 2. Logarithm of the relative error versus order of the Gauss-Chebyshev quadrature (M) for the $f_2$ function with $a = 5$ at argument $\rho = 5$ using $\alpha = 10$. Different curves correspond to different orders of Gauss-Laguerre quadratures; black (N=5); blue (N=15); red (N=25); green (N=35).

This was done because the method performs poorly (with this order of quadrature) when the result is very small ($\leq 10^{-10}$). Even with the use of $a = 1$, one can observe that the values for $\rho = 20$, and especially $\rho = 50$, are extremely poor since the values are small. Note also $\rho = 50$ for $f_6$. Thus the errors for these points are not reported.

$f_6$ proved to be a more difficult function to evaluate thus a higher order Gauss-Chebyshev quadrature was used. The method performs nicely for the $f_5$ function throughout the presented range of $\rho$. It should be noted that the last value is with $a = 2$ due to difficulty with the numerical calculation of the analytical result.

Table III contains the results for the same functions reported in Table II but now with higher orders of quadrature. Comparing the values in the two tables, one appreciates that more accurate results are attainable with the use of higher order quadratures. In particular, the result for $f_4$ at $\rho = 10$, and especially at $\rho = 20$, are now more reasonable in contrast to those in Table II. Also, $f_6$ at $\rho = 10, 20$ shows considerable improvement.

We also tested $f_4$ and $f_6$ with increased (60,120) order quadratures at $\rho = 20$. The values of Log[E] are -6.3 and -7.6 respectively, which illustrates that better approximations at larger arguments can be obtained by increasing the order of quadrature.

Next, we turn our attention to a study of the convergence with respect to the orders of the quadratures. Figures 1 and 2 present plots of the logarithm of the relative error for $f_1$ and $f_2$ as a function of the order of the Gauss-Chebyshev quadrature (M). Each curve represents a different order of Gauss-Laguerre quadrature (N).

The plots show that increasing the order of the Gauss-Chebyshev quadrature yields more accurate results up to a certain point or plateau. Thereafter, increasing the order of the Gauss-Chebyshev quadrature does not increase the accuracy of the result. On comparing the behavior of the different curves, one observes that increasing the order of the Gauss-Laguerre quadrature yields that the plateau is attained at a higher precision of the result.

Figure 2 for the $f_2$ function shows that the convergence with respect to the order of the Gauss-Chebyshev is by no means monotonic. It exhibits oscillatory behavior especially for smaller orders of quadrature.

We now address the behavior of the method as a function of the parameter, $\alpha$. Figures 3 and 4 present plots of $f_2$ and $f_6$ for different values of the argument. These results were obtained with $(N, M)$=(45,100) and (60,100) quadratures which are larger than those used to generate Tables I and II.

Important to note is that there exists an optimal value of $\alpha$ (smallest error) which is centered around a range of $\alpha$ for each value of $\rho$. Second, the plots for different values of $\rho$ are very similar among them and are shifted to the right as $\rho$ increases. Thus optimal values of $\alpha$ increase with $\rho$. This corroborates the use of $\alpha = 2\rho$ for $f_2$ and $f_6$ in Tables 1-3.

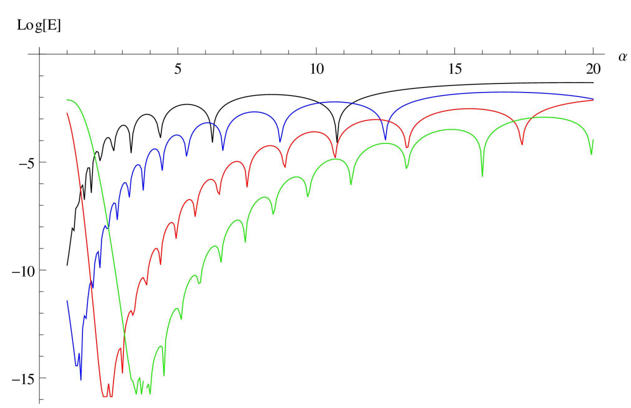


FIGURE 3. Plot of the logarithm of the relative error as a function of parameter, $\alpha$ for the $f_2$ function with quadrature order (45,100), $a = 5$, at different values of $\rho$; black (0.5); blue (1.0); red (2.0); green (3.0).

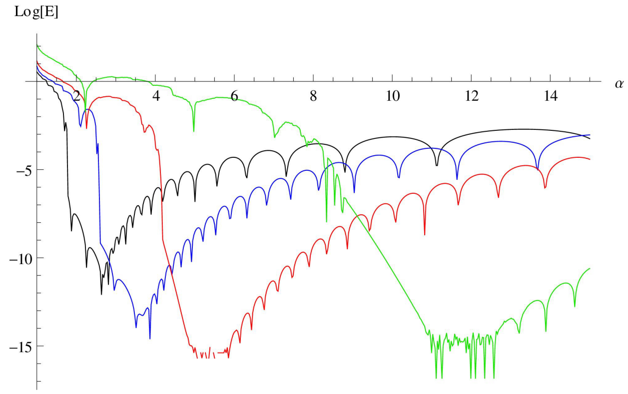


FIGURE 4. Plot of the logarithm of the relative error as a function of parameter, $\alpha$ for the $f_6$ function with quadrature order (60,100), $a = 5$, at different values of $\rho$; black (0.5); blue (1.0); red (2.0); green (3.0).

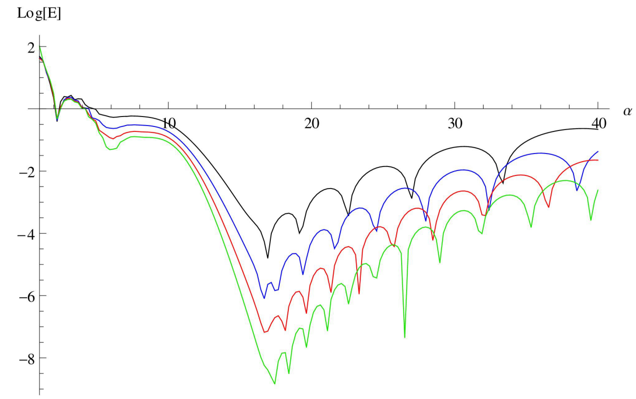


FIGURE 5. Plot of the logarithm of the relative error as a function of parameter, $\alpha$ for the $f_2$ function at $\rho = 5$ with quadrature order: black (25,25), blue (35, 35), red (45,45), green (55,55).

Figure 5 gives the dependence of the relative error with different orders of quadratures for the $f_2$ function. All curves display a similarity in their behavior. However, what is most important to observe is that the range of $\alpha$'s for which decent results are obtained increases with the order of the quadrature. Thus, the quality of the result obtained from higher-order quadratures is less dependent on the selection of a particular value of $\alpha$.

## 2.2. First-order Bessel function

In this section we present the results for integrals of the form in Eq. (1) with $\nu = 1$. The corresponding Gauss-Chebyshev quadratures of the second kind have closed-form solutions with

$$t_k = \cos\left(\frac{k}{M+1}\pi\right) \quad \text{and} \quad v_k = \frac{\pi}{M+1}\sin^2\left(\frac{k}{M+1}\pi\right).$$

The results for six trial functions are included in Tables 4 and 5. These functions were selected based on their behavior and the availability of analytical solutions. These results show that the method performs well even for larger values of the argument with a judicious choice of parameter $\alpha$. The error for the $f_1$ and $f_3$ functions at $\rho = 50$ are not reported since the calculated values are not close to the analytical results. However, it should be emphasized that these results for $f_1$ were obtained with a relatively small order of quadrature [(15,15)]. The (25,50) order quadrature yielded a virtually exact result for $\rho = 20$ and a Log[E] value of -6.3 for $\rho = 50$. On the other hand, using a (60,120) order quadrature for the $f_3$ function gave Log[E] values of -4.4 at $\rho = 20$ and -1.8 at $\rho = 50$.

TABLE IV. Results for $\nu = 1$ using (N,M)=(15,15) order quadratures for $f_1$ and (N,M)=(25,50) for the other functions with $a$=5 in all functions. The first entry is the numerical approximation while the second is obtained from the analytical result.

| $\rho$ | $f_1(x) = e^{-ax^2}$, $\alpha$=20 | | $f_2(x) = e^{-ax^3}$, $\alpha$=40 | | $f_3(x) = e^{-a\sqrt{x}}$, $\alpha$=10 | |
|---|---|---|---|---|---|---|
| | Calculation | Log[E] | Calculation | Log[E] | Calculation | Log[E] |
| 0.5 | $2.48443990 \cdot 10^{-2}$ | -11.1 | $3.83209123 \cdot 10^{-2}$ | -11.4 | $4.62000350 \cdot 10^{-3}$ | -4.5 |
| | $2.48443990 \cdot 10^{-2}$ | | $3.83209123 \cdot 10^{-2}$ | | $4.61984160 \cdot 10^{-3}$ | |
| 1.0 | $4.87705755 \cdot 10^{-2}$ | -10.9 | $7.50420832 \cdot 10^{-2}$ | -11.4 | $8.45346240 \cdot 10^{-3}$ | -4.7 |
| | $4.87705755 \cdot 10^{-2}$ | | $7.50420832 \cdot 10^{-2}$ | | $8.45328920 \cdot 10^{-3}$ | |
| 2.0 | $9.06346235 \cdot 10^{-2}$ | -10.8 | 0.13802496 | -11.6 | $1.35523270 \cdot 10^{-2}$ | -4.6 |
| | $9.06346235 \cdot 10^{-2}$ | | 0.13802496 | | $1.35519560 \cdot 10^{-2}$ | |
| 5.0 | 0.14269904 | -11.8 | 0.19831227 | -12.0 | $1.85546210 \cdot 10^{-2}$ | -4.4 |
| | 0.14269904 | | 0.19831227 | | $1.85538560 \cdot 10^{-2}$ | |
| 10.0 | $9.93262053 \cdot 10^{-2}$ | -10.3 | 0.10368762 | -9.9 | $1.83166310 \cdot 10^{-2}$ | -5.4 |
| | $9.93262053 \cdot 10^{-2}$ | | 0.10368762 | | $1.83167120 \cdot 10^{-2}$ | |
| 20.0 | $5.00408351 \cdot 10^{-2}$ | -3.1 | $5.00937021 \cdot 10^{-2}$ | -8.9 | $1.49203923 \cdot 10^{-2}$ | -2.1 |
| | $4.99999999 \cdot 10^{-2}$ | | $5.00937022 \cdot 10^{-2}$ | | $1.50463891 \cdot 10^{-2}$ | |
| 50.0 | 0.19 | - | $2.00024004 \cdot 10^{-2}$ | -7.7 | $6.45115243 \cdot 10^{-3}$ | - |
| | $2.0 \cdot 10^{-2}$ | | $2.00024000 \cdot 10^{-2}$ | | $9.39105075 \cdot 10^{-3}$ | |

TABLE V. Results for $\nu = 1$ using (N,M)=(60,100) order quadratures with $a$=5 in all functions. The first entry is the numerical approximation while the second is obtained from the analytical result.

| $\rho$ | $f_4(x) = \frac{1-e^{-ax}}{x}$, $\alpha = 2\rho$ | | $f_5(x) = \frac{1}{\sqrt{a^2+x^2}}$, $\alpha = 2\rho$ | | $f_6(x) = \frac{1}{x^2+a^2}$, $\alpha = 2\rho$ | |
|---|---|---|---|---|---|---|
| | Calculation | Log[E] | Calculation | Log[E] | Calculation | Log[E] |
| 0.5 | 0.95012438 | -10.8 | 0.36716600 | -10.4 | $6.52218367 \cdot 10^{-2}$ | -12.1 |
| | 0.95012438 | | 0.36716600 | | $6.52218367 \cdot 10^{-2}$ | |
| 1.0 | 0.90098049 | -10.7 | 0.19865241 | -10.1 | $3.91910773 \cdot 10^{-2}$ | -11.5 |
| | 0.90098049 | | 0.19865241 | | $3.91910773 \cdot 10^{-2}$ | |
| 2.0 | 0.80741760 | -10.7 | $9.99954600 \cdot 10^{-2}$ | -9.8 | $1.99962702 \cdot 10^{-2}$ | -11.0 |
| | 0.80741760 | | $9.99954600 \cdot 10^{-2}$ | | $1.99962702 \cdot 10^{-2}$ | |
| 5.0 | 0.58578644 | -10.6 | $4.00000000 \cdot 10^{-2}$ | -9.4 | $8.00000000 \cdot 10^{-3}$ | -10.1 |
| | 0.58578644 | | $4.00000000 \cdot 10^{-2}$ | | $8.00000000 \cdot 10^{-3}$ | |
| 10.0 | 0.38196601 | -10.4 | $2.00000000 \cdot 10^{-2}$ | -9.1 | $4.00000000 \cdot 10^{-3}$ | -9.6 |
| | 0.38196601 | | $2.00000000 \cdot 10^{-2}$ | | $4.00000000 \cdot 10^{-3}$ | |
| 20.0 | 0.21922359 | -10.1 | $9.99999999 \cdot 10^{-3}$ | -8.8 | $2.00000000 \cdot 10^{-3}$ | -9.1 |
| | 0.21922359 | | $1.00000000 \cdot 10^{-2}$ | | $2.00000000 \cdot 10^{-3}$ | |
| 50.0 | $9.50124379 \cdot 10^{-2}$ | -9.7 | $3.99999999 \cdot 10^{-3}$ | -8.7 | $7.99999999 \cdot 10^{-4}$ | -8.8 |
| | $9.50124379 \cdot 10^{-2}$ | | $4.00000000 \cdot 10^{-3}$ | | $8.00000000 \cdot 10^{-4}$ | |

The method performs better for $f_2$ as compared to $f_3$ for the given order of quadrature. Even at $\rho = 50$, the approximate value for $f_2$ is a good one.

Note also that the results for the slower decaying functions in Table V have been obtained using a larger order quadrature, (60,100), as compared to Table IV [(15,15) and (25,50)]. Overall, these results demonstrate that the method yields good results for functions with and without exponential dependence.

Furthermore, one could also expect better results for the same order of quadrature, if the $z$ factor in Eq. (6) ($\nu = 1$) were incorporated into the weight function. Thus one could use generalized Gauss-Laguerre quadratures from the weight function $ze^{-z}$ in Eq. (7).

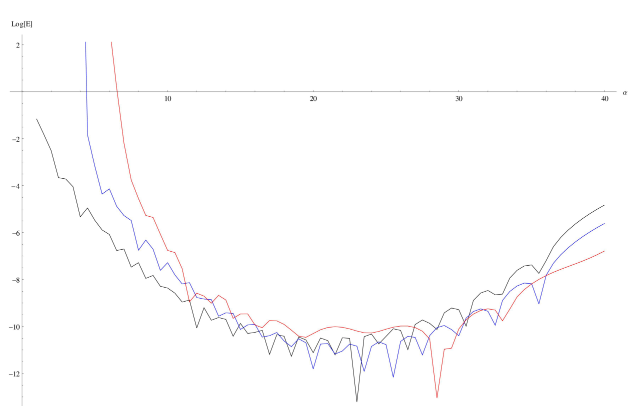

FIGURE 6. Plot of the logarithm of the relative error as a function of parameter, $\alpha$ for the $f_1$ function with quadrature order (15,15), $a = 5$, at different values of $\rho$; black (0.5); blue (5.0); red (10.0).

The results from the method as a function of $\alpha$ can be compared and contrasted for the $f_1$ and $f_6$ functions in Figs. 6 and 7 (for different orders of quadrature). Fig. 6 for the $f_1$ gaussian function shows that the general behavior is somewhat independent of the particular value of $\rho$ and that there is a wide range of $\alpha$-values which yield good results ($\log E \leq$ -10).

On the other hand, Fig. 7 for the $f_6$ function illustrates that the optimal values of $\alpha$ are shifted as $\rho$ increases and that the range of $\alpha$-values which give good results is now narrower in comparison to the $f_1$ function. Ideally, one would wish for a broad range of $\alpha$-optimal values so that the method is not overly sensitive to a slight variation in the parameter. Thus, $f_1$ fulfills these requirements to a larger extent than $f_6$.

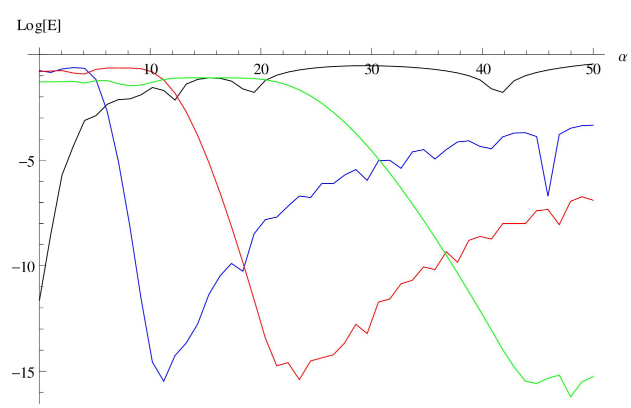


FIGURE 7. Plot of the logarithm of the relative error as a function of parameter, $\alpha$ for the $f_6$ function with quadrature order (60,100), $a = 5$, at different values of $\rho$; black (0.5); blue (5.0); red (10.0); green(20.0).

The convergence of the method as a function of the order of the quadrature is not presented for brevity since these plots are similar to those presented in the previous section for the zero-order Bessel function.

## 3. Conclusions

A method for the numerical evaluation of Bessel function integrals is presented and studied for zero- and first-order Bessel functions. This method uses the integral representation of the Bessel function to formulate the Bessel function integral as a double integral; one which is amenable to Gauss-Chebyshev quadrature while the other is a cosine transform. This transform can be calculated with Gauss-Laguerre quadratures if one uses a transformation into the complex plane. The introduction of a parameter enables one to correctly model the behavior of the function to be integrated. The results for various trial functions with both exponential and non-exponential dependence shows that the method performs well with relatively small orders of quadrature and a judicious choice of the parameter. Notably, the method performs the best for the gaussian-type trial function. A numerical study of the convergence with respect to the orders of quadrature is presented and the dependence of the method on the parameter is also analyzed. A particular benefit is that the method is based on standard Gaussian quadrature which are readily available and easily implemented. The method can also be applied to integrals with Bessel functions of higher order with the use of the appropriate Gauss-Jacobi quadrature.

## 4. Acknowledgment

JLL thanks SNI-CONACyT for a student fellowship.


1. P. Linz, *Math. Comput.* **26** (1972) 509.
2. J.D. Talman, *J. Comput. Phys.* **29** (1978) 35.
3. S.M. Candel, *Comput. Phys. Commun.* **23** (1981) 343.
4. A.V. Oppenheim, G.V. Frisk, and D.R. Martinez, *J. Acoust. Soc. Am.* **68** (1980) 523.
5. R. Piessens, *Comput. Phys. Commun.* **25** (1982) 289.
6. R. Piessens and M. Branders, *J. Comput. Appl. Math.* **11** (1984) 119.
7. M. Puoskari, *J. Comput. Phys.* **75** (1988) 334.
8. J.D. Secada, *Comput. Phys. Commun.* **116** (1999) 278.
9. R. Piessens, *The Transforms and Applications Handbook*, 2nd ed. Ed. A.D. Poularikis,(CRC Press LLC, Boca Raton, 2000) Ch. 9.
10. L. Knockaert, *IEEE Trans. Signal Proc.* **48** (2000) 1695.
11. M. Guizar-Sicairos, J.C. Gutiérrez-Vega, *J. Opt. Soc. Am. A* **21** (2004) 53.
12. V.K. Singh, O.P. Singh, and R.K. Pandey, *Comput. Phys. Commun.* **179** (2008) 424.
13. V.K. Singh, O.P. Singh, and R.K. Pandey, *Comput. Phys. Commun.* **179** (2008) 812.
14. R.P. Sagar, H. Schmider, and V.H. Smith Jr., *J. Phys. A: Math. Gen.* **25** (1992) 189.
15. Wolfram Research, *Inc.*, *Mathematica*, Version 8.0.4.0, (Champaign, IL, USA, 2011).